习题课 1

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2021.10.17

基本概念：

一、一些数学结论

1.坐标系/积分、∇算符

除了直角坐标系，还有两个常用正交曲线坐标系：柱坐标系、球坐标系；其他：环坐标系（可控核聚变 Tokamak）、椭球坐标系（Landau 椭球面电荷面密度）

①微分：（坐标：(q_i, q_j, q_k)，对应方向矢量$\vec{\boldsymbol{e}}_i, \vec{\boldsymbol{e}}_j, \vec{\boldsymbol{e}}_k$）

(1)弧元、面元、体积元表达式

$$dl_i = h_i dq_i$$

（垂直$\vec{\boldsymbol{e}}_i$的面元记为dS_i）

$$d\vec{\boldsymbol{S}}_i = dS_i\vec{\boldsymbol{e}}_i = dl_j dl_k \vec{\boldsymbol{e}}_i = h_j h_k dq_j dq_k \vec{\boldsymbol{e}}_i$$

$$dV = dl_i dl_j dl_k = h_i h_j h_k dq_i dq_j dq_k$$

(2)Lamé 系数，以直角坐标系为跳板

$$h_i = \left|\frac{\partial \vec{\boldsymbol{l}}}{\partial q_i}\right| = \left|\frac{\partial x}{\partial q_i}\hat{\boldsymbol{x}} + \frac{\partial y}{\partial q_i}\hat{\boldsymbol{x}} + \frac{\partial z}{\partial q_i}\hat{\boldsymbol{x}}\right| = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}$$

(3)∇算符：

梯度$\vec{\boldsymbol{E}}=-\nabla U$、散度$\begin{cases}\nabla\cdot\vec{\boldsymbol{E}}=\frac{\rho_e}{\varepsilon_0}\\ \nabla\cdot\vec{\boldsymbol{P}}=\rho_e'\\ \nabla\cdot\vec{\boldsymbol{D}}=\rho_{e0}\end{cases}$、旋度$\nabla\times\vec{\boldsymbol{E}}=\vec{\boldsymbol{0}}$

a)梯度

$$\nabla U=\sum_i\frac{\partial U}{\partial r_i}\vec{\boldsymbol{e}}_i=\sum_i\frac{1}{h_i}\frac{\partial U}{\partial q_i}\vec{\boldsymbol{e}}_i$$

b)散度。定义（邻域的通量）：

$$\nabla\cdot\vec{\boldsymbol{E}}\overset{\text{def}}{\equiv}\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_y}{\partial y}$$

$$\overset{\text{def}}{\equiv}\lim_{V\to 0}\frac{1}{V}\oiint_{\partial V}\vec{\boldsymbol{E}}\cdot\mathrm{d}\vec{\boldsymbol{S}}=\lim_{V\to 0}\frac{1}{V}\oiint_{\partial V}E_i\cdot\mathrm{d}S_i$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3E_1)}{\partial q_1}+\frac{\partial(h_3h_1E_2)}{\partial q_2}+\frac{\partial(h_1h_2E_3)}{\partial q_3}\right]$$

c)旋度

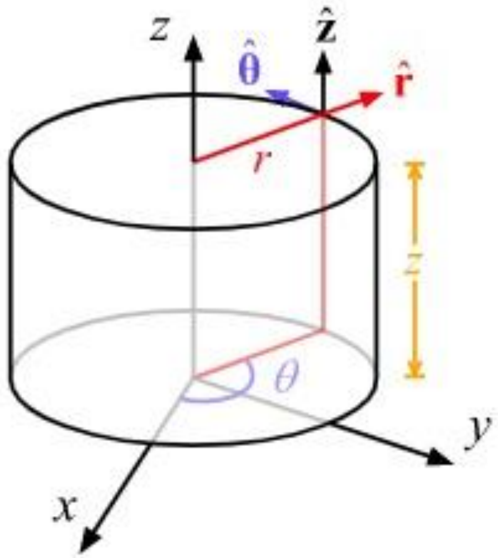
$$\nabla\times\vec{\boldsymbol{E}}\overset{\text{def}}{\equiv}\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x}\\ E_x & E_y & E_z\end{vmatrix}$$

$$\overset{\text{def}}{\equiv}\lim_{S\to 0}\frac{1}{S}\oint_{\partial S}\vec{\boldsymbol{E}}\cdot\mathrm{d}\vec{\boldsymbol{l}}=\lim_{S\to 0}\frac{1}{S}\oint_{\partial S}E_i\cdot\mathrm{d}l_i$$

$$=\frac{1}{h_2h_3}\frac{\partial(h_1E_1)}{\partial q_1}+\frac{1}{h_3h_1}\frac{\partial(h_2E_2)}{\partial q_2}+\frac{1}{h_1h_2}\frac{\partial(h_3E_3)}{\partial q_3}$$

②特殊正交曲线坐标系：

(1)柱坐标系：



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Lamé 系数

$$\begin{cases} h_r = 1 \\ h_\theta = r \\ h_z = 1 \end{cases}$$

柱面上的积分：

$$\iint_S \sigma dS_r = \int_0^{2\pi}\int_0^z \sigma r d\theta dz$$

圆盘上的积分：

$$\iint_S \sigma dS_z = \int_0^{2\pi}\int_0^z \sigma r dr d\theta$$

体积分：

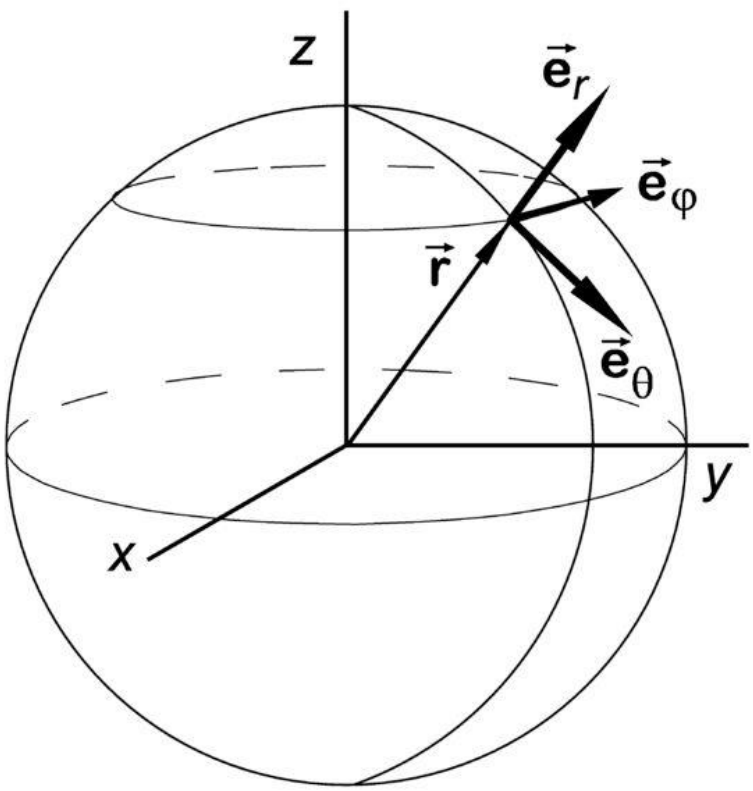
$$\iiint\limits_V \rho dV = \int_0^r \int_0^{2\pi} \int_0^z \sigma r dr d\theta dz$$

∇算符

$$-\nabla U = -\frac{\partial U}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{z}}$$

$$\nabla \cdot \vec{\boldsymbol{P}} = \frac{1}{r}\left[\frac{\partial(rP_r)}{\partial r} + \frac{\partial P_\theta}{\partial \theta} + \frac{\partial(rP_z)}{\partial z}\right] = \frac{1}{r}\frac{\partial(rP_r)}{\partial r} + \frac{1}{r}\frac{\partial P_\theta}{\partial \theta} + \frac{\partial P_z}{\partial z}$$

(1)球坐标系：注意θ为纬度，与柱坐标有区别



$$\begin{cases} x = r\cos\theta\cos\phi \\ y = r\cos\theta\sin\phi \\ z = r\sin\theta \end{cases}$$

Lamé 系数

$$\begin{cases} h_r = 1 \\ h_\theta = r \\ h_z = r\sin\theta \end{cases}$$

球面上的积分：

$$\oiint\limits_S \sigma dS_r = \int_0^\pi \int_0^{2\pi} \sigma r^2 \sin\theta\, d\theta d\varphi$$

体积分：

$$\iiint\limits_V \rho dV = \int_0^r \int_0^\pi \int_0^{2\pi} \sigma r^2 \sin\theta\, dr d\theta d\varphi$$

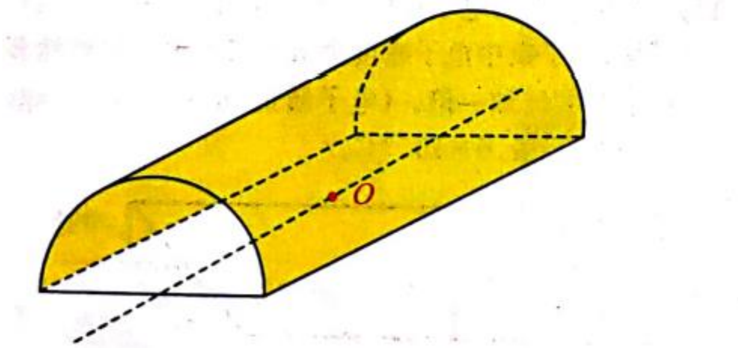
∇算符

$$-\nabla U = -\frac{\partial U}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}\hat{\boldsymbol{\varphi}}$$

$$\nabla\cdot\vec{\boldsymbol{P}} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta\, P_r)}{\partial r} + \frac{\partial(r\sin\theta\, P_\theta)}{\partial\theta} + \frac{\partial\left(r^2 P_\varphi\right)}{\partial\varphi}\right]$$
$$= \frac{1}{r^2}\frac{\partial(r^2 P_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, P_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial P_\varphi}{\partial\varphi}$$

1.13（柱面积分和换元）

半径为 R 的无限长半圆柱薄筒，均匀带电，单位面积的电量为 σ，求半圆柱轴线上一点 O 的场强.



以O为原点，建立适当的柱坐标系。由对称性，点O处的电场强度方向竖直向下，大小为

$$E = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} dz \int_0^{\pi} \frac{\sigma}{R^2+z^2}\frac{R\sin\theta}{\sqrt{R^2+z^2}} R d\theta = \frac{\sigma R^2}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dz}{(R^2+z^2)^{\frac{3}{2}}}$$

换元积分：$z \in (-\infty,+\infty)$，记$z = R\tan\varphi$，$\varphi \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$，则$\frac{1}{R^2+z^2} = \frac{\cos^2\varphi}{R^2}$，$dz = \frac{Rd\varphi}{\cos^2\varphi}$

$$E = \frac{\sigma R^2}{2\pi\varepsilon_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\varphi\, d\varphi}{R^2} = \frac{\sigma}{\pi\varepsilon_0}$$

例题 1：（球面积分）某导体球壳面电荷分布满足$\sigma_e = 3\varepsilon_0 E_0\cos\theta$（球坐标系）。求$\hat{\boldsymbol{z}}$方向上各点的电场强度、电势

记导体球半径为R。先求电势：

$$U = \frac{1}{4\pi\varepsilon_0}\oiint_{\partial V}\frac{\sigma_e}{(r^2+R^2-2Rr\cos\theta)^{\frac{1}{2}}} dS$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^{\pi} \frac{3\varepsilon_0 E_0 \cos\theta}{(r^2+R^2-2Rr\cos\theta)^{\frac{1}{2}}} R^2 \sin\theta\, d\theta$$

$$= \frac{3E_0R^2}{2}\int_0^{\pi} \frac{\cos\theta}{(r^2+R^2-2Rr\cos\theta)^{\frac{1}{2}}} - d\cos\theta$$

$$= \frac{3E_0R^2}{2b^2}\int_1^{-1} \frac{(a-bt)-a}{(a-bt)^{\frac{1}{2}}} d(bt)$$

$$= \frac{3E_0R^2}{2b^2}\left[-\frac{2}{3}(a-bt)^{\frac{3}{2}}\Big|_1^{-1} + 2a(a-bt)^{\frac{1}{2}}\Big|_1^{-1}\right]$$

$$= \frac{E_0R^2}{4R^2r^2}\{-[(r+R)^3-(r-R)^3]+3(r^2+R^2)[r+R-(r-R)]\}$$

$$= \frac{E_0}{4r^2}\{-[6Rr^2+2R^3]+3(2Rr^2+2R^3)\}$$

$$= \frac{E_0R^3}{r^2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\oiint_{\partial V} \frac{\sigma_e}{r^2+R^2-2Rr\cos\theta}\frac{r-R\cos\theta}{(r^2+R^2-2Rr\cos\theta)^{\frac{1}{2}}} dS$$

$$= \left(1+\frac{2{r_0}^3}{r^3}\right)E_0$$

作为参考，空间任一点的电势：

$$U = \frac{1}{4\pi\varepsilon_0}\oiint_{\partial V'} \frac{3\varepsilon_0 E_0 \cos\theta'}{(r^2+R^2-2Rr[\cos\theta\cos\theta'\cos(\varphi-\varphi')+\sin\theta\sin\theta'])^{\frac{1}{2}}} dS'$$

$$= \frac{E_0R^3}{r^2}\cos\theta$$

例题 2：（球面积分）库伦定律$\delta \neq 0$时，均匀带电球壳的电场强度和电势分布。

点电荷的电场强度和电势分布易得：

$$E=\frac{Q}{4\pi\varepsilon_0 r'^{2+\delta}}$$

$$U=\frac{Q}{4\pi\varepsilon_0(1+\delta)r'^{1+\delta}}$$

球壳$r>R$时：

$$U=\oiint_{\partial V}\frac{\sigma_e}{4\pi\varepsilon_0(1+\delta)r'^{1+\delta}}dS$$

$$=\frac{Q}{8\pi\varepsilon_0(1+\delta)}\int_0^\pi\frac{1}{(r^2+R^2-2Rr\cos\theta)^{\frac{1+\delta}{2}}}\sin\theta\,d\theta$$

$$=\frac{Q}{8\pi\varepsilon_0(1+\delta)}\frac{1}{Rr}\left[\frac{1}{1-\delta}(r^2+R^2-2Rr\cos\theta)^{\frac{1-\delta}{2}}\Big|_0^\pi\right]$$

$$=\frac{Q}{8\pi\varepsilon_0}\frac{(r+R)^{1-\delta}-(r-R)^{1-\delta}}{(1-\delta^2)Rr}$$

$$\approx\frac{Q}{4\pi\varepsilon_0}\frac{1-\frac{(r+R)\ln\left(1+\frac{R}{r}\right)-(r-R)\ln\left(1-\frac{R}{r}\right)}{2R}\delta}{r^{1+\delta}}$$

$$E=\oiint_{\partial V}\frac{\sigma_e}{4\pi\varepsilon_0 r'^{2+\delta}}\cdot\frac{r-R\cos\theta}{r'}dS$$

$$=\frac{Q}{8\pi R^2\varepsilon_0}\int_0^\pi\frac{r-R\cos\theta}{(r^2+R^2-2Rr\cos\theta)^{\frac{3+\delta}{2}}}R^2\sin\theta\,d\theta$$

2.简谐振动

若物体受力

$$\vec{\boldsymbol{F}}=-kx\hat{\boldsymbol{x}}$$

k为常数。则物体运动形式为一维简谐振动，振动周期

$$T = 2\pi\sqrt{\frac{m}{k}}$$

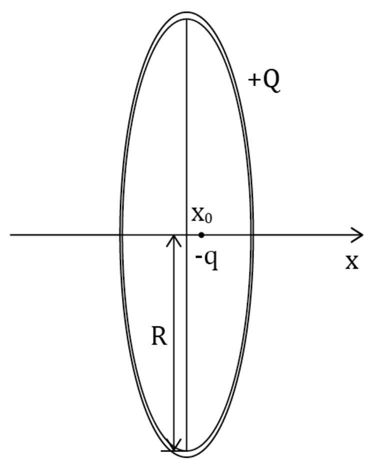
位移满足

$$x = A\cos\left(\sqrt{\frac{k}{m}}t + \delta\right)$$

A、δ由初始条件$\begin{cases} x|_{t=0} = x_0 \\ \left.\frac{dx}{dt}\right|_{t=0} = v_0 \end{cases}$求得

1.7（柱面积分，简谐振动）

半径为 R 的细金属环带电量为 $+Q$，小球质量为 m，电量为 $-q$，它可以沿着穿过环中心的一根细轨道无摩擦地运动，如果把小球放置在与环中心距离为 $x_0 \ll R$ 处无初速地释放，求小球的运动规律。



细金属环的电荷线密度为：

$$\lambda = \frac{Q}{2\pi R}$$

建立适当的柱坐标系。由对称性，$\vec{\boldsymbol{F}} \parallel \hat{\boldsymbol{x}}$，且应当与小球的位置矢量$\vec{\boldsymbol{x}}$方向相反。

$$\vec{\boldsymbol{F}}=\frac{1}{4\pi\varepsilon_0}\oint_{\partial S}\frac{-q\lambda}{R^2+x^2}\frac{x}{\sqrt{R^2+x^2}}\widehat{\boldsymbol{x}}dl=-\frac{1}{8\pi^2\varepsilon_0 R}\widehat{\boldsymbol{x}}\int_0^{2\pi}\frac{qQx}{(R^2+x^2)^{\frac{3}{2}}}Rd\theta$$

$$=-\frac{1}{4\pi\varepsilon_0}\frac{qQx}{(R^2+x^2)^{\frac{3}{2}}}\widehat{\boldsymbol{x}}\approx-\frac{qQx}{4\pi\varepsilon_0 mR^3}\widehat{\boldsymbol{x}}$$

因此小球的运动形式为一维简谐振动：

$$x=A\cos(\omega t+\delta)$$

初始条件$x|_{t=0}=x_0$，$\left.\frac{dx}{dt}\right|_{t=0}=0$得到

$$x=x_0\cos\left(\sqrt{\frac{qQ}{4\pi\varepsilon_0 mR^3}}t\right)$$

3.泰勒展开

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+o(x^2)$$

二、一些物理结论（黑字部分适当说明可以直接使用）

1.电场、电势（除电偶极子需说明使用高斯定理；揣摩出题人意图，特定情形电偶极子结论可以直接使用）

(1)点：

点电荷：$\vec{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$，$U = \frac{q}{4\pi\varepsilon_0 r}$

电偶极子：$\vec{\boldsymbol{E}} = -\frac{\vec{\boldsymbol{p}}}{4\pi\varepsilon_0 r^3} + \frac{3(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}}{4\pi\varepsilon_0 r^3}$，$U = \frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{r}}}{4\pi\varepsilon_0 r^2}$

(2)直线：$\vec{\boldsymbol{E}} = \frac{\lambda_e}{2\pi\varepsilon_0 r}\hat{\boldsymbol{r}}$，$U = \frac{\lambda_e}{2\pi\varepsilon_0}\ln r$（以距直线单位长度位置为参考点）

(3)面：

无穷大平面、不考虑边缘效应的平板、面电荷元在邻域上产生的电场（z_+平面）：$\vec{\boldsymbol{E}} = \frac{\sigma_e}{2\varepsilon_0}\hat{\boldsymbol{z}}$，$U = \frac{\sigma_e}{2\varepsilon_0}(x - x_0)$

无穷长柱面：$\vec{\boldsymbol{E}} = \frac{\sigma_e R}{\varepsilon_0 r}\hat{\boldsymbol{r}}$，$U = \frac{\sigma_e R}{\varepsilon_0}\ln\frac{r}{R}$（以$r = R$位置为参考点）

球面：$\vec{\boldsymbol{E}} = \begin{cases} \frac{\sigma_e R^2}{\varepsilon_0 r^2}\hat{\boldsymbol{r}}, & r > R \\ 0, & r \le R \end{cases}$，$U = \begin{cases} \frac{\sigma_e R^2}{\varepsilon_0 r}, & r > R \\ \frac{\sigma_e R}{\varepsilon_0}, & r \le R \end{cases}$

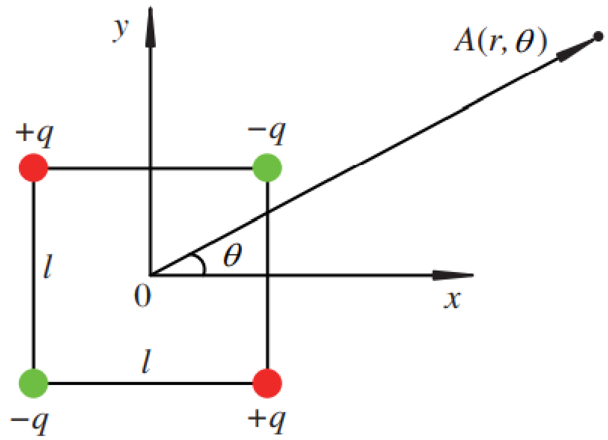
(4)体：

无穷长柱体：$\vec{\boldsymbol{E}} = \frac{\rho_e R^2}{2\varepsilon_0 r}\hat{\boldsymbol{r}}$，$U = \frac{\rho_e R^2}{2\varepsilon_0 r}\ln\frac{r}{R}$（以$r = R$位置为参考点）

球体：$\vec{\boldsymbol{E}} = \begin{cases} \frac{\rho_e R^3}{3\varepsilon_0 r^2}\hat{\boldsymbol{r}}, & r > R \\ \frac{\rho_e r}{3\varepsilon_0}\hat{\boldsymbol{r}}, & r \le R \end{cases}$，$U = \begin{cases} \frac{\rho_e R^3}{3\varepsilon_0 r}, & r > R \\ \frac{\rho_e(3R^2 - r^2)}{6\varepsilon_0}, & r \le R \end{cases}$

1.15（电偶极子，泰勒展开，球坐标劈形算符，无明显对称性考虑电势）（拓展：空间内任意一点）

面电四极子如习题 1.17 图所示，点 $A(r,\theta)$ 与电四极子共面，极轴（$\theta=0$）通过正方形中心并与两边平行。设 $r \gg l$，求面电四极子在点 A 处产生的电场强度。



习题 1.17 图

视为两个相反方向的电偶极子。球坐标系(r,θ,φ)中

$$U_1=\frac{1}{4\pi\varepsilon_0}\frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}}}{r_1^2}=\frac{pr\sin\theta\cos\varphi}{4\pi\varepsilon_0}\frac{1}{r_1^3}$$

$$U_2=-\frac{pr\sin\theta\cos\varphi}{4\pi\varepsilon_0}\frac{1}{r_2^3}$$

$$U=U_1+U_2=-\frac{pr\sin\theta\cos\varphi}{4\pi\varepsilon_0}\left(\frac{1}{r_2^3}-\frac{1}{r_1^3}\right)\approx-\frac{pr\sin\theta\cos\varphi}{4\pi\varepsilon_0 r^3}\frac{3l}{r}\sin\varphi$$
$$=-\frac{3ql^2\sin\theta\sin 2\varphi}{8\pi\varepsilon_0 r^3}$$

其中

$$\frac{1}{r_2^3}-\frac{1}{r_1^3}=\frac{1}{\left(r^2+\frac{l^2}{4}-lr\sin\theta\right)^{\frac{3}{2}}}+\frac{1}{\left(r^2+\frac{l^2}{4}+lr\sin\theta\right)^{\frac{3}{2}}}$$
$$=\frac{1}{r^3}\left[\frac{1}{\left(1+\frac{l^2}{4r^2}-\frac{l}{r}\sin\theta\right)^{\frac{3}{2}}}+\frac{1}{\left(1+\frac{l^2}{4r^2}+\frac{l}{r}\sin\theta\right)^{\frac{3}{2}}}\right]$$

$$=\frac{1}{r^3}\left\{\left[1-\frac{3}{2}\left(\frac{l^2}{4r^2}-\frac{l}{r}\sin\theta\right)+o\left(\frac{l}{r}\right)\right]-\left[1-\frac{3}{2}\left(\frac{l^2}{4r^2}+\frac{l}{r}\sin\theta\right)+o\left(\frac{l}{r}\right)\right]\right\}$$

$$=\frac{1}{r^3}\left\{\frac{3l}{r}\sin\theta+o\left(\frac{l}{r}\right)\right\}\approx\frac{1}{r^3}\left\{\frac{3l}{r}\sin\theta\right\}$$

$$\therefore\ \vec{\boldsymbol{E}}=-\nabla U=-\frac{\partial U}{\partial r}\hat{\boldsymbol{r}}-\frac{1}{r}\frac{\partial U}{\partial\theta}\hat{\boldsymbol{\theta}}-\frac{1}{r\sin\theta}\frac{\partial U}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

$$=-\frac{9ql^2\sin\theta\sin 2\varphi}{8\pi\varepsilon_0 r^4}\hat{\boldsymbol{r}}+\frac{3ql^2\cos\theta\sin 2\varphi}{8\pi\varepsilon_0 r^4}\hat{\boldsymbol{\theta}}+\frac{3ql^2\cos 2\varphi}{4\pi\varepsilon_0 r^4}\hat{\boldsymbol{\varphi}}$$

本题讨论xOy平面，令$\theta=\frac{\pi}{2}$得到

$$\vec{\boldsymbol{E}}=-\frac{9ql^2\sin 2\varphi}{8\pi\varepsilon_0 r^4}\hat{\boldsymbol{r}}+\frac{3ql^2\cos 2\varphi}{4\pi\varepsilon_0 r^4}\hat{\boldsymbol{\varphi}}$$

2.电容器

平行板电容器：$C=\frac{\varepsilon_0 S}{d}$

柱形电容器：$C=\frac{2\pi\varepsilon_0 L}{\ln\left(\frac{R_2}{R_1}\right)}$

球形电容器：$C=\frac{4\pi\varepsilon_0 R_1 R_2}{R_2-R_1}$

3.回路电压定律（第 3 章）

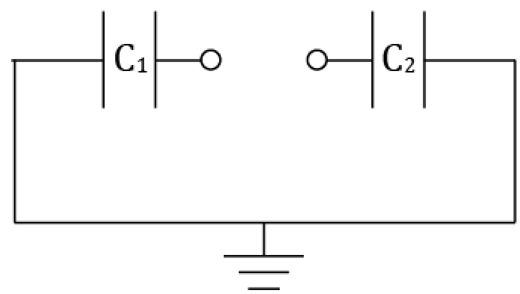
回路逆（顺）时针电压降之和为 0。2.11、2.12 用到

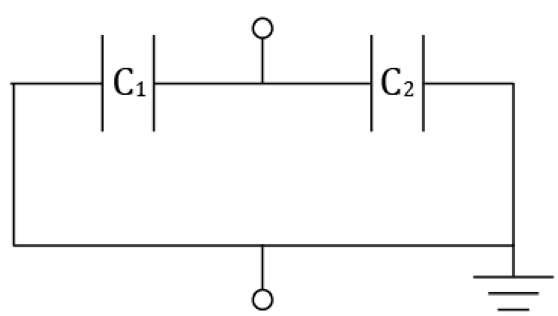
4.电容的串联、并联

充当正极板的导体板与充当正极板的导体板连接，属于并联。

2.8（串并联，无穷远点）

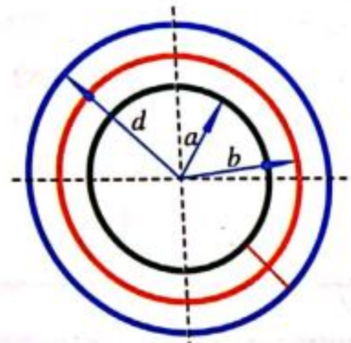
把地球和月球均当作导体球，地球半径为 a，月球半径为 b，它们之间的距离远大于它们各自的半径（即可视为孤立导体球），则地球与月球之间的电容值为多少？如果地球与月球之间用一根细导线接通，则此时地球与月球之间的电容为多少？



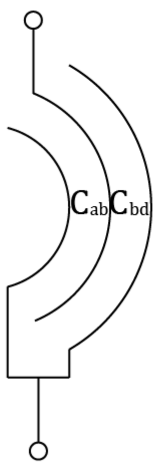


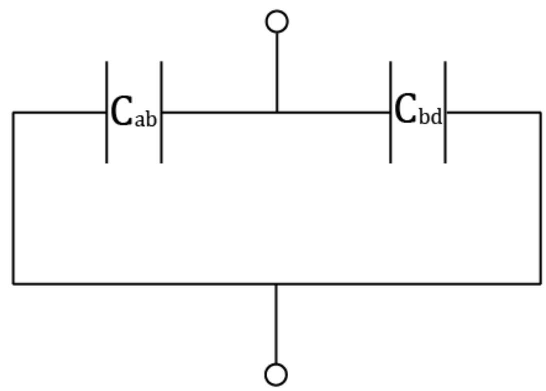
2.8（串并联）

一个球形电容器由三个很薄的同心导体壳组成，它们的半径分别为 a，b，d。一根绝缘细导线通过中间壳层的一个小孔把内外球壳连接起来。忽略小孔的边缘效应。(1) 求此系统的电容；(2) 若在中间球壳上放置任意电量 Q，确定中间球壳内外表面上的电荷分布。



（1）若中间球壳带正电，则内球壳外表面和外球壳内表面均带负电。只考虑作为电容的部分，内球壳和外球壳均为负极板。可见导线连接在相同的极板上，因此为并联。





$$C = C_{ab} + C_{bc} = 4\pi\varepsilon_0 b\left(\frac{a}{b-a} + \frac{c}{c-b}\right)$$

5.边界（高斯定理、环路定理）

边界处法向分量关系（高斯定理）：$\begin{cases} \hat{\boldsymbol{n}} \cdot \left(\vec{\boldsymbol{E}}_2 - \vec{\boldsymbol{E}}_1\right) = \frac{\sigma_e}{\varepsilon_0} \\ \hat{\boldsymbol{n}} \cdot \left(\vec{\boldsymbol{P}}_2 - \vec{\boldsymbol{P}}_1\right) = \sigma_e' \\ \hat{\boldsymbol{n}} \cdot \left(\vec{\boldsymbol{D}}_2 - \vec{\boldsymbol{D}}_1\right) = \sigma_{e0} \end{cases}$

边界处切向分量关系（环路定理）：$\begin{cases} \hat{\boldsymbol{n}} \times \left(\vec{\boldsymbol{E}}_2 - \vec{\boldsymbol{E}}_1\right) = \vec{\boldsymbol{0}} \\ \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{P}}_2}{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{P}}_1} = \frac{\boldsymbol{\varepsilon_2}}{\boldsymbol{\varepsilon_1}} \\ \frac{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{D}}_2}{\hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{D}}_1} = \frac{\boldsymbol{\varepsilon_2}}{\boldsymbol{\varepsilon_1}} \end{cases}$

作业题

1.8（特别的积分）

真空中有两个点电荷 q_1 和 q_2，它们的质量为 m_1 和 m_2，位置为 r_1 和 r_2，相距为 r_0，只考虑库仑相互作用，求两个电荷相遇的时间。

易知两点电荷异号，令q_1、q_2表示电荷量的绝对值。记点电荷q_1到q_2的距离为$\vec{\boldsymbol{r}}$。

$$\vec{\boldsymbol{F}}_{21} = m_1 \frac{d^2\overrightarrow{\boldsymbol{r_1}}}{dt^2} = \frac{kq_1q_2}{r^2}\hat{\boldsymbol{r}}$$

$$\vec{\boldsymbol{F}}_{12} = m_2 \frac{d^2\overrightarrow{\boldsymbol{r_2}}}{dt^2} = -\frac{kq_1q_2}{r^2}\hat{\boldsymbol{r}}$$

$$\therefore \frac{d^2\vec{\boldsymbol{r}}}{dt^2} = \frac{d^2(\overrightarrow{\boldsymbol{r_2}} - \overrightarrow{\boldsymbol{r_1}})}{dt^2} = -\frac{kq_1q_2}{r^2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{\boldsymbol{r}}$$

方法一：可降阶的二阶微分方程（$y'' = f(y, y')$）

$$\frac{d^2r}{dt^2} = -\frac{kq_1q_2}{r^2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \triangleq -\frac{\lambda^2}{r^2}$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{dr}\frac{dr}{dt} = \dot{r}\frac{d\dot{r}}{dr}$$

$$\dot{r}\frac{d\dot{r}}{dr} = -\frac{\lambda^2}{r^2}$$

边界条件$\dot{\boldsymbol{r}}|_{t=0} = 0$，$r|_{t=0} = r_0$。积分得

$$\dot{r}^2 = 2\lambda^2\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$\dot{r}$应当为负值，因此

$$\frac{dr}{dt}=-\sqrt{2\lambda^2\left(\frac{1}{r}-\frac{1}{r_0}\right)}$$

$$dt=-\frac{dr}{\sqrt{\frac{2\lambda^2\left(1-\frac{r}{r_0}\right)}{r}}}=-\sqrt{\frac{r_0^3}{2\lambda^2}}\frac{2x^2dx}{\sqrt{1-x^2}}$$

积分得

$$t=-\sqrt{\frac{r_0^3}{2\lambda^2}}\left(-\frac{r}{r_0}\sqrt{1-\left(\frac{r}{r_0}\right)^2}+\arcsin\frac{r}{r_0}\right)\Bigg|_{r_0}^{0}=\frac{\pi}{2}\sqrt{\frac{r_0^3}{2\lambda^2}}=\frac{\pi}{2}\sqrt{\frac{2\pi\varepsilon_0 r_0^3}{q_1q_2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}}$$

方法二：退化椭圆

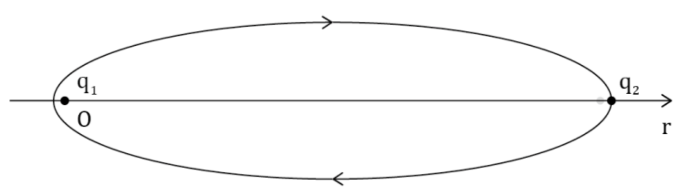
该方程不易求解。可以考虑以点电荷q_1为圆心做非惯性参考系。则点电荷q_2有惯性加速度

$$\overrightarrow{\boldsymbol{a_i}}=-\frac{\overrightarrow{\boldsymbol{F}}_{21}}{m_1}=-\frac{kq_1q_2}{m_1r^2}\hat{\boldsymbol{r}}$$

因此非惯性系中q_2的受力情况为

$$\overrightarrow{\boldsymbol{F}}'_{12}=m_2\overrightarrow{\boldsymbol{a_i}}+\overrightarrow{\boldsymbol{F}}_{12}=-\frac{kq_1q_2m_2}{r^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\hat{\boldsymbol{r}}$$

类比引力情形，点电荷的运动相当于引力为$-\frac{GMm_2}{r^2}=-\frac{kq_1q_2m_2}{r^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$，半长轴$a=\frac{r_0}{2}$，半短轴$b=0$，半焦距$a=\frac{r_0}{2}$的行星运动（如图）。



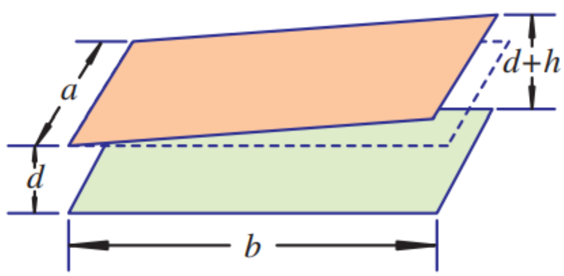
由开普勒第三定律：

$$T=2\pi\sqrt{\frac{a^3}{GM}}=2\pi\sqrt{\frac{\left(\frac{r_0}{2}\right)^3}{kq_1q_2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}}=\pi r_0\sqrt{\frac{2\pi\varepsilon_0r_0^3}{q_1q_2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}}$$

因此从静止到相遇时间为$t=\frac{T}{2}=\frac{\pi r_0}{2}\sqrt{\frac{2\pi\varepsilon_0r_0}{q_1q_2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)}}$

2.9（并联，题目有瑕疵）

两块长与宽均为 a 和 b 的导体平板在制成平行板电容器时稍有偏斜，使两板间距一端为 d，另一端为 $d+h$，且 $h\ll d$，求该电容器的电容。



方法一*：平行导体板的叠加

将稍有倾斜的导体板电容器视为无数长a宽$\mathrm{d}x$相距$d+\frac{x}{b}h$的平行导体板电容器的并联。注意无法视为导线的叠加，否则计算中引入无穷大，而这要求$h\ll b$。由此：

$$C=\int dC=\int_0^b\frac{\varepsilon_0a\mathrm{d}x}{d+\frac{x}{b}h}=\frac{\varepsilon_0ab}{h}\ln\left(1+\frac{h}{d}\right)$$

宽相对极板距离过窄的情况下，电容器一般无法忽略边缘效应。因此要求$h\ll d\ll dx\ll b$

方法二*：计算电荷量

导体板的面电荷未必是均匀分布的，但电势一定是定值。可以给定两极板的电势差值，计算导体板的总电荷，从而得到电容。

两极板延长线的交点O为原点，建立柱坐标系。下极板上表面的位置为$\theta = 0$。简单计算可得下极板左边缘位置为$r_e = \frac{bd}{h}$

假定下极板电势为0，上极板电势为U。利用高斯定理配合电势进行计算，由对称性（忽略边缘效应），$U = U(r, \theta)$。

$$\nabla \cdot \vec{\boldsymbol{E}} = -\nabla^2 U = 0$$

边界条件为：

$$U|_{\theta=0} = 0$$

$$U|_{\theta=\arctan\frac{h}{b}} = U$$

得到 Laplace 方程边值问题

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial U}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2} = 0 \\ U|_{\theta=0} = 0 \\ U|_{\theta=\arctan\frac{h}{b}} = U \end{cases}$$

分离变量$U = R(r)\Theta(\theta)$，分离得常微分方程组：

$$\begin{cases} \Theta'' + \mu\Theta = 0 \\ r(rR')' - \mu R = 0 \end{cases}$$

Θ的常微分方程解得

$$\mu_n = n^2,\ \ n = 0,1,2,\dots$$

$$\Theta_m(\theta) = \begin{cases} A_n + B_n\theta, & n = 0 \\ A_n \cos n\theta + B_n \sin n\theta\,, & n = 1,2,3,\dots \end{cases}$$

对于R的常微分方程，设$r = e^t$。则

$$\frac{d^2R}{dt^2}-n^2R=0$$

解得

$$R_n(\theta)=\begin{cases}C_n+D_n\ln r\,, & n=0\\ C_nr^n+D_nr^{-n}, & n=1,2,3,\dots\end{cases}$$

$$\therefore\ U(r,\theta)=A_0'+B_0\theta+(C_0'+D_0'\theta)\ln r$$
$$+\sum_{n=1}^{\infty}[(A_n'\cos n\theta+B_m'\sin n\theta)r^n+(C_n'\cos n\theta+D_n'\sin n\theta)r^{-n}]$$

代入边值条件$U|_{\theta=0}=0$：

$$A_0'+(C_0'+D_0'\theta)\ln r+\sum_{n=1}^{\infty}[A_n'r^n+C_n'r^{-n}]=0$$

得到$A_n'=0$；$C_n'=0$；$D_0'=0$。代入边值条件$U|_{\theta=\arctan\frac{h}{b}}=U$：

$$B_0\arctan\frac{h}{b}+\sum_{n=1}^{\infty}\left[B_n'\sin n\arctan\frac{h}{b}r^n+D_n'\sin n\arctan\frac{h}{b}r^{-n}\right]=U$$

得到$B_0=\frac{U}{\arctan\frac{h}{b}}$，$B_n'=0\ (n\neq 1)$；$D_n'=0\ (n\neq 1)$。因此：

$$U(r,\theta)=U(\theta)=\frac{U\theta}{\arctan\frac{h}{b}}$$

$$\vec{\boldsymbol{E}}=-\nabla U=-\frac{1}{r}\frac{\partial U}{\partial\theta}\widehat{\boldsymbol{\theta}}=-\frac{U}{r\arctan\frac{h}{b}}\widehat{\boldsymbol{\theta}}$$

下极板上表面$(r_0,0,z)$处取一微小柱形高斯面，高平行$\hat{\boldsymbol{\theta}}$方向，上下底面面积为$dS$。可得

$$-\frac{U}{r\arctan\dfrac{h}{b}}dS=\frac{\sigma_e(r)dS}{\varepsilon_0}$$

$$\therefore\ \sigma_e(r)=-\frac{\varepsilon_0 U}{r\arctan\dfrac{h}{b}}$$

$$\therefore\ Q=\iint_S\sigma_e(r)dS=\int_0^a dz\int_{r_e}^{r_e+b}-\frac{\varepsilon_0 U}{r\arctan\dfrac{h}{b}}dr=-\frac{\varepsilon_0 Ua\ln\left(1+\dfrac{h}{d}\right)}{\arctan\dfrac{h}{b}}$$

$$\therefore\ C=\frac{Q}{0-U}=\frac{\varepsilon_0 a\ln\left(1+\dfrac{h}{d}\right)}{\arctan\dfrac{h}{b}}\approx\frac{\varepsilon_0 ab}{h}\ln\left(1+\frac{h}{d}\right)$$

2.15（球面电荷的等效电偶极矩。题目有瑕疵）

水分子是有极分子，一个水分子的电偶极矩为 0.61×10^{-30} C•m，若所有的水分子电矩都朝同一方向，(1) 试估算水的极化强度；(2) 直径为 1 mm 的水滴的电偶极矩有多大？距水 10 cm 处的电场强度有多大？

（1）水的摩尔质量$M=18.0152\ \mathrm{g}$，密度$\rho\approx1.00\times10^3\ \mathrm{g/m^3}$

$$P=np=\frac{\rho N_A}{M}p=2.04\times10^{-2}\ \mathrm{C/m^2}$$

（2）该处电场为所有电偶极子产生电场的叠加（同方向可直接叠加，相反方向可能为多极子）。总电偶极矩p_d为：

$$p_d=VP=\frac{\pi d^3}{6}P=1.07\times10^{-11}\ \mathrm{C\cdot m}$$

题目未给出电偶极矩与位置矢量的方向，而电偶极子产生的电场与方向有

关。若10 cm位置垂直电偶极矩方向，则

$$E=\frac{p_d}{4\pi\varepsilon_0 r^3}=9.62\times 10\ \mathrm{C/m^2}$$

若10 cm位置平行电偶极矩方向，则

$$E=\frac{p_d}{2\pi\varepsilon_0 r^3}=1.92\times 10^2\ \mathrm{C/m^2}$$

计算总电偶极矩p_d时，也可用球面电荷计算等效电偶极矩，但这只会徒增计算量：

$$\sigma_e=\hat{\boldsymbol{r}}\cdot(\vec{\boldsymbol{0}}-\vec{\boldsymbol{P}})=-P\cos\theta$$

$$\vec{\boldsymbol{p}}_d=\oiint_{\partial V}\sigma_e 2R\cos\theta\,dS\,(-\hat{\boldsymbol{z}})=-\int_0^{2\pi}d\varphi\int_0^{\pi}\sigma_e 2R\cos\theta\,R^2\sin\theta\,d\theta\,\hat{\boldsymbol{z}}$$

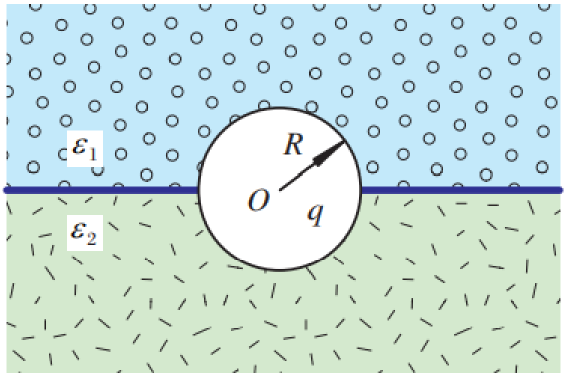
$$=2\pi R^3\vec{\boldsymbol{P}}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta=\frac{4\pi R^3}{3}\vec{\boldsymbol{P}}$$

$$=1.07\times 10^{-11}\ \mathrm{C\cdot m}$$

（

2.22（介质面平行电场线）

2.36 如图所示，一导体球外充满两半无限电介质，介电常量分别为 ε_1 和 ε_2，介质界面为通过球心的无限平面。设导体球半径为 R，总电荷为 q，求空间电场分布和导体球表面的自由面电荷分布。



习题 2.36 图

杨国桢《电磁学与电动力学（上）》P57）介质界面与（无介质时的）电场线重合，极化电荷只可能分布在介质与导体的边界面上。从结果上说，介质与导体的边界面的总电荷分布与无介质时导体边界面上的电荷分布相同

介质界面与（无介质时的）电场线垂直，从结果上来说，有

$$\vec{\boldsymbol{E}}_i = \frac{\varepsilon_0 \vec{\boldsymbol{E}}_0}{\varepsilon_i}$$

$\vec{\boldsymbol{E}}_0$是无介质情形的电场。但没必要使用该结论，直接先计算$\vec{\boldsymbol{D}}$再计算$\vec{\boldsymbol{E}}_i$即可（2.16）

介质界面处，电场强度的切向分量相等。而电场平行于介质界面，因此界面两侧电场强度相等。可见电场强度应当仍然满足一维对称分布，边界面总电荷均匀分布。由电位移矢量的高斯定理：

$$2\pi r^2 D_1 + 2\pi r^2 D_2 = 2\pi r^2 \varepsilon_1 E + 2\pi r^2 \varepsilon_2 E = q$$

$$\therefore\ \vec{E} = \frac{q}{2\pi(\varepsilon_1 + \varepsilon_2) r^2} \hat{r}$$

[2.1] 第一章习题（1）

2019年9月24日 16:49

数+基 [同类量]: $1\text{kg} = 1000\text{g} \xrightarrow[(全)]{代换} 1\text{kg} \to 5000\text{g}$

数 [不同类量]: $1[m] = 5[kg] \xrightarrow[(仅数)]{代换} 1[kg] \to 10^{-3}[g]$

△. 第一章: [公式分为同类量与非同类量.]

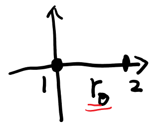
1.2. SI: 量: $F = k\frac{q_1 q_2}{r^2}$ $1[kg\ m/s^2] = 9\times10^9 \frac{[C^2]}{[m^2]}$

$\therefore 1[C^2] = \frac{1[kg\ m^3]}{9\times10^9 [s^2]}$ 设 $1C = x\,esu$. ✓

$\frac{1}{x^2} = \frac{10^{-9}}{9\times10^9}$ $1C \to x\,esu$ $\to \frac{1}{x}[esu]$

$\therefore x = 3\times10^9$.

1.8 两体RL, $\begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} = \mu\frac{d^2 r}{dt^2}, \\ r(0) = r_0,\ v(0) = 0. \end{cases}$ Gm_1m_2



$\therefore \boxed{\ddot r + \beta\cdot\frac{1}{r^2} = 0}$, 其中, $\beta = \frac{q_1 q_2}{4\pi\varepsilon_0\mu}$

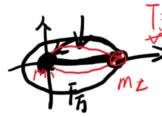
两体问题 ⇒ 单体问题 ($\mu = \frac{m_1 m_2}{m_1+m_2}$)

由 $\frac{d\dot r}{dt} = \frac{d\dot r}{dr}\cdot\dot r$, $\frac{d\dot r}{dr}\cdot\dot r = -\beta\frac{1}{r^2}$

$\therefore \dot r d\dot r = -\beta\frac{dr}{r^2} \Rightarrow \frac{1}{2}\dot r^2 = \beta\frac{1}{r} + C_1$

$\therefore 0 = \beta\frac{1}{r_0} + C_1 \Rightarrow C_1 = -\beta\frac{1}{r_0}$

$\therefore \dot r = -[2\beta(\frac{1}{r} - \frac{1}{r_0})]^{\frac{1}{2}}$ $r(t)$?



$T = 2\pi\sqrt{\frac{a^3 m_2}{GM m_2}}$ $T = 2\pi\sqrt{\frac{a^3 m_2}{\frac{1}{4\pi\varepsilon_0}q_1 q_2}}$ ✓

$a = \frac{r_0}{2}$

$\therefore t = \frac{T}{2}$. ✓

$\therefore dt = -\frac{dr}{[2\beta(\frac{1}{r} - \frac{1}{r_0})]^{\frac{1}{2}}}$ $(0 < r < r_0)$ $\Rightarrow t = \frac{1}{\sqrt{2\beta}}\cdot\frac{\pi}{2}r_0^{\frac{3}{2}}$

类比 退化椭圆: $-\frac{GMm}{r^2} = m\ddot r \Rightarrow \boxed{\ddot r + GM\frac{1}{r^2} = 0}$. $T = 2\pi\sqrt{\frac{a^3}{GM}}$ ✓

则 此题中, $GM \to \beta$, $a \to \frac{r_0}{2}$.

$\therefore t = \frac{T}{2} = \pi\sqrt{\frac{4\pi\varepsilon_0 r_0^3\mu}{8 q_1 q_2}} = \pi\sqrt{\frac{\pi\varepsilon_0 r_0^3\mu}{2 q_1 q_2}}$ ✓

1.10

$r<R$, $E\cdot 2\pi r h = \frac{\iiint\rho dV}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_0^r\int_0^{2\pi}\int_0^h (ar - br^3)\, r\,dr\,d\theta\,dz$

$= \frac{2\pi h}{\varepsilon_0}\int_0^r (ar^2 - br^4)\,dr$

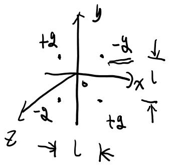
$= \frac{2\pi h}{\varepsilon_0}\cdot(\frac{a}{3}r^3 - \frac{b}{5}r^5)$

$\therefore E = -\frac{b}{5\varepsilon_0}r^4 + \frac{a}{3\varepsilon_0}r^2$

$$\therefore E = -\frac{b}{5\varepsilon_0} r^4 + \frac{a}{3\varepsilon_0} r^2$$

$$r > R: E = \left(-\frac{b}{5\varepsilon_0} R^5 + \frac{a}{3\varepsilon_0} R^3\right) \frac{1}{r}.$$

1.15. 法一： $Q=0$



$$\vec{p}=\sum_{i=1}^{4} q_i\vec{r}_i = -q\left(\tfrac{l}{2},\tfrac{l}{2},0\right)+q\left(-\tfrac{l}{2},\tfrac{l}{2},0\right)-q\left(-\tfrac{l}{2},-\tfrac{l}{2},0\right)+q\left(\tfrac{l}{2},-\tfrac{l}{2},0\right)$$

$$=0 \checkmark$$

$$D=\sum_{i=1}^{4} q_i(3\vec{r}_i\vec{r}_i-r_i^2 I) = -q\left[\frac{3l^2}{4}(\hat{x}\hat{x}+\hat{x}\hat{y}+\hat{y}\hat{x}+\hat{y}\hat{y})-\frac{l^2}{4}I\right]$$

$$+q\left[\frac{3l^2}{4}(\hat{x}\hat{x}-\hat{x}\hat{y}-\hat{y}\hat{x}+\hat{y}\hat{y})-\frac{l^2}{4}I\right]$$

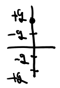
$$-q\left[\frac{3l^2}{4}(\hat{x}\hat{x}+\hat{x}\hat{y}+\hat{y}\hat{x}+\hat{y}\hat{y})-\frac{l^2}{4}I\right]$$

$$+q\left[\frac{3l^2}{4}(\hat{x}\hat{x}-\hat{x}\hat{y}-\hat{y}\hat{x}+\hat{y}\hat{y})-\frac{l^2}{4}I\right]$$

$$=\frac{3ql^2}{4}(-4\hat{x}\hat{y}-4\hat{y}\hat{x})$$

$$=-3ql^2(\hat{x}\hat{y}+\hat{y}\hat{x}) \quad ☆$$

☆ "电四极子" "电偶极子"，不唯一.



$$\therefore \varphi=\frac{1}{8\pi\varepsilon_0 r^3}D:\hat{r}\hat{r} = -\frac{3ql^2}{8\pi\varepsilon_0}\cdot\frac{1}{r^3}(\hat{x}\hat{y}:\hat{r}\hat{r}+\hat{y}\hat{x}:\hat{r}\hat{r})$$

$$=-\frac{3ql^2}{4\pi\varepsilon_0}\cdot\frac{1}{r^3}\cdot\sin^2\theta\sin\varphi\cos\varphi$$

$$\therefore \vec{E}=-\nabla\varphi=\frac{3ql^2}{4\pi\varepsilon_0}\left(\frac{\partial}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right)\left(\frac{1}{r^3}\sin^2\theta\sin\varphi\cos\varphi\right)$$

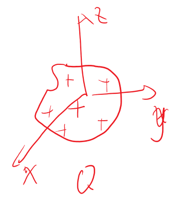
$$=\frac{3ql^2}{4\pi\varepsilon_0}\left[-\frac{3}{r^4}\sin^2\theta\sin\varphi\cos\varphi\,\hat{r}+\frac{2}{r^4}\sin\varphi\cos\varphi\sin\theta\cos\theta\,\hat{\theta}+\frac{1}{r^4}\sin\theta\cos2\varphi\,\hat{\varphi}\right]$$

$$=\frac{3ql^2}{4\pi\varepsilon_0}\cdot\frac{1}{r^4}\left[-\frac{3}{2}\sin^2\theta\sin2\varphi\,\hat{r}+\frac{1}{2}\sin2\theta\sin2\varphi\,\hat{\theta}+\sin\theta\cos2\varphi\,\hat{\varphi}\right]$$

☆ 取 $\theta=\frac{\pi}{2}$，则 xoy 平面：

$$\varphi_{xoy}=-\frac{3ql^2}{8\pi\varepsilon_0}\frac{1}{r^3}\sin2\varphi$$

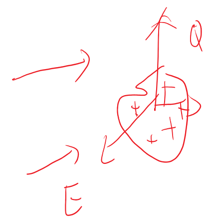
$$\vec{E}_{xoy}=\frac{3ql^2}{4\pi\varepsilon_0}\frac{1}{r^4}\left[-\frac{3}{2}\sin2\varphi\,\hat{\rho}+\cos2\varphi\,\hat{\varphi}\right]$$



Q

$\vec{p}$

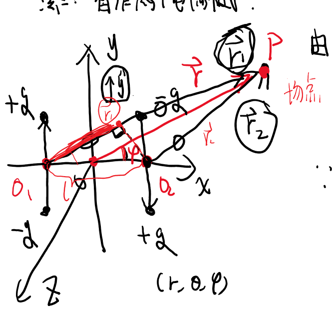
D



$F=EQ$

E

法二：看作两个电偶极子.



源点（'）

由图，$\boxed{\vec{r}_1=\vec{r}+\frac{l}{2}\hat{x}}$，$\vec{p}_1=ql\hat{y}$

$\boxed{\vec{r}_2=\vec{r}-\frac{l}{2}\hat{x}}$，$\vec{p}_2=-ql\hat{y}$

$\hat{\ } = \vec{e}$

$$\because \varphi_1(\vec{r}_1)=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}_1\cdot\vec{r}_1}{r_1^3}$$

$$\varphi_2(\vec{r}_2)=\frac{1}{4\pi\varepsilon_0}\cdot\frac{\vec{p}_2\cdot\vec{r}_2}{r_2^3}$$

$$\therefore \varphi(\vec{r})=\varphi_1(\vec{r}_1)+\varphi_2(\vec{r}_2)$$

$$=\frac{ql}{4\pi\varepsilon_0}\left(\frac{\hat{y}\cdot\vec{r}_1}{r_1^3}-\frac{\hat{y}\cdot\vec{r}_2}{r_2^3}\right)$$

$$=\frac{qly}{4\pi\varepsilon_0}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)$$

$$=\frac{qly}{4\pi\varepsilon_0}\cdot\frac{r_2^3-r_1^3}{r_1^3r_2^3}$$

$(r_2-r_1)(\underbrace{r_2^2+r_1r_2+r_1^2}_{3r^2})$

$$\approx\frac{qly}{4\pi\varepsilon_0}\frac{3r^2(r_2-r_1)}{r^6}=\frac{3ql}{4\pi\varepsilon_0}\cdot\frac{y}{r^4}(r_2-r_1)=\frac{3ql}{4\pi\varepsilon_0}\cdot\frac{1}{r^3}\cdot\frac{y}{r}\cdot(r_2-r_1)$$

$y=r\sin\theta\sin\varphi$

$$\approx \frac{qly}{4\pi\varepsilon_0}\frac{3r^2(r_2-r_1)}{r^6} = \frac{3ql}{4\pi\varepsilon_0}\cdot\frac{y}{r^4}(r_2-r_1) = \frac{3ql}{4\pi\varepsilon_0}\cdot\frac{1}{r^3}\cdot\frac{y}{r}\cdot(r_2-r_1)$$

$y = r\sin\theta\sin\varphi$

$\frac{y}{r} = \sin\theta\sin\varphi$ ✓

法二：直接。

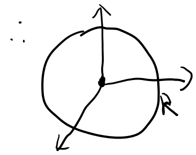
$$= -\frac{3ql^2}{4\pi\varepsilon_0}\cdot\frac{1}{r^3}\sin^2\theta\sin\varphi\cos\varphi \quad (xoy:\ 取\ \theta=\frac{\pi}{2})$$

$r_2 - r_1 = -l\cos\varphi$

$\vec{E} = -\nabla\varphi$

1.21 $\because \rho = \rho_0\frac{e^{-kr}}{r}$.

$\therefore$



$$E\cdot 4\pi R^2 = \frac{\iiint_V \rho dV}{\varepsilon_0} = \frac{\rho_0}{\varepsilon_0}\int_0^R\int_0^\pi\int_0^{2\pi}\frac{e^{-kr}}{r}\cdot r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\int_0^R re^{-kr}dr$$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\left[-\frac{1}{k}re^{-kr}\Big|_0^R + \frac{1}{k}\int_0^R e^{-kr}dr\right]$$

$$= \frac{4\pi\rho_0}{k\varepsilon_0}\left[-Re^{-kR} - \frac{1}{k}(e^{-kR}-1)\right]$$

$$\therefore \vec{E}(\vec{R}) = \frac{\rho_0}{k\varepsilon_0}\left[-\frac{e^{-kR}}{R} - \frac{1}{k}\cdot\frac{e^{-kR}-1}{R^2}\right]\hat{R}$$

($\nabla\times\vec{E}=0 \Rightarrow \vec{E} = -\nabla\varphi$)

1.24. 无旋场有势.

$\begin{cases}\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}\\ \nabla\times\vec{E}=0\\ \vec{E}=-\nabla\varphi\end{cases}$

$\because \vec{E} = \frac{kq}{4\pi\varepsilon_0}\cdot\frac{1}{r^3}\hat{r}$

$$\therefore \nabla\times\vec{E} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi}\\ \partial_r & \partial_\theta & \partial_\varphi\\ \frac{kq}{4\pi\varepsilon_0}\frac{1}{r^3}\hat{r} & 0 & 0\end{vmatrix}$$

$= 0$

$\therefore$ 必有势. 设 $\vec{E} = -\nabla\varphi$.

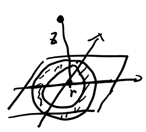
$\varphi_1 - \varphi_2 = \int_{r_1}^{r_2}\vec{E}\cdot d\vec{l}$

$$\therefore \varphi(r) = \int_r^{+\infty}\vec{E}\cdot d\vec{r} = \int_r^{+\infty}\frac{kq}{4\pi\varepsilon_0 r^3}dr = \frac{kq}{8\pi\varepsilon_0 r^2}$$ ✓ (定义 $\varphi(+\infty)=0$)

此时 Gauss 定律不适用. $\left[\nabla\cdot\frac{1}{|\vec{r}-\vec{r}'|} = 4\pi\delta(\vec{r}-\vec{r}')\right]$

$$\therefore \varphi(\vec{r}) = \iint_{S'}\frac{k\sigma(\vec{r}-\vec{r}')}{8\pi\varepsilon_0|\vec{r}-\vec{r}'|^3}\cdot d\vec{s}' \quad (一圈圈积)$$

$$= \frac{k\sigma}{8\pi\varepsilon_0}\int_0^{+\infty}\frac{z\, 2\pi r\, dr}{(r^2+z^2)}$$



$$= \frac{k\sigma}{8\varepsilon_0}\int_0^{+\infty}\frac{d(r^2+z^2)}{r^2+z^2}$$

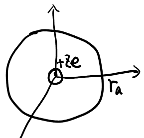
$$= \frac{k\sigma}{8\varepsilon_0}\ln(r^2+z^2)\Big|_0^{+\infty}$$

$$= \frac{k\sigma}{8\varepsilon_0}(\ln+\infty - \ln z^2)$$

$$= \frac{k\sigma}{4\varepsilon_0}\ln z + \frac{k\sigma}{8\varepsilon_0}\ln(+\infty)$$

(不影响 $\vec{E}$)

1.30. 电子电荷密度 $\rho_e = -\frac{Ze}{\frac{4}{3}\pi r_a^3}$



$$\therefore r<r_a 时,\ \varphi = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{r} + \frac{1}{4\pi\varepsilon_0}\frac{\rho\cdot\frac{4}{3}\pi r^3}{r} + \frac{1}{4\pi\varepsilon_0}\iiint_{r<r'<r_a}\frac{\rho(r')}{|\vec{r}-\vec{r}'|}dV' \quad (一圈圈积)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{Ze}{r} - \frac{Zer^2}{r_a^3} + \frac{Ze}{\frac{4}{3}\pi r_a^3}\cdot\int_r^{r_a}\int_0^\pi\int_0^{2\pi}\frac{r'^2\sin\theta'\, dr'd\theta'd\varphi'}{[r^2+r'^2-2rr'(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'))]^{\frac{1}{2}}}\right)$$

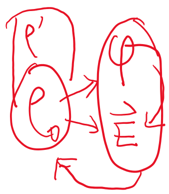
$$= \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{r^2}{r_a^3} + \frac{3}{2r_a}\right) \qquad 流程$$

$$\therefore \vec{E} = -\nabla\varphi = \frac{Ze}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{r_a^3}\right)\vec{e}_r.$$

☆ PPT, 例 1-2-5, 电像法. (题目有问题?)

☆ PPT, 例 1-2-4, 导体给出 ρ_0 后, 不管它了..

☆ PPT，例 1-2-4，导体给出 ρ_0 后，不管它了、、



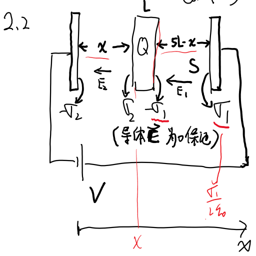
☆ 带电体：

(1) 产生电场：$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{\overleftrightarrow{D}:\hat{r}\hat{r}}{2r^3} + \cdots\right)$

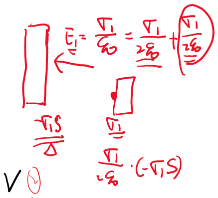
(2) 外电场中：$E_p = Q\varphi(\vec{r}_0) - \vec{p}\cdot\vec{E} - \frac{1}{6}\overleftrightarrow{D}:\nabla\vec{E} + \cdots$

△. 第二章

2.2

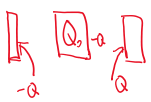


(法一): $\begin{cases} Q=\dfrac{\varepsilon_0 S}{6L}V \\ (\sigma_2-\sigma_1)S=Q \\ \dfrac{\sigma_1}{\varepsilon_0}(5L-x)+\dfrac{\sigma_2}{\varepsilon_0}x=V \end{cases}$



$\therefore \begin{cases} \sigma_1=-\dfrac{\varepsilon_0 V}{30L^2}x+\dfrac{\varepsilon_0 V}{5L} \\ \sigma_2=-\dfrac{\varepsilon_0 V}{30L^2}x+\dfrac{11\varepsilon_0 V}{30L} \end{cases}$ (净电荷在导线上)

① Q怎么分布？



$\therefore F_Q=-\dfrac{\sigma_1}{2\varepsilon_0}\cdot(-\sigma_1 S)-\dfrac{\sigma_2}{2\varepsilon_0}\cdot\sigma_2 S$ (自身的电场对自身无作用)

$=\dfrac{\varepsilon_0 SV^2}{180L^3}x-\dfrac{17\varepsilon_0 SV^2}{360L^2}$

$\therefore W_{F外}=-W_{F_Q}=-\int_L^{3L}F_Q dx=\dfrac{13\varepsilon_0 SV^2}{180L}$

(法二): $dW_{F外}+dW_{电源}=dE_p$ （不可行，因电势改变）

$E_p=\frac{1}{2}\cdot\frac{5L-x}{\varepsilon_0 S}(\sigma_1 S)^2+\frac{1}{2}\cdot\frac{x}{\varepsilon_0 S}(\sigma_2 S)^2$

$=\dfrac{\varepsilon_0 SV^2}{1800L^2}\left[(5L-x)\left(-\frac{x}{L}+6\right)^2+x\left(-\frac{x}{L}+11\right)^2\right]$

$=\dfrac{\varepsilon_0 SV^2}{1800L^2}\left(-\frac{5}{L}x^2+25x+180L\right)$

$=-\dfrac{\varepsilon_0 SV^2}{360L^3}(x^2-5Lx-36L^2)$

$\therefore F_Q'=-\frac{\partial}{\partial x}E_p=\dfrac{\varepsilon_0 SV^2}{360L^3}(2x-5L)\neq F_Q$

$\left[(5L-x)\dfrac{\varepsilon_0^2 V^2}{900L^2}\left(-\frac{x}{L}+6\right)^2+x\cdot\dfrac{\varepsilon_0^2 V^2}{900L^2}\left(-\frac{x}{L}+11\right)^2\right]$

$\dfrac{S}{2\varepsilon_0}\cdot\dfrac{\varepsilon_0^2 V^2}{900L^2}$

$\dfrac{\varepsilon_0 SV^2}{1800L^2}$

$E_p(L)$
$=-\dfrac{\varepsilon_0 SV^2}{360L^3}$
$(L^2-5L^2-36L^2)$
$=\dfrac{\varepsilon_0 SV^2}{9L}$
$E_p(3L)$
$=-\dfrac{\varepsilon_0 SV^2}{360L^3}$
$(9L^2-15L^2-36L^2)$
$=\dfrac{7\varepsilon_0 SV}{360L}$ 60

$\dfrac{40+21}{180}$

$\dfrac{S}{2\varepsilon_0}\left[(5L-x)\sigma_1^2+x\sigma_2^2\right]$

$=\dfrac{S}{2\varepsilon_0}[$

$(5L-x)\left(\frac{x^2}{L^2}-\frac{12x}{L}+36\right)$

$=-\frac{1}{L^2}x^3+\frac{12}{L}x^2-36x+\frac{5}{L}x^2-60x+180L$

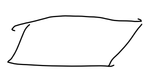
$=-\frac{1}{L^2}x^3+\frac{17}{L}x^2-96x+180L$

$x\left(\frac{1}{L^2}x^2-\frac{22}{L}x+121\right)$

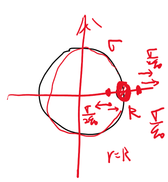
$=\frac{1}{L^2}x^3-\frac{22}{L}x^2+121x$

$\therefore -\frac{5}{L}x^2+25x+180L$

☆ ①



$E=\dfrac{\sigma}{2\varepsilon_0}$



$E=\begin{cases} 0, & r<R \\ \dfrac{4\pi R^2\sigma}{4\pi\varepsilon_0 r^2}=\dfrac{R^2\sigma}{\varepsilon_0}\dfrac{1}{r^2}, & r>R \end{cases}$

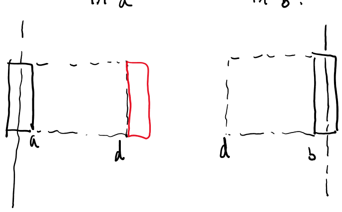
$r=R$, $E=\dfrac{\sigma}{2\varepsilon_0}$ (除去 r=R 的情况)

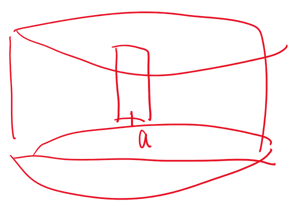
$\longrightarrow \begin{cases} F \\ E_p \end{cases}$

$E=\begin{cases} \dfrac{\sigma}{\varepsilon_0}, & r>R \\ 0, & r<R \end{cases}$

2.5. 看做两个电容器并联

$C_1=\dfrac{2\pi\varepsilon_0}{\ln\frac{d}{a}}$, $C_2=\dfrac{2\pi\varepsilon_0}{\ln\frac{d}{b}}$.

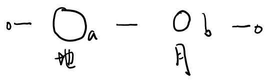




$\therefore C=\left[C_1^{-1}+C_2^{-1}\right]^{-1}=\dfrac{2\pi\varepsilon_0}{\ln\frac{d^2}{ab}}$

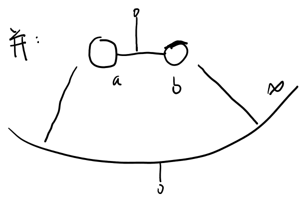
2.6 $C_1=4\pi\varepsilon_0 a$, $C_2=4\pi\varepsilon_0 b$.

(1) 串：



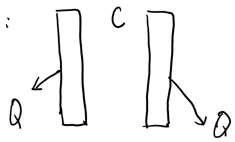
$$C=\left[\frac{1}{4\pi\varepsilon_0 a}+\frac{1}{4\pi\varepsilon_0 b}\right]^{-1}=4\pi\varepsilon_0\frac{ab}{a+b}$$

(2) 并：



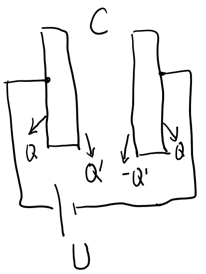
$C=4\pi\varepsilon_0(a+b)$

2.7 充电前：



内极板上无电荷，内无电场.

充电后：

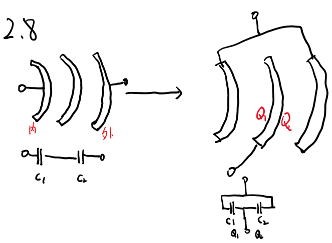


$Q'=CU$

$\therefore$ 总电量：左 $Q+CU$

右 $Q-CU$

2.8



$\therefore C=C_1+C_2=4\pi\varepsilon_0\left[\frac{ab}{b-a}+\frac{bd}{d-b}\right]$

当 $Q_1+Q_2=Q$ 时，

有 $\frac{Q_1}{C_1}=\frac{Q_2}{C_2}$

$\therefore Q_1=\frac{a(d-b)}{d(b-a)+a(d-b)}Q$ （内）

$Q_2=\frac{d(b-a)}{d(b-a)+a(d-b)}Q$ （外）

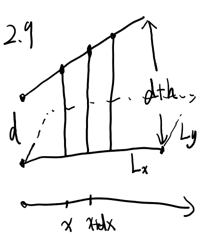
$Q_2=Q-Q_1$

$$\frac{Q_1}{Q-Q_1}=\frac{a\cancel{b}}{b-a}\cdot\frac{d-b}{\cancel{b}d}=\frac{a(d-b)}{d(b-a)}$$

$\therefore d(b-a)Q_1=a(d-b)(Q-Q_1)$

$\therefore Q_1=\frac{a(d-b)}{d(b-a)+a(d-b)}Q$

2.9

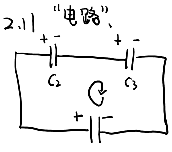


$$C=\int_0^{L_x}dC=\int_0^{L_x}\frac{\varepsilon_0 L_y}{d+\frac{h}{L_x}x}dx$$

$$=\frac{\varepsilon_0 L_x L_y}{h}\ln\left(d+\frac{h}{L_x}x\right)\Big|_0^{L_x}$$

$$=\frac{\varepsilon_0 L_x L_y}{h}\ln\frac{d+h}{d}$$

2.11 "电路"、



$$\therefore\begin{cases}Q_1+Q_2=C_1V_0\\-Q_1-Q_3=-C_1V_0\\-\frac{Q_1}{C_1}+\frac{Q_2}{C_2}+\frac{Q_3}{C_3}=0\end{cases}$$

$\therefore Q_1=\frac{2C_1^2V_0}{2C_1+C_2}$

$Q_2=Q_3=\frac{C_1C_2V_0}{2C_1+C_2}$

$Q_2=Q_3$

$\frac{Q_1}{C_1}=\frac{2Q_2}{C_2}$

$Q_1=\frac{2C_1}{C_2}Q_2$

$\left(1+\frac{2C_1}{C_2}\right)Q_2=C_1V_0$

$Q_2=\frac{C_1C_2V_0}{2C_1+C_2}$

$$\therefore Q_1 = \frac{\quad}{2C_1+C_2}$$

$$Q_2 = Q_3 = \frac{C_1C_2V_0}{2C_1+C_2}$$

$$\left(1+\frac{2C_1}{C_2}\right)Q_2 = C_1V_0$$

$$Q_2 = \frac{C_1C_2V_0}{2C_1+C_2}$$

$$Q_1 = \frac{2C_1^2V_0}{2C_1+C_2}$$

第二章习题（2）

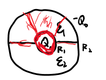
2020年4月13日　9:53

2.14.
$$\begin{cases} \dfrac{Q_1}{\varphi_1}\varphi_1' + C\varphi_3 = Q_1 \\ \dfrac{Q_2}{\varphi_2}\varphi_2' - C\varphi_3 = Q_2 \end{cases} \Rightarrow U = \varphi_1' - \varphi_2' = \frac{Q_1Q_2(\varphi_1-\varphi_2)}{Q_1Q_2 + C\varphi_1Q_2 + C\varphi_2Q_1}$$

2.16、2.20、2.21、2.22 分区均匀介质.

$\begin{cases} \parallel 分界面：用 \vec{E} \\ \perp 分界面：用 \vec{D} \end{cases}$ （判断 ρ 的均匀性）　$[\ \rho(总电荷) = \underbrace{\rho_0}_{(自由电荷)} + \underbrace{\rho'}_{(极化电荷)}\ ]$

(1)



求 $\vec{D}$、$\vec{E}$.（$R_1<r<R_2$）

由对称性，$\vec{E}$均匀，ρ_0不均匀

$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$\nabla\cdot\vec{D} = \rho_0$

$\therefore E\cdot 2\pi r^2 = \frac{Q_{01}}{\varepsilon_1}$　　$E\cdot 2\pi r^2 = \frac{Q}{\varepsilon_0}$

$E\cdot 2\pi r^2 = \frac{Q_{02}}{\varepsilon_2}$

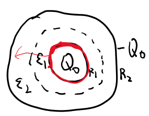
$\therefore (\varepsilon_1+\varepsilon_2)E\cdot 2\pi r^2 = Q_0$　$\varepsilon_1 E$　$D\cdot 2\pi r^2 = Q_{10}$

$\therefore \vec{E} = \frac{Q_0}{2\pi(\varepsilon_1+\varepsilon_2)}\frac{1}{r^2}\hat{r}$.（$R_1<r<R_2$）

$\therefore \vec{D}_1 = \varepsilon_1\vec{E} = \frac{\varepsilon_1 Q_0}{2\pi(\varepsilon_1+\varepsilon_2)}\frac{1}{r^2}\hat{r}$

$\vec{D}_2 = \varepsilon_2\vec{E} = \frac{\varepsilon_2 Q_0}{2\pi(\varepsilon_1+\varepsilon_2)}\frac{1}{r^2}\hat{r}$

(2)



求 $\vec{D}$、$\vec{E}$？

由对称性，ρ_0均匀。

$\therefore D\cdot 4\pi r^2 = Q_0$

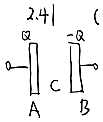
$\therefore \vec{D} = \frac{Q_0}{4\pi}\frac{1}{r^2}\hat{r}$

$\therefore \vec{E}_1 = \frac{\vec{D}}{\varepsilon_1} = \frac{Q_0}{4\pi\varepsilon_1}\frac{1}{r^2}\hat{r}$

$\vec{E}_2 = \frac{\vec{D}}{\varepsilon_2} = \frac{Q_0}{4\pi\varepsilon_2}\frac{1}{r^2}\hat{r}$

2.28.☆ $\begin{cases} \varepsilon_0 时 \\ \varepsilon 时 \\ 导体球 \end{cases}$

2.41



(1) $Q = CV_{AB}$.

隧穿前后，能量须减小。

① e^-：$B\to A$

$\Delta W = \frac{1}{2C}(Q-e)^2 - \frac{1}{2C}Q^2 < 0$

$\therefore\ 2Q - e > 0$

即 $Q > \frac{e}{2}$

$\therefore V_{AB} > \frac{e}{2C}$

② e^-：$A\to B$

$\Delta W = \frac{1}{2C}(Q+e)^2 - \frac{1}{2C}Q^2 < 0$

$\therefore Q < -\frac{e}{2}$

即 $V_{AB} < -\frac{e}{2C}$

由①、②，$V_{AB} < -\frac{e}{2C}$ 或 $V_{AB} > \frac{e}{2C}$ 时可隧穿.

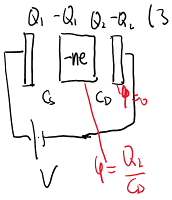
故当 $-\frac{e}{2C} \le V_{AB} \le \frac{e}{2C}$ 时，发生库仑阻塞.（Coulomb blockade）

(2) $\because \frac{e}{2C} = 10\,mV$.

$\therefore C = 8e{-18}\,F$

电子：$W = \frac{1}{2}QU$

(3)



$$\begin{cases} \dfrac{Q_1}{C_S} + \dfrac{Q_2}{C_D} = V \\ -Q_1 + Q_2 = -ne \end{cases}$$

$\therefore Q_1 = \frac{C_SC_D}{C_S+C_D}V + \frac{C_S ne}{C_S+C_D}$

$Q_2 = Q_1 - ne = \frac{C_SC_D}{C_S+C_D}V - \frac{C_D ne}{C_S+C_D}$

（法一）：$W = \frac{1}{2}(-ne)\cdot\frac{Q_2}{C_D} = \frac{ne(ne - C_SV)}{2(C_S+C_D)}$（与V有关）

$W = \frac{1}{2}\iiint_V \rho_0\varphi\, dV$

$W = \frac{1}{2}\iiint_V \vec{D}\cdot\vec{E}\, dV$

（修正）$Q_2 = Q_1 - ne$.

$\frac{1}{C_S}Q_1 + \frac{1}{C_D}Q_1 - \frac{ne}{C_D} = V$

$Q_1 = \left(V + \frac{ne}{C_D}\right)\frac{C_SC_D}{C_D+C_S}$

(法一): $W=\frac{1}{2}(-ne)\cdot\frac{Q_2}{C_D}=\frac{ne(ne-C_SV)}{2(C_S+C_D)}$ (与V有关)

$W_总=W+\frac{1}{2}Q_1V=\frac{n^2e^2+C_SC_DV^2}{2(C_S+C_D)}$

(法二): $V_S=\frac{Q_1}{C_S}=\frac{C_DV+ne}{C_S+C_D}$, $V_D=\frac{Q_2}{C_D}=\frac{C_SV-ne}{C_S+C_D}$

$W_总=\frac{1}{2}C_SV_S^2+\frac{1}{2}C_DV_D^2$

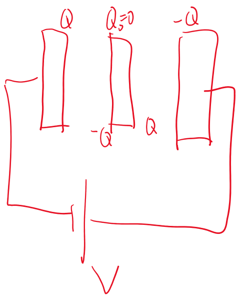
$=\frac{n^2e^2+C_SC_DV^2}{2(C_S+C_D)}$

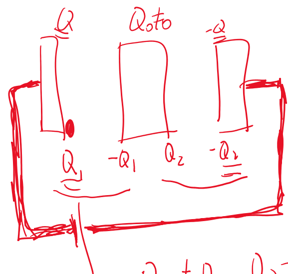
$$\frac{C_S}{2}\frac{(C_DV+ne)^2}{(C_S+C_D)^2}+\frac{C_D}{2}\frac{(C_SV-ne)^2}{(C_S+C_D)^2}$$

$$=\frac{C_S(C_D^2V^2+n^2e^2+2C_DVne)+C_D(C_S^2V^2+n^2e^2-2neC_SV)}{2(C_S+C_D)^2}$$

$$=\frac{C_SC_DV^2(C_D+C_S)+(C_S+C_D)n^2e^2}{2(C_S+C_D)^2}$$

$$=\frac{C_SC_DV^2+n^2e^2}{2(C_S+C_D)}$$





$Q_1\neq Q$, $Q_2\neq Q$

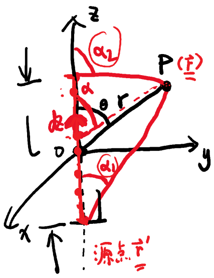
$Q_1\neq Q_2$

☆ 若定义 $W'=W_总-W_C=\frac{n^2e^2}{2(C_S+C_D)}$

[2.4] 第四章习题（1）

2020年4月9日 19:30

4.1 (见 PPT)



$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{I\hat{z}\times(\vec{r}-z\hat{z})}{|\vec{r}-z\hat{z}|^3}dz$$

$$=\hat{\theta}\frac{\mu_0 I r}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dz}{[z^2-2r\cos\theta\cdot z+r^2]^{\frac{3}{2}}}$$

$z=\frac{L}{2}-r\cos\alpha$

亦可对换解：(将 $z\to\alpha$)

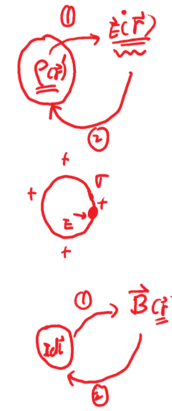
$$\left[=\hat{\theta}\frac{\mu_0 I r}{4\pi}\int_{\alpha_1}^{\alpha_2}\frac{(r\sin\theta/\sin^2\alpha)\,d\alpha}{(r\sin\theta/\sin\alpha)^3}=\hat{\theta}\frac{\mu_0 I}{4\pi r\sin\theta}(\cos\alpha_1-\cos\alpha_2)\right]$$

积分：

$$\therefore \vec{B}(\vec{r})=\frac{\mu_0 I}{4\pi r\sin\theta}(\cos\alpha_1-\cos\alpha_2)\,\hat{\theta}$$

$\therefore$ 当 $r\ll L$ 时，$\alpha_1\approx 0$，$\alpha_2\approx\pi$.

$$\therefore \vec{B}(\vec{r})=\frac{\mu_0 I}{2\pi r\sin\theta}$$

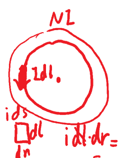


B　$I=\lambda v$　E Q

密度 {线：$I\,d\vec{l}$　面：$\vec{i}\,ds$　体：$\vec{j}\,dV$}　[λdl　σds　ρdV]

$j=\rho v$

4.4 (1) 源 $\vec{r}'$：$r'\in[a,b]$，$\theta'\in[0,2\pi]$



$$\therefore \vec{B}_{(0,0,0)}=\frac{\mu_0}{4\pi}\iint_{S'}\frac{i\,dr'\,d\vec{l}'\times\hat{r}}{r'^2},\quad i=\frac{N}{b-a}I$$

$$=\hat{z}\frac{\mu_0}{4\pi}\frac{N}{(b-a)}I\int_a^b\int_0^{2\pi}\frac{r'\,dr'\,d\theta'}{r'^2}$$

$$=\hat{z}\frac{\mu_0 NI}{2(b-a)}\ln\frac{b}{a}.$$

$i\,dl\,dr=I\,dl$　$i=\frac{NI}{b-a}$

[亦可一圈圈算：每 r' 处面电流环：$dB=\frac{\mu_0 i_0\,dr}{2r}\Rightarrow \vec{B}=\int_a^b d\vec{B}$]

(2) $$\vec{B}(0,0,z)=\frac{\mu_0}{4\pi}\iint_{S'}\frac{i\,dr'\,d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

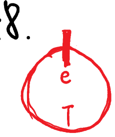
$\vec{r}'\to(r',\theta',z)$　$\vec{r}\to(x,y,z)$

(同4.4(1)) $$=\hat{z}\frac{\mu_0 NI}{4\pi(b-a)}\int_0^{2\pi}\int_a^b\frac{r'\,dr'\,d\theta'}{r'^2+z^2}\cdot\frac{r'}{\sqrt{r'^2+z^2}}$$

$$=\frac{\mu_0 NI}{2(b-a)}\left[\ln\frac{\sqrt{b^2+z^2}+b}{\sqrt{a^2+z^2}+a}+\frac{a}{\sqrt{a^2+z^2}}-\frac{b}{\sqrt{b^2+z^2}}\right]$$

[亦可一圈圈算]

4.8



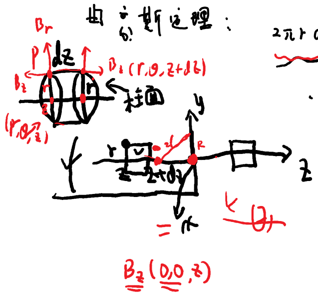
$$I=\frac{\Delta Q}{\Delta t}=\frac{e}{T}=\frac{ev}{2\pi r}$$

则 $$B=\frac{\mu_0 I}{2r}=\frac{\mu_0 ev}{4\pi r^2}=12.52\,T.$$

4.16 由 PPT，

$$B_z(0,0,z)=\frac{\mu_0 nI}{2}(\cos\theta_1-\cos\theta_2)\approx\frac{\mu_0 nI}{2}\left(1-\frac{z}{R}\right)$$

由高斯定理：

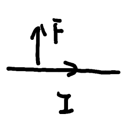


$$2\pi r\,dz\cdot B_r(r,\theta,z)+\pi r^2\,dB_z(r,\theta,z)=0$$

$$\therefore B_r(r,\theta,z)=-\frac{r}{2}\frac{dB_z(r,\theta,z)}{dz}=\frac{r}{2}\frac{dB_z(0,0,z)}{dz}=\frac{\mu_0 nIr}{4R}.$$

[暗由 B_z 与 $\nabla\cdot\vec{B}=0$ 推出]

4.17



$F=ILB$，Δt 内跳起．

$\therefore ILB\cdot\Delta t=mv$

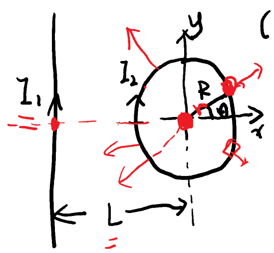
且 $mgh=\frac{1}{2}mv^2$

$$\therefore \quad \quad m\sqrt{2gh}$$

且 $mgh = \frac{1}{2}mv^2$

$\therefore v = \sqrt{2gh}$, $q = I\Delta t = \frac{m\sqrt{2gh}}{Bl}$

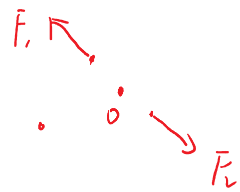
4.18.



(1) I_2 对 I_1 的力 $\vec{F}_{21} = -\vec{F}_{12}$.

$\therefore d\vec{F}_{12} = \underbrace{\frac{\mu_0 I_1}{2\pi(L+R\cos\theta)}}_{dB} \underbrace{I_2 R d\theta}_{Idl} \hat{r}$

（可用对称性简化：$F_{12y} = 0$.）



$dF_{12x} = |d\vec{F}_{12}|\cos\theta$

$\therefore \vec{F}_{12} = \hat{x}\int_0^{2\pi} dF_{12x}\, dx = \hat{x}\mu_0 I_1 I_2 \left(\frac{L}{\sqrt{L^2-R^2}} - 1\right)$

$\vec{F}_{21} = -\vec{F}_{12}$

(2) $\vec{M}_{12}$：以O为心，由于 $d\vec{F}_{12} \propto \hat{r}$

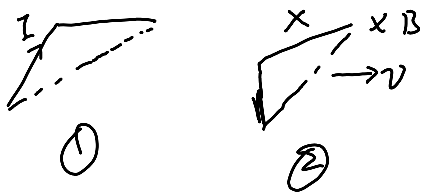
$\therefore d\vec{M}_{12} = \hat{r} \times d\vec{F}_{12} = 0$

$\therefore \vec{M}_{12} = 0$.

第四章习题（2）

2020年4月29日 12:45

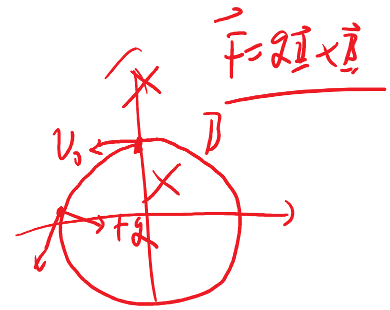
4.23



4.32. $\vec{B} = B\hat{x}$

$q\vec{v}\times\vec{B} = m\vec{a}$

$\therefore \begin{cases} \dot{x} = 0 \ ✓ \\ \dot{y} = \frac{qB}{m}\dot{z} \\ \dot{z} = -\frac{qB}{m}\dot{y} \end{cases}$ ，且 $\begin{cases} x(0)=0 \\ y(0)=0 \\ z(0)=\frac{mv_0\sin\beta}{qB} \end{cases}$ $\begin{cases} \dot{x}(0)=v_0\cos\beta \\ \dot{y}(0)=v_0\sin\beta \\ \dot{z}(0)=0. \end{cases}$



此外可用通解法（特征根）、代入法、猜解法.

$\therefore \begin{cases} x = v_0 t\cos\beta \\ y = \frac{mv_0\sin\beta}{qB}\sin\frac{qB}{m}t \\ z = \frac{mv_0\sin\beta}{qB}\cos\frac{qB}{m}t. \end{cases}$

$\dot{z} = \frac{m}{qB}\ddot{y}$

$\dot{z} = \frac{m}{qB}\ddot{y}$

$\boxed{\frac{m}{qB}\dddot{y} = -\frac{qB}{m}\dot{y}}$

$v = \dot{y}(t)$

$\ddot{v} = -\Omega^2 v$

[2.5] 第五章习题

2020年4月9日 19:31

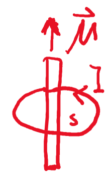
5.4 (1) $N=2\times\frac{m}{M}\cdot N_A=\frac{2N_A\rho V}{M}=1.58\times10^{24}$

(2) 电子磁矩为 $\mu_e=\frac{1}{2}\hbar\mu_B$ (物理常量)

$\therefore \mu=N\mu_e=8.33\times10^{-11}\,A\cdot m^2$

(3) 由线圈的磁矩: $\mu=IS=i\cdot l\cdot S$

$\therefore i=\frac{\mu}{Sl}=8.84\times10^{-6}\,A\cdot m^{-1}$



(4) 看做无限长螺线管:

$B=\mu_0 i=1.11\times10^{-11}\,T.$

5.6 $Q=CU=4\pi\varepsilon_0 RU$

[也可以看做多个线圈磁矩]

$\therefore \vec{i}(\vec{r})=\sigma\vec{v}(\vec{r})=\frac{4\pi\varepsilon_0 RU}{4\pi R^2}\cdot\omega R\quad \hat{z}\times\hat{r}$

$=\varepsilon_0 U\omega\quad(\sin\theta\cos\varphi\,\hat{y}-\sin\theta\sin\varphi\,\hat{x})$

$\therefore \vec{m}=\frac{1}{2}\iint_S \vec{r}\times\boxed{\vec{i}(\vec{r})\,dS}$ $Id\vec{l}$ $\quad d\vec{m}=\frac{1}{2}\vec{r}\times(Id\vec{l})$

$=\varepsilon_0\omega U\int_0^{2\pi}\int_0^{\pi}R\hat{r}\times(\sin\theta\cos\varphi\,\hat{y}-\sin\theta\sin\varphi\,\hat{x})R^2\sin\theta\,d\theta\,d\varphi$

$=\varepsilon_0 UR^3\omega\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\,(\cos\varphi\,\hat{r}\times\hat{y}-\sin\varphi\,\hat{r}\times\hat{x})\,d\theta\,d\varphi$

$=\varepsilon_0 UR^3\omega\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\left[\cos\varphi(\sin\theta\cos\varphi\,\hat{z}-\cos\theta\,\hat{x})-\sin\varphi(-\sin\theta\sin\varphi\,\hat{z}+\cos\theta\,\hat{y})\right]d\theta\,d\varphi$

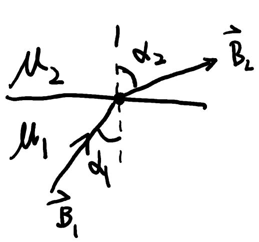
$=\varepsilon_0 UR^3\omega\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\left(\sin\theta\,\hat{z}-\cos\theta\cos\varphi\,\hat{x}-\cos\theta\sin\varphi\,\hat{y}\right)d\theta\,d\varphi$

$=2\pi\varepsilon_0 UR^3\omega\int_0^{\pi}\sin^3\theta\,d\theta\quad\hat{z}$

$=\frac{4\pi}{3}\varepsilon_0\omega UR^3\quad\hat{z}$

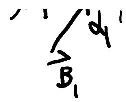
5.10 $\because \vec{n}\cdot(\vec{B}_2-\vec{B}_1)=0$

$\vec{n}\times(\vec{H}_2-\vec{H}_1)=0$



$\therefore B_1\cos\alpha_1=B_2\cos\alpha_2$

$\frac{B_1}{\mu_0}\sin\alpha_1=\frac{B_2}{\mu_0\mu_2}\sin\alpha_2$

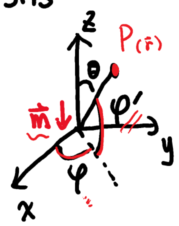


$$\frac{B_1}{\mu_0}\sin\alpha_1=\frac{B_2}{\mu_0\mu_2}\sin\alpha_2$$

$$\therefore \tan\alpha_1=\frac{\tan\alpha_2}{\mu_2}$$

$$\therefore \alpha_2 \approx 0.09^\circ$$

5.13



磁偶极子在球坐标下： $\vec{m}=-m\hat{z}$

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi r^3}\left[-\vec{m}+3(\vec{m}\cdot\hat{r})\hat{r}\right]$$

$$=\frac{\mu_0}{4\pi r^3}\left[m\hat{z}-3m\cos\theta\,\hat{r}\right]$$

$$=\frac{\mu_0 m}{4\pi r^3}\left(\sin\theta\,\hat{\theta}-2\cos\theta\,\hat{r}\right)$$

$$=\frac{\mu_0 m}{4\pi r^3}\left(\cos\varphi'\,\hat{\varphi}'-2\cos\theta\,\hat{r}\right).$$

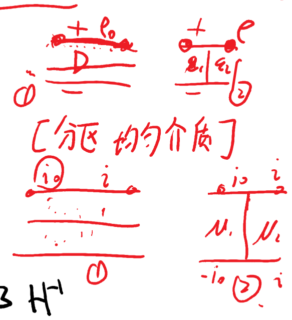
$\therefore \tan\beta=\frac{B_y}{B_x}$

$$\therefore \tan\beta=\frac{2\sin\varphi'}{\cos\varphi'}=2\tan\varphi'.$$

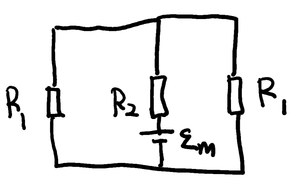
5.14. 5.16. 判断 $\vec{i}_0$ 均匀还是 $\vec{i}$ 均匀

（根据对称性，均为 $\vec{i}_0$ 均匀）

[分区 均匀介质]



5.20. 磁路图：



$$R_1=\frac{L}{\mu S}=5411.3\ H^{-1}$$

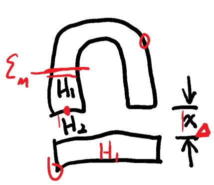
$$R_2=5411.3\ H^{-1}$$

$$\varepsilon_m=NI$$

$$\therefore \Phi=\frac{\varepsilon_m}{R_{总}}$$

$$\therefore \varepsilon_m=NI=BS\left(\frac{R_1}{2}+R_2\right)=94583A.$$

5.21.



$$H_1L+2H_2x=2NI \qquad 且\ L=2l+\pi\cdot\left(a+\frac{a}{2}\right)+2a+d=3l+\left(\frac{3}{2}\pi+2\right)a=0.64m.$$

$$且\ \vec{n}\cdot(\vec{B}_2-\vec{B}_1)=0$$

$$\therefore \mu_0H_2=\mu H_1$$

$$\frac{1}{2}\cdot 2NI.$$

$$\therefore B_1=\frac{2\mu\mu_0NI}{\mu_0L+2\mu x}.$$

$$\therefore W_m=\frac{1}{2}I\Phi_m=\frac{2\mu\mu_0I^2N^2a^2}{\mu_0L+2\mu x}$$

$\mu_0 L T \ 2\mu^2 x$

$$\therefore F = \frac{1}{2}\left(\frac{\partial W_m}{\partial x}\right)_{x=0} = -\frac{2\mu^2 N^2 I^2 a^2}{\mu_0 L^2} = -99.5\ N.$$

☆ 小磁体在远处:

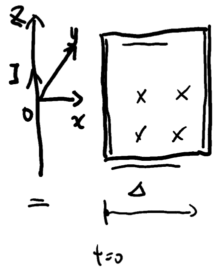
(1) 产生磁场: $\vec{B}(\vec{r}) = \frac{\mu}{4\pi}\left[\left(-\frac{\vec{m}}{r^3} + \frac{3(\vec{m}\cdot\hat{r})\hat{r}}{r^3}\right)+\cdots\right]$

(2) 外场中: $E_p = \vec{m}\cdot\vec{B}_{外} + \cdots$

[2.6] 第六章习题

2020年4月29日 12:45

6.3



(1) $\because \vec{B}=\dfrac{\mu_0 I}{2\pi r}\hat{\varphi}$

$\therefore \Phi=\iint_S \vec{B}\cdot d\vec{S}=\dfrac{\mu_0 I_0 l\sin\omega t}{2\pi}\int_a^b \dfrac{1}{r}dr$

$=\dfrac{\mu_0 I_0 l\sin\omega t}{2\pi}\ln\dfrac{b}{a}.$

$\therefore \mathcal{E}=-\dfrac{d\Phi}{dt}=-\dfrac{\mu_0 I_0 l\omega\cos\omega t}{2\pi}\ln\dfrac{b}{a}$

(2) $\Phi=\dfrac{\mu_0 I_0 l\sin\omega t}{2\pi}\int_{a+vt}^{b+vt}\dfrac{1}{r}dr$

$=\dfrac{\mu_0 I_0 l\sin\omega t}{2\pi}\cdot\ln\dfrac{b+vt}{a+vt}$

$\therefore \mathcal{E}=-\dfrac{d\Phi}{dt}.\Leftrightarrow$ 动生+感生

(3) $I=\dfrac{\mathcal{E}}{R}$

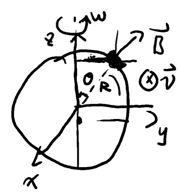
$F=B_1IL-B_2IL$

① 源 → 场
(q, $Id\vec{l}$) (E, B)
(ε, …)

② 场 → 源

6.4. $\mathcal{E}=\int_L \vec{v}\times\vec{B}\cdot d\vec{l}$ $\quad \vec{B}=V.$

$=-\dfrac{\mu_0 M\omega}{4\pi R}$ (L定向)



$\vec{m}=-m\hat{z}$; $\vec{B}=\dfrac{\mu_0}{4\pi}\left[-\dfrac{\vec{m}}{R^3}+\dfrac{3(\vec{m}\cdot\vec{R})\vec{R}}{R^5}\right]=\dfrac{\mu_0}{4\pi}\left[\dfrac{m}{R^3}\hat{z}-\dfrac{3m\cos\theta}{R^3}\hat{R}\right]=\dfrac{\mu_0 M}{4\pi R^3}\left[\hat{z}-3\cos\theta\hat{R}\right]=-\dfrac{\mu_0 M}{4\pi R^3}\left[\sin\theta\hat{\theta}+2\cos\theta\hat{R}\right]$

$\hat{z}=\hat{R}\cos\theta-\hat{\theta}\sin\theta$

$\vec{v}=R\omega\sin\theta\hat{\varphi}$, $d\vec{l}=Rd\theta\hat{\theta}$

$\therefore \mathcal{E}=\int_L \vec{v}\times\vec{B}\cdot d\vec{l}$

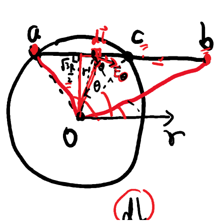
$=-\int\left[R\omega\sin\theta\hat{\varphi}\times\dfrac{\mu_0 m}{4\pi R^3}\left[\sin\theta\hat{\theta}+2\cos\theta\hat{R}\right]\right]\cdot Rd\theta\hat{\theta}$

$=-\dfrac{\mu_0 m\omega}{4\pi R}\int_L\left[-\hat{R}+\hat{\theta}\right]\sin\theta\cdot d\theta\hat{\theta}$

$=-\dfrac{\mu_0 m\omega}{4\pi R}\int_0^{\pi}\sin\theta d\theta$

$=-\dfrac{\mu_0 m\omega}{2\pi R}$

6.14.



(法一)：求出 E. $\because \vec{E}=E_\theta(r)\hat{\theta}$

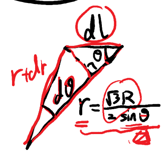
$\therefore \mathcal{E}=\oint_C \vec{E}\cdot d\vec{r}=E_\theta(r)\cdot 2\pi r$.

又 $\because \mathcal{E}=-\dfrac{d\Phi}{dt}=\begin{cases}-k\cdot\pi r^2, & r\le R\\ -k\cdot\pi R^2, & r>R\end{cases}$

$\therefore E_\theta(r)=\begin{cases}-\dfrac{k}{2}r, & r\le R\end{cases}$

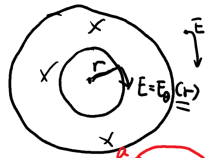
$= \cdots \begin{cases} \cdots \\ -k\cdot\pi R^2, \quad r>R \end{cases}$

$\therefore E_\theta(r) = \begin{cases} -\dfrac{k}{2} r, & r \le R \\ -\dfrac{kR^2}{2}\dfrac{1}{r}, & r > R. \end{cases}$

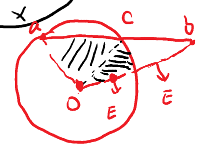


$\because U_{ab} = \int_{ab} \vec{E}\cdot d\vec{l} = \int_{ac} \frac{k}{2} r\sin\theta \cdot \frac{r\,d\theta}{\sin\theta} + \int_{cb} \frac{kR^2}{2}\cdot\frac{1}{r}\sin\theta\cdot\frac{r\,d\theta}{\sin\theta}$ ✓

且 $r = \frac{\sqrt{3}R}{2\sin\theta}$.



$\therefore U_{ab} = \frac{k}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{3R^2}{4\sin^2\theta}\,d\theta + \frac{kR^2}{2}\theta\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \left(\frac{\sqrt{3}}{4} + \frac{\pi}{12}\right)kR^2$



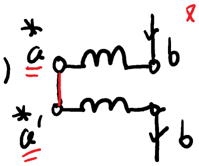
(法二)： $\mathcal{E}_{ab} = \mathcal{E}_{oabo}$

$= \left(\frac{\sqrt{3}}{4} + \frac{\pi}{12}\right)kR^2$

$\mathcal{E} = -\frac{d\Phi}{dt} = S\cdot k$

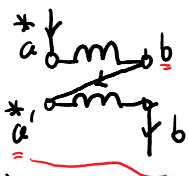
又 $\because \mathcal{E} = \mathcal{E}_{ab} + \mathcal{E}_{bo} + \mathcal{E}_{oa} = \mathcal{E}_{ab}$.

6.27. (1)



异名端串： $L = L_1 + L_2 - 2M = L_1 + L_2 - 2\sqrt{L_1L_2} = 0$

(2)



同名端串： $L = L_1 + L_2 + 2M = 0.2\,\text{H}$

6.31. 代入公式证明.