重整化的数学方法和几何应用
Renormalization: Mathematical Structures and Geometric Applications

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"Anyone who is not shocked by quantum theory has not understood it."—Niels Bohr

"I think I can safely say that nobody understands quantum mechanics."—Richard Feynman

The theory of the interaction of quantum fields of radiation (photons) and Dirac fields (electrons and positrons) formulated in the early 1930s is known as quantum electrodynamics (QED).

Puzzle: the first QED approximation (e.g. for Compton scattering) produces a reasonable result, while the second, involving vacuum-polarization effects, yields an infinite contribution.

Renormalization was first developed in quantum electrodynamics (QED) to make sense of infinite integrals in perturbation theory.

The puzzle was resolved in the late 1940s, mainly by **Bethe**, **Tomonaga**, **Feynman**, **Schwinger** and **Dyson**.

$$\Downarrow$$

Covariant and gauge-invariant formulations of QED that allow computations of observables at any order of perturbation theory.

Tomonaga, **Schwinger** and **Feynman**: 1965 Nobel Prize.

With a series of important developments, the perturbative construction of quantum field theory was essentially complete.

(**Bogoliubov**, **Parasiuk** and **Hepp** (BPH); **Zimmermann**, **Epstein**, **Glaser**, **Steinmann**, **Faddeev**, **'t Hooft**, **Veltman**; **Becchi**, **Rouet**, **Stora** (BRS); **Gell-Mann**, **Low**, **Kadanoff**, etc).

Wilson's theory of renormalization group, as a "theory of theories" , connects the microscopic with the macroscopic. The work earned Wilson a Nobel Prize.

Renormalization has subsequently become one of the fundamental aspects of quantum physics and a criterion for acceptability.

- Statistical mechanics
- Condensed-matter/solid-state physics
- Standard Model
- Non-perturbatively renormalizable QFTs
- Asymptotic freedom/safety
- ...

Quantum field theory deals with "∞-dimensional geometry", which lies behind many of its nontrivial consequences and predictions.

Typically (but not always) a physics system is described by a map

$$S : \mathcal{E} \to \mathbb{R}.$$

- $\mathcal{E}$: *space of fields*.
- S: *action functional*.

Typical examples

- Scalar field theory

$$\mathcal{E} = C^\infty(X)$$

- Gauge theory

$$\mathcal{E} = \{\text{connections on } V \to X\}$$

- σ-model

$$\mathcal{E} = \mathrm{Map}(\Sigma, X)$$

- Gravity

$$\mathcal{E} = \{\text{metrics on } X\}$$

Path integral

- Classical physics is decribed by the critical locus (equation of motion, eg: Laplace equation, Yang-Mills equation, etc)

$$\mathrm{Crit}(S) = \{\delta S = 0\}.$$

- Quantum physics can be described by Feynman's "path integral"

$$\langle \mathcal{O} \rangle := \int_{\mathcal{E}} \mathcal{O} e^{iS/\hbar}$$

$\mathcal{O}$: quantum observable. $\langle \mathcal{O} \rangle$: correlation function.
  - Mathematical challenge for such ∞-dim integral.
  - Asymptotic analysis leads to - renormalization theory.

We are mainly interested in "integrals"

$$\int f$$

We rarely compute integrals by definition (Riemann/Lebesgue).
Instead, we use symmetries and differential equations.

Example: Gaussian integral

Gaussian integral

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}}\ e^{-\frac{1}{2}x^2} = 1$$

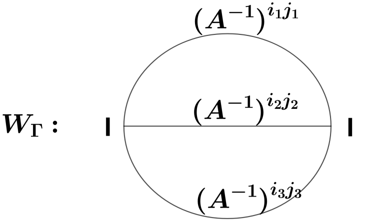
or more generally

$$\int_{\mathbb{R}^n} \prod_i \frac{dx^i}{\sqrt{2\pi}}\ e^{-\frac{1}{2}A(X)} = \frac{1}{\sqrt{\det A}}, \quad \text{where} \quad A(x) = \sum_{i,j} A_{ij}x^i x^j$$

$A = (A_{ij})$ is a positive definite matrix.

Feynman diagram expansion

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{dx^i}{\sqrt{2\pi}} e^{-\frac{1}{2}A(x)+I(x)} \sim \frac{1}{\sqrt{\det(A)}} \exp\left(\sum_{\Gamma:\text{conn}} \frac{W_\Gamma}{|Aut(\Gamma)|} \right)$$



Combinatorial formula via the inverse matrix A^{-1} and I.

A^{-1} : propagator

In quantum field theory, we can use Feynman's formula to model the ∞-dim integral asymptotically.

Example (ϕ^4-theory)

$$\int_{\mathcal{E}=C^\infty(X)} [D\phi]\ e^{-\frac{1}{\hbar}S[\phi]}, \qquad S[\phi] = \frac{1}{2}\int_X \phi\Delta\phi + \lambda \int_X \phi^4.$$

where Δ is the Laplacian operator ($\Delta = -\sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}$ on flat space).

The inverse Δ^{-1} is

$$\text{Green's function} \qquad G(x,y) \sim \Delta^{-1}$$

The index i, j is replaced by points x, y on X.

The Green's function is singular along the diagonal

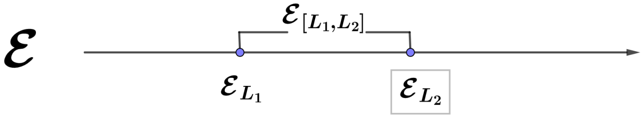
$$G(x, y) \sim \frac{1}{|x - y|^{d-2}}, \qquad x \to y.$$

In Feynman diagrams, we will encounter integrals where we multiply many G's together. They are divergent in general!

This is called the UV divergence in QFT, due to the nature of ∞-many degrees of freedom.

Basic idea of renormalization (we use **Wilson**'s viewpoint): we set a scale and cut the full degrees of freedom

$$\mathcal{E} = \bigcup \mathcal{E}_L, \qquad \mathcal{E}_{L_2} = \mathcal{E}_{L_1} \oplus \mathcal{E}_{[L_1,L_2]}.$$



On each $\mathcal{E}_L$, we have an effective action S_L. They are related by

$$e^{\frac{i}{\hbar}S_{L_1}} = \int_{\mathcal{E}_{[L_1,L_2]}} e^{\frac{i}{\hbar}S_{L_2}}.$$

Renormalization group flow.

There are many ways we can cut:

- Momentum cut
- Distance cut
- Energy/ Eigenvalue cut
- ...

To construct such S_L, we can have

- Scale dependence of the coupling constants
- Running under renormalization group flow
- Renormalizable theories: perturbation computation
- ...

Some examples of renormalization method in QFT

Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) approach

- A scheme for subtracting UV divergence in Feynman integral
- Locality of subtractions (divergent counter-terms).
- Normalization conditions (finite counter-terms).

Connes-Kreimer: BPHZ Renormalization as a Birkhoff decomposition over the dual Hopf algebra of Feynman graphs.

Costello: Homotopic renormalization in BV formalism.

Why QFT has rich structures?

$$\text{Spacetime} : X \quad \Longrightarrow \quad \text{Fields} : \mathcal{E} = \Gamma(X, E).$$

- $\mathcal{E}$ is the space (called fields) where we will do calculus $\int_{\mathcal{E}}$.
- Topology of X leads to new structures in ∞-dim geometry

When X =point, $\mathcal{E} = \mathbb{R}^n$. We arrive at the usual calculus.

$$\boxed{\text{Calculus} = \text{0-dim QFT}}.$$

When $\dim X > 0$, the geometry and topology of X come in!

One algebraic structure associated to the topology of X is

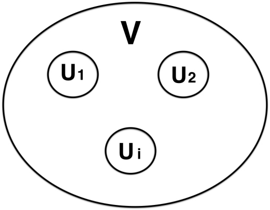
observables=functions on fields

Given an open subset $U \subset X$, we can talk about

$$Obs(U) = \text{observables supported in U}$$

Example: δ-function.

Observables form an algebraic structure as follows: given disjoint open subset U_i contained in an open V: $\coprod_i U_i \subset V$



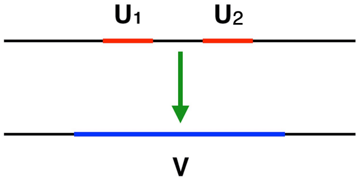
we have a factorization product for observables

$$\bigotimes_i Obs(U_i) \to Obs(V).$$

- Physics: OPE (operator product expansion)
- Mathematics: factorization algebra.
  - Origin: **Beilinson-Drinfeld** in 2d CFT
  - **Costello-Gwilliam**: (perturbative renormalized) QFT.

Example: $\dim X = 1$ (topological quantum mechanics)

QFT in $\dim = 1$ is quantum mechanics.



In the topological case, for any contractible open U, $Obs(U) = A$.

The factorization product doesn't depend on the location and size:

$$A \otimes A \to A \qquad \Longrightarrow \qquad \text{associative algebra}.$$

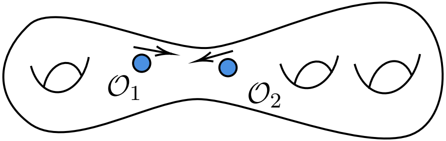
The operations of left and right multiplication are characterized by

$$H_\bullet(\mathbb{R} - \{0\}, \mathbb{Z}) = \mathbb{Z}\ \text{Left} \oplus \mathbb{Z}\ \text{Right}.$$

Example: $\dim X = 2$ (chiral conformal field theory)

The factorization product of 2d chiral theory is holomorphic.

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



which is the 2d analogue of "associative product". We find ∞-many binary operations $\mathcal{O}_{1(n)} \cdot \mathcal{O}_2$!

In this case, observable algebra forms a vertex algebra.

The binary operations operations are parametrized by

$$H^{0,\bullet}_{\bar\partial}(\mathbb{C} - \{0\}) = \text{Span}\,\{z^n\}_{n\in\mathbb{Z}}\,.$$

An important class of quantities are correlation functions of observables. They capture "global" information of the theory.

- Local correlation

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\rangle\,, \quad x_i \in X.$$

  It is singular when points collide, hence a function on

$$\mathrm{Conf}_n(X) := \{x_1, \cdots, x_n \in X | x_i \neq x_j \text{ for } i \neq j\}.$$

- Non-local correlation

$$\int_{\mathrm{Conf}_n(X)} \langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\rangle$$

  which might be divergent and require further renormalization.

Example: abelian Chern-Simons and Linking

We consider the abelian Chern-Simons theory on S^3.

$$CS[A] = \frac{1}{2}\int_{S^3} A \wedge dA, \qquad A: \text{ 1-form on } S^3$$

Let C, C' be two disjoint circles inside S^3. Consider

$$\left\langle \int_C A \int_{C'} A \right\rangle = \int [DA] e^{iCS[A]} \left(\int_C A\right)\left(\int_{C'} A\right)$$

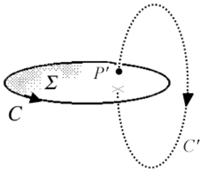
The propagator is d^{-1}.

In a suitable interpretation (gauge), this correlation function is

$$\langle [C'], d^{-1}([C]) \rangle = \int_{C'} d^{-1}([C])$$

$=$fill C by a disk Σ and intersect with C'

$=$Linking number of C and C'.



We find the Gauss Linking formula

$$\mathrm{Link}(C, C') = \frac{1}{4\pi} \int_C \int_{C'} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \cdot (d\vec{r}_1 \times d\vec{r}_2).$$

Example: Iterated integral and quantum mechanics

Let $LX = \mathrm{Map}(S^1, X)$ free loop space of X. Consider

$$\begin{array}{ccc} \mathrm{Conf}_n(S^1) \times LX & \xrightarrow{ev} & X^n \\ \downarrow{\scriptstyle \pi} & & \\ LX & & \end{array}$$

where ev sends $(p_1, \cdots, p_n) \times \gamma \to (\gamma(p_1), \cdots, \gamma(p_n))$. Then

$$\pi_* ev^* = \int_{\mathrm{Conf}_n(S^1)} ev^*(-) : (\Omega(X))^{\otimes n} \to \Omega(LX)$$

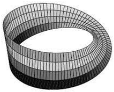
defines a quasi-isomorphism [**K.T. Chen**]

$$\mathrm{Hochschild}(\Omega(X)) \quad \to \quad \Omega(LX).$$

This can be viewed as correlation functions in quantum mechanical model, detecting the topology of the free loop space LX.

Geometry enhanced by QFT

One main object in geometry and topology is the vector bundle

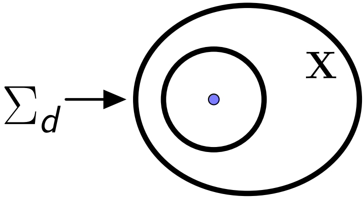


$$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

This is fibered by $\mathbb{R}^n$, which can be viewed as 0-dim QFT.

$$\begin{array}{c} QFT_0 \\ \downarrow \\ X \end{array}$$

In generel, a QFT of σ-model $\Sigma_d \to X$



will produce a geometry of

$$\begin{array}{c} QFT_d \\ \downarrow \\ X \end{array}$$

We get a large class of new geometries enhanced by QFT.

Geometric Application: Quantum Mechanics and Index Theory

Topological quantum mechanics (TQM)

TQM leads to a path integral on the loop space

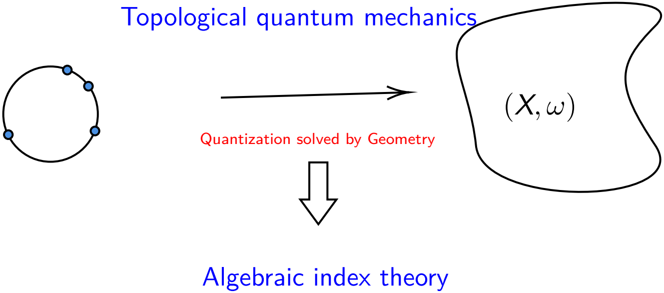
$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \quad \overset{\hbar\to 0}{\Longrightarrow} \quad \int_X (\text{curvatures})$$

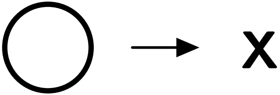
Topological nature implies the exact semi-classical limit $\hbar \to 0$, which localizes the path integral to constant loops.

- LHS= the analytic index expressed in physics
- RHS= the topological index.

This is the physics "derivation" of **Atiyah-Singer** Index Theorem.

In [**Grady-Li-L 2017**, **Gui-L-Xu, 2020**], a rigorous connection between the effective BV quantization for topological quantum mechanics and the algebraic index theorem.





They glue [**Fedosov**] on X to give a bundle of Weyl algebras

$$\begin{array}{c} \mathcal{W}(X) \\ \downarrow \\ X \end{array}$$

- [**Grady-Li-L**, **Gui-L-Xu**]: Quantization of TQM. Correlation function of non-local observables $\int_{\mathrm{Conf}(S^1)}$ gives

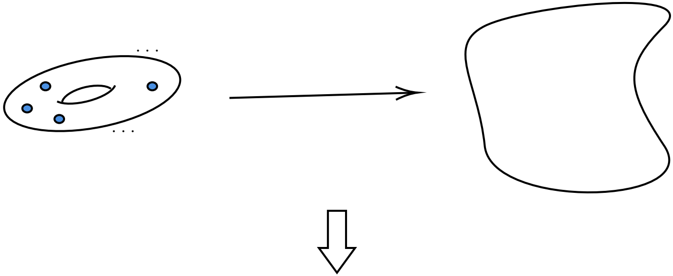
$$\langle 1 \rangle = \int_X e^{\omega_\hbar/\hbar} \hat{A}(X).$$

This is the simplest version of algebraic index theorem which was first formulated by **Fedosov** and **Nest-Tsygan** as the algebraic analogue of Atiyah-Singer index theorem.

Geometric Application: 2d Chiral CFT and Chiral Index

Witten's "Index Theorem" on loop space

Replace S^1 by an elliptic curve E. (**Witten**: index of Dirac operators on loop space).

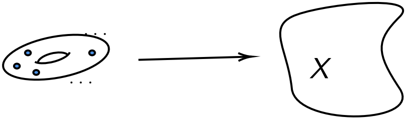


2d Chiral analogue of algebraic index?

Example: 2d Chiral CFT

A chiral σ-model

$$\varphi : E \to X$$



will produce a bundle $\mathcal{V}(X)$ of chiral vertex algebras

$$\begin{array}{c}\mathcal{V}(X)\\ \downarrow \\ X\end{array}$$

The quantization/renormalization leads to a flat gluing [**L**].

The issue of singular integral and renormalization

Correlation function of local observables

$$\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$$

is very singular along diagonals and this integral requires renormalization. We need to understand the meaning of

$$\int_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle \text{"="?}$$

- Geometric renormalization by regularized integrals [**L-Zhou**].
- Elliptic chiral algebraic index [**Gui-L**].

Regularized integral (L-Zhou 2020)

Let us first consider the integral of a 2-form ω on Σ with meromorphic poles of arbitrary orders along a finite subset $D \subset \Sigma$. Locally we can write $\omega = \frac{\eta}{z^n}$ where η is smooth 2-form and $n \in \mathbb{Z}$.

We can decompose ω into

$$\omega = \alpha + \partial\beta$$

where α is a 2-form with at most logarithmic pole along D, β is a $(0,1)$-form with arbitrary order of poles along D, and $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham. We define the regularized integral

$$\boxed{\fint_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial\Sigma} \beta}$$

This does not depend on the choice of the decomposition.

$⨍_\Sigma$ is invariant under conformal transformations. The conformal geometry of Σ gives an intrinsic regularization of the integral $\int_\Sigma \omega$.

The regularized integral can be viewed as a "homological integration" by the holomorphic de Rham ∂

$$⨍_\Sigma \partial(-) = \int_{\partial\Sigma} (-).$$

The $\bar{\partial}$-operator intertwines the residue

$$⨍_\Sigma \bar{\partial}(-) = \mathrm{Res}(-).$$

In general, we can define

$$\int_{\Sigma^n} (-) := \int_{\Sigma} \int_{\Sigma} \cdots \int_{\Sigma} (-) \, .$$

This gives a rigorous and intrinsic definition of

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \int_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle .$$

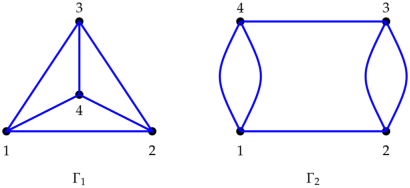


Fig. 9. Two trivalent graphs Γ_1, Γ_2 with 4 vertices each

$$\widehat{I}_{\Gamma_2} = \int_{E_\tau^4} \left(\prod_{i=1}^{4} \frac{d^2 z_i}{\operatorname{im} \tau} \right) \Phi_{\Gamma_2} = \frac{(2\pi i)^{12}}{2^{10} \cdot 3^7} (-3\widehat{E}_2^6 + 6\widehat{E}_2^4 E_4 + 4\widehat{E}_2^3 E_6 - 3\widehat{E}_2^2 E_4^2 - 12\widehat{E}_2 E_4 E_6 + 4E_4^3 + 4E_6^2).$$

More Examples

Example: $\dim X = 4$ (holomorphic theory)

We consider 4d holomorphic theory on $X = \mathbb{C}^2$. The algebraic structures that lie behind the factorization products will contain

$$\mathsf{H}^{\bullet}_{\bar\partial}(\mathbb{C}^2 - \{0\}) = \mathsf{H}^0_{\bar\partial} \oplus \mathsf{H}^1_{\bar\partial}.$$

By Hartogs's extension theorem, $\mathsf{H}^0_{\bar\partial} = \mathbb{C}[z_1, z_2]$ while

$$\mathsf{H}^1_{\bar\partial} = \mathbb{C}\left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right].$$

So it will predict degree one OPEs indexed by $\mathsf{H}^1_{\bar\partial}$.

What are they in physics?

Example: Mirror symmetry

Mirror symmetry is about a duality between

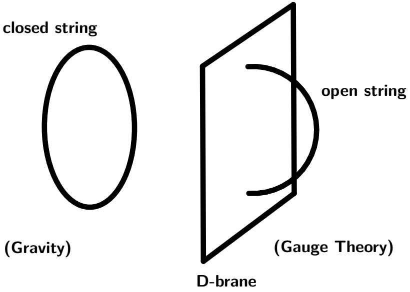
$$\boxed{\text{symplectic geometry}}\ (\text{A-model}) \Longleftrightarrow \boxed{\text{complex geometry}}\ (\text{B-model})$$

$$\begin{array}{ccc}
\int_{\mathrm{Map}(\Sigma_g, X)} (\text{A-model}) & \overset{\text{Fourier transform}}{=\!=\!=\!=\!=} & \int_{\mathrm{Map}(\Sigma_g, X')} (\text{B-model}) \\
\Big\downarrow \text{localize} & & \text{localize} \Big\downarrow \\
\int_{\text{Holomorphic maps}(\Sigma_g, X)} & \longleftarrow - - - - \longrightarrow & \int_{\text{Constant maps}(\Sigma_g, X')} \ ??? \\
\Big\Downarrow & & \Big\downarrow \\
\text{Gromov-Witten Theory} & & \text{Hodge theory}
\end{array}$$

The B-model can be viewed as a suitable mysterious way to "count constant surfaces", which will be related to the variation of Hodge structures and its quantization.

Example: Gauge/Gravity duality

Gauge theory at large N $\Longrightarrow$ Dynamics of Gravity



[**Costello-L**]: Renormalization theory for

- Twisted supergravity
- Open-closed string field theory in the large N.
- Koszul duality

New structures of mathematics blow up from QFT and yet to come!

Thank you!

I have taught a course in spring 2022 on aspects of this talk. If you are interested in, you can find lecture information at https://sili-math.github.io/teachingpage.htm.

Hyperbolicity Ubiquitous

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September 26, 2022

Hyperbolicity: its structures and implications

Hyperbolic fixed point

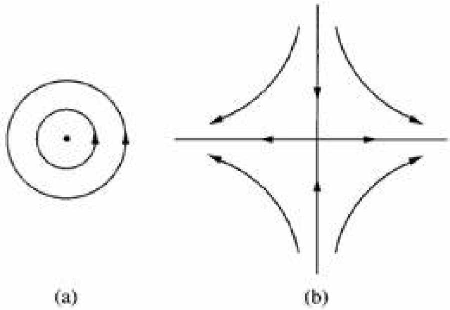


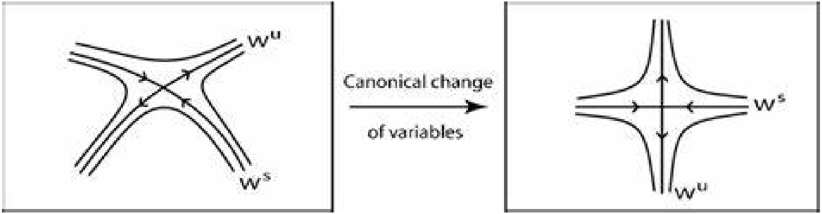
Figure: Phase portraits for elliptic and hyperbolic fixed points

ODEs:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Instability and structural stability

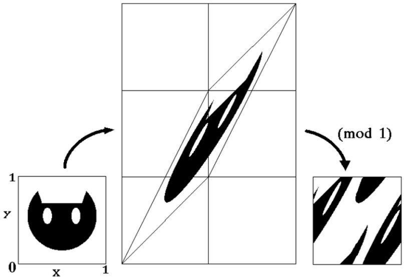
- A hyperbolic fixed point is **unstable**: for almost all points close to the fixed point, its forward evolution will escape.
- However, it also enjoys the **structural stability**: If we perturb the second equation $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, the phase portrait will be deformed, but can be brought back to the standard one.



Arnold cat map

A more interesting example is the Arnold cat map, discovered by Anosov.
Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, consider

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \ \mathbb{T}^2 \to \mathbb{T}^2.$$



Properties of the cat map

- The matrix A has $\det A = 1$ and $\operatorname{tr} A = 3$, so two eigenvalues

$$\lambda_1 = \frac{1}{2}(3+\sqrt{5}) > 1 > \frac{1}{2}(3-\sqrt{5}) = \lambda_2$$

  and corresponding eigenvectors

$$v_1 = \left(1, \frac{1}{2}(-1+\sqrt{5})\right), \quad v_2 = \left(\frac{1}{2}(-1+\sqrt{5}), -1\right).$$

- The set of lines parallel to v_1 (respectively v_2) are preserved by the map A. Each line has irrational slope, when projected to the torus $\mathbb{T}^2$, it winds around the torus densely.
- The two sets of lines projects to two transversely intersecting **foliations** on $\mathbb{T}^2$, one is expanded by A and the other is contracted.

Ergodicity

Hyperbolicity is a mechanism that implies some important properties of a dynamical system. Let $f: M \to M$ be a map preserving the volume vol on M.

Definition (Ergodicity)

The map f is called **ergodic**, if for any set S with $\mathrm{vol}(S) > 0$ and $\mathrm{vol}(S \Delta f^{-1}(S)) = 0$, we have $\mathrm{vol}(S) = 0$ or $\mathrm{vol}(S) = \mathrm{vol}(M)$.

Theorem (Birkhoff ergodic theorem)

Let f be ergodic, then for a.e. point $x \in M$, and any $\phi \in L^1$, we have as $n \to \pm\infty$

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^k(x) \to \frac{1}{\mathrm{vol}(M)} \int \phi(x)\, d\mathrm{vol}.$$

Hopf Argument

Theorem

The Arnold cat map is ergodic with respect to volume.

Proof.

- Choose any continuous function ϕ, and we consider $S_n^{\pm}(x) := \frac{1}{n}\sum_{k=0}^{n-1} \phi \circ f^{\pm k}(x)$.
- If two points x and y are on the same stable leaf, i.e. line parallel to v_2, then we have $|A^n x - A^n y| \leq C\lambda_2^n,\ n > 0$, thus $\lim_n S_n^+(x) = \lim_n S_n^+(y)$.
- Similarly, if two points x and y are on the same unstable leaf, i.e. line parallel to v_1, then we have $|A^n x - A^n y| \leq C\lambda_1^n, n < 0$, thus $\lim_n S_n^-(x) = \lim_n S_n^-(y)$.
- **Modulo a zero measure set**, we can connect two points x, y by stable/unstable leaves. Thus, $\lim_n S_n^{\pm}(x) = \lim_n S_n^{\pm}(y)$.

□

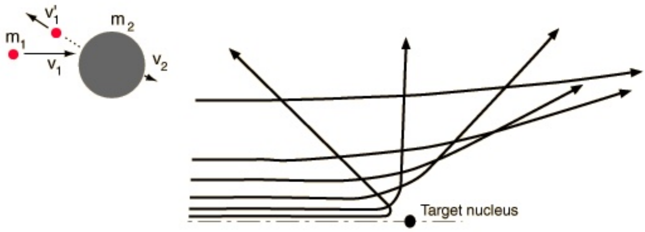
The idea of deterministic chaos

- The form

$$\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ f^k(x)\to\frac{1}{\mathrm{vol}(M)}\int\phi(x)d\mathrm{vol}$$

  in Birkhoff ergodic theorem is similar to the **law of large numbers** in probability theory, e.g. coin tossing.
- It is a general principle that a system with sufficient hyperbolicity is ergodic. In other words, the system, even though deterministic, after sufficiently many iterations, the dynamics behaves so chaotic that it looks like a random variable. We can even pursue central limit theorem like results.

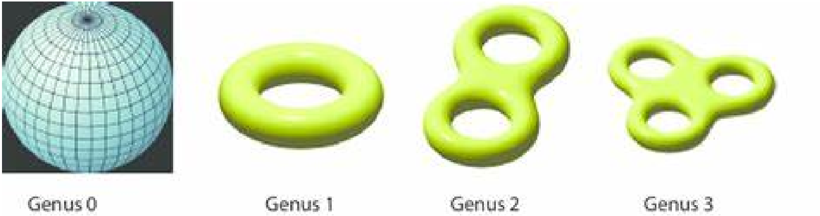
Some natural philosophy



In the scattering process of two particles, a small change of initial positions leads to a huge change of the outgoing velocities. This is a form of hyperbolicity. The outgoing velocities are almost random. When the number of particles is large, the system tends to be ergodic, and the process becomes not reversible. This is the origin of time arrow, characterized by the increasing of entropy.

Hyperbolic surfaces

Classification of closed surfaces



Arnold: this is a theorem as significant as Columbus' discovery of America.

Most surfaces are hyperbolic

- On $\mathbb{S}^2$, we can give it a metric of constant curvature 1;
- On $\mathbb{T}^2$, we can give it a metric of constant curvature 0;
- On $\Sigma_g,\ g > 1$, we can give it a metric of constant curvature -1.

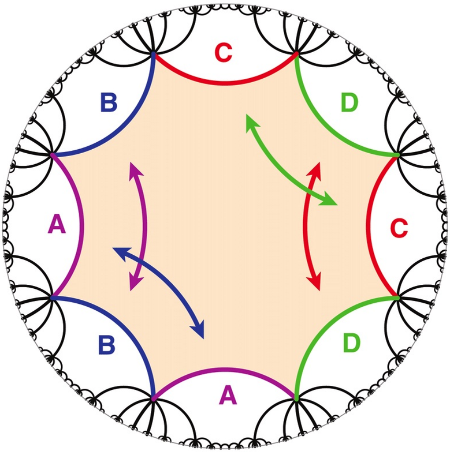
The last case is called hyperbolic, for the following reason.
If we study the distance between nearby geodesics, we get the **Jacobi field equation**

$$x'' + Kx = 0,$$

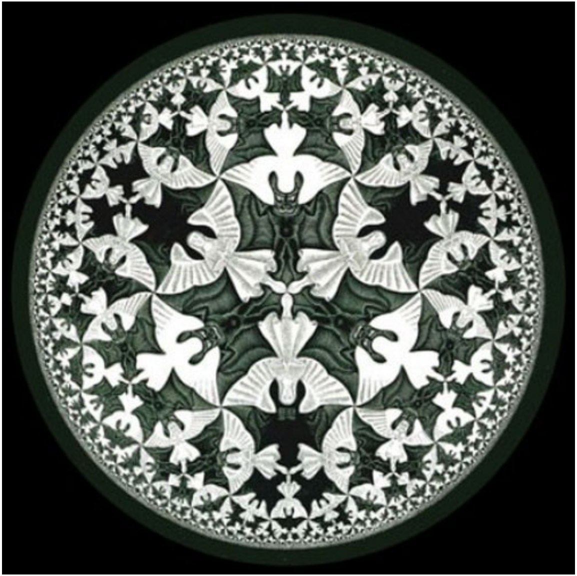
where K is curvature.

- when $K = 1$, the solution oscillates $x = A\cos t + B\sin t$;
- when $K = 0$, the solution grows linearly $x = A + Bt$;
- when $K = -1$, the solution grows exponentially $x = Ae^t + Be^{-t}$.

Hyperbolicity and negative curvature



Angels and Demons by Escher



Fundamental groups

The elements of the fundamental group $\pi_1(X)$ of the pointed topological space (X, x_0) are the homotopy classes of closed paths in X.

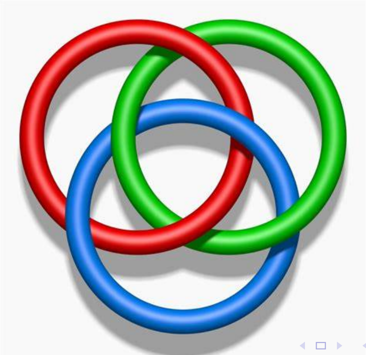
- $\pi_1(\mathbb{S}^2) = \{*\}$;
- $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$;
- $\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid b_g^{-1} a_g^{-1} b_g a_g \dots b_1^{-1} a_1^{-1} b_1 a_1 = 1\rangle$.

The fundamental group of a hyperbolic surface is also very complicated, almost a free group. They fits into the general theory of hyperbolic group of Gromov. It is known that **most groups are hyperbolic.**

Most 3-manifolds are hyperbolic

It was known to Thurston that

- Every closed 3-manifold is obtained from the 3-sphere $\mathbb{S}^3$ by Dehn surgery along some link whose complement is hyperbolic.
- Suppose $L \subset M^3$ is a link such that $M \setminus L$ has a hyperbolic structure, then most manifold obtained from M by a Dehn surgery along L has hyperbolic structures.



Thurston's theory on mapping class groups

Classification of mapping class groups

The mapping class group of a surface is its group of homeomorphisms modulo isotopy.
For example, the MCG of $\mathbb{T}^2$ is given by $\mathrm{SL}_2\mathbb{Z}/\{\pm I\}$. It can be classified into ($A \in \mathrm{SL}_2\mathbb{Z}/\{\pm I\}$)

- $\operatorname{tr} A = 0, 1$, **periodic**, $\exists p \in \mathbb{N}$ s.t. $A^p = I$, e.g. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$;
- $\operatorname{tr} A = 2$, **reducible**, A^n grows linearly, e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$;
- $\operatorname{tr} A > 2$, **Anosov**, A^n grows exponentially, e.g. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

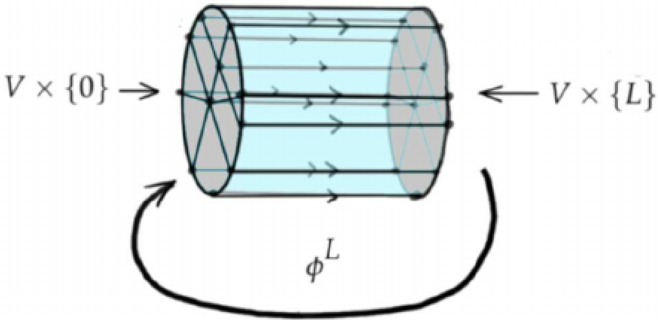
It is not hard to see that **most mapping classes are Anosov.**

Thurston's geometrization program

3-manifolds are modeled on the following 8 types:

- Spherical $\mathbb{S}^3$;
- Euclidean $\mathbb{R}^3$;
- Hyperbolic $\mathbb{H}^3$;
- $\mathbb{S}^2 \times \mathbb{R}$;
- $\mathbb{H}^2 \times \mathbb{R}$;
- $T_1\mathbb{H}^2$;
- Nilmanifold;
- Solv-manifold.

Mapping torus construction



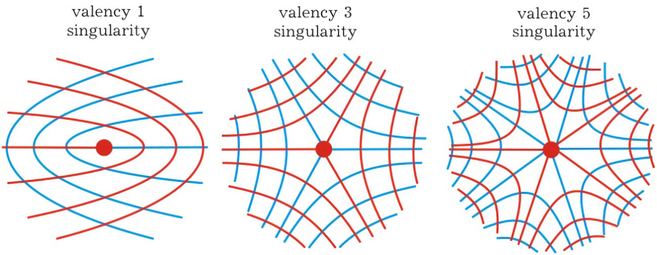
Let $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism. We have the mapping torus construction giving a 3-manifold M_ϕ: We glue the boundary $\{0\} \times \mathbb{T}^2$ and $\{1\} \times \mathbb{T}^2$ of the product space $[0, 1] \times \mathbb{T}^2$ via $(0, \phi x) \sim (1, x)$.

3-manifolds from the mapping torus construction

- If ϕ is periodic, then M_ϕ is finitely covered by $\mathbb{T}^3$.
- If ϕ is reducible, then M_ϕ is a nilmanifold.
- If ϕ is Anosov, then M_ϕ is a solv-manifold.

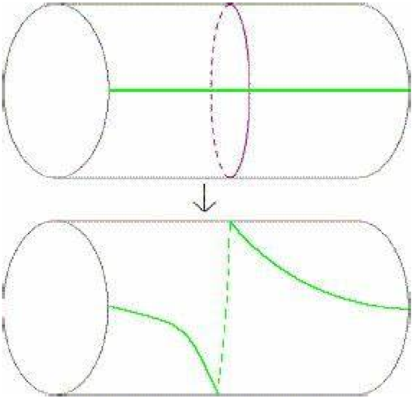
Pseudo Anosov elements

For surfaces of genus greater than 1, similarly, the mapping class group can be classified into three classes: periodic, reducible and pseudo-Anosov.



For a pseudo-Anosov element, the surface admits two foliations that intersect transversely, except for finitely many singular points. The map expands one foliation and contracts the other.

Dehn twist



Composing Dehn twists produces pseudo-Anosov elements

- Note that we have the observation

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

  which means that the cat map is the composition of two Dehn twists along two intersecting "essential" circles of the torus.
- This is also true in general. The MCG is generated by Dehn twists.

A higher dimensional generalization

We are interested in pursuing a higher dimensional generalization of Thurston's theory.
In particular, we are interested in generalizing the principle of **composing Dehn twists produces hyperbolicity.**
The problem is in general very difficult, and it makes sense to study manifolds and maps with some additional structures.

Symplectic manifold and symplectic maps

Definition

- A symplectic manifold is a manifold M^{2n} of dimension $2n$ endowed with a sympletic form ω, that is a closed nondegenerate 2-form.
- A diffeomorphism $\phi : M \to M$ is called a symplectic map if $\phi^*\omega = \omega$. The set of all symplectic maps are denoted by $\text{Symp}(M, \omega)$.

In dimension 2, i.e. $n = 1$, a symplectic map is the same as an area preserving diffeormorphism.

Arnold-Seidel Dehn twist

We define the model Dehn twist τ on $T^*\mathbb{S}^n$ to be

$$\tau(x,v) = \begin{cases} \sigma(r(|v|))(x,v) & |v| > 0 \\ (A(x),0) & |v| = 0 \end{cases}$$

for $x \in \mathbb{S}^n$, $v \in T_x^*\mathbb{S}^n$, where

- $\sigma(r(|v|))(x,v)$ means starting from the point $x \in \mathbb{S}^n$ we travel along the geodesic in the direction v for time $r(|v|)$,
- and $r(t) \in C^\infty(\mathbb{R},\mathbb{R})$ is a smooth function such that $r(-t) + r(t) = 2\pi$, $r(t) = 0$ for $t \geq \epsilon$, and $\frac{d^k r}{dt^k}(0) = 0, \forall\ k \geq 1$.

The twist τ reduces to the standard Dehn twist when $n = 1$.

Symplectic Dehn twist as a nontrivial element in $\mathrm{Symp}(M, \omega)$

The symplectic Dehn twist is a remarkable symplectically nontrivial map, but it is isotopic to identity in the group of diffeomorphisms (Seidel).

It is also known in many cases the generator of all symplectic mapping classes.

Composition of symplectic Dehn twists

Theorem (Wenmin Gong, Zhijing Wang, X.)

Let S_1 and S_2 be two Lagrangian spheres in a symplectic manifold M, intersecting transversely at a single point. Then there exist symplectic Dehn twists τ_1 and τ_2 of S_1 and S_2, such that for any $k, \ell \in \mathbb{Z}, k\ell < 0$, the topological entropy of the composition $\tau = \tau_1^k \tau_2^\ell$ of symplectic Dehn twists is positive, i.e. $h_{top}(\tau_1^k \tau_2^\ell) > 0$.

Remark

- *Lagrangian submanifold $L^n \subset M^{2n}$ is a submanifold such that $\omega|_{TL} = 0$.*
- *Positive topological entropy means expansion, and we also have a structure generalizing the singular foliations.*

Symplectic growth rate

The last result shows certain hyperbolicity of the composite symplectic Dehn twists as **diffeomorphisms**. To take into account of the symplectic aspects, we need to study the behavior of **symplectic invariants**.

Definition (Symplectic growth rate)

Let (M, ω) be a symplectic manifold, and ϕ a symplectomorphism of M. Let (L_1, L_2) be a pair of connected compact Lagrangian submanifolds of M. When the Lagrangian Floer cohomology group $\mathrm{HF}(L_1, \phi^n(L_2))$ are well defined for all $n \in \mathbb{N}$, then the symplectic growth rate of the triple (ϕ, L_1, L_2) is defined by

$$\Gamma(\phi, L_1, L_2) = \liminf_{n\to\infty} \frac{\log \operatorname{rank} \mathrm{HF}(L_1, \phi^n(L_2))}{n}.$$

Exponential growth of Floer homology groups

We next show that the exponential growth can actually be achieved by composite symplectic Dehn twists.

Theorem (Wenmin Gong, Zhijing Wang, X.)

Let M be an exact symplectic manifold which is equal to a symplectization of a contact manifold near infinity. Let S_1 and S_2 be two Lagrangian spheres of M intersecting transversely at a single point, and τ_1 and τ_2 be two symplectic Dehn twists along the spheres S_1 and S_2 respectively. Then for any $k, \ell \in \mathbb{Z}$ with $k\ell \neq 0, 1, 2, 3, 4$, we have that $\Gamma(\tau_1^k \tau_2^\ell, S_1, S_2) > 0$;

The standard map conjecture

- It is conjectured that the following standard map $\phi_a : \mathbb{T}^2 \to \mathbb{T}^2$

$$\phi_a(x, y) = (x + y + a \sin x, y + a \sin x)$$

has positive Lyapunov exponent $\lambda(x) := \lim \frac{1}{n} \log D\phi_a^n(x)$ for x in a positive measure set of $\mathbb{T}^2$.

- The problem is asking if a map in a reducible mapping class $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has dynamics similar to an Anosov class.
- We may ask similar problem for composite symplectic Dehn twist, which is isotopic to identity, but symplectically Anosov.

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Thank you very much for your attention!