PROBLEM SET 1, PART 1: TOPOLOGY (H)
DUE: FEBRUARY 28, 2022, BEFORE CLASS

(1) [Topology of symbols]
Classify the following symbols according to the topology of their pictures:

$$\Pi, \Sigma, \Psi, \Phi, \Gamma, \Upsilon, \Omega, \Theta, \Xi, \heartsuit$$
$$\phi, \varphi, \pi, \theta, \alpha, \beta, \gamma, \mu, \tau, \delta, \epsilon$$
$$+, \times, \otimes, \nabla, \cup, \sim, \propto, \to$$

(2) [Fake soccer!]
In Figure 1 you can see a soccer ball that I found from the internet. Obviously the careless designer never learned topology. Explain why.

FIGURE 1. Fake soccer ball

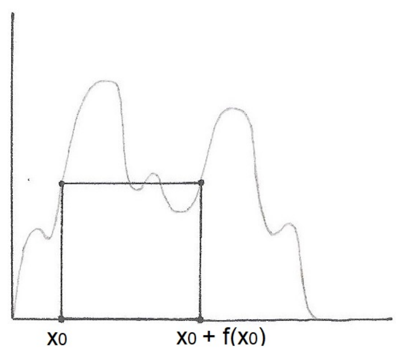


FIGURE 2. Inscribed square

(3) [Inscribed square problem: a simple case]
Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1) = 0$. Consider the simple closed curve C that consists of the graph of f and the line segment of the x-axis from $x = 0$ to $x = 1$. Prove: One can find four points on C that are the vertices of a square. [Hint: Consider the function $g(x) = f(x) - f(x + f(x))$. See Figure 2.]

(4) [Weierstrass's counterexample to Dirichlet principle]
For any $u \in \mathcal{A} = \{C^1([-1,1]) \mid u(-1) = 0, u(1) = 1\}$, define

$$F(u) = \int_{-1}^{1} |xu'(x)|^2 dx.$$

(a) Prove: For each $n \in \mathbb{N}$, the function

$$u_n(x) := \left(\sin \frac{n\pi x}{2}\right)^2 \chi_{[0,1/n]}(x) + \chi_{(1/n,1](x)}$$

is an element in $\mathcal{A}$ (where $\chi_A(x)$ is the characteristic function of the set A).

(b) Prove: $\lim_{n\to\infty} F(u_n) = 0$.

(c) Prove: There is no function $u \in \mathcal{A}$ that attains the minimum of F.

TOPOLOGY (H), PROBLEM SET 1, PART 2
DUE: FEBRUARY 28, 2022, BEFORE CLASS

(1) [Pseudo-metric]

A *pseudo-metric* on a set X is a map $d : X \times X \to [0, +\infty)$ that satisfies

- $d(x, x) = 0$. [Note: this is weaker than being a metric.]
- $d(x, y) = d(y, x)$.
- $d(x, y) + d(y, z) \geq d(x, z)$.

Let (X, d) be a pseudo-metric space. Define an *equivalence relation* on X via

$$x \sim y \iff d(x, y) = 0.$$

Let $\overline{X} = X/\sim$ be the quotient (i.e. the set of equivalent classes), and let $p : X \to \overline{X}$ be the quotient map. Prove: there is a unique metric $\bar{d}$ on $\overline{X}$ so that

$$d(x, y) = \bar{d}(p(x), p(y)).$$

(2) [Metric-preserving functions]

Let $f : [0, +\infty) \to [0, +\infty)$ be a function (which need not be continuous). We say f is a *metric-preserving function* if for any metric space (X, d), the map $\widetilde{d} : X \times X \to \mathbb{R}$ defined by $\widetilde{d}(x, y) := f(d(x, y))$ is a metric on X.

(a) Prove: If f is a metric-preserving function, then $f^{-1}(\{0\}) = \{0\}$ and f is sub-additive:

$$f(\alpha + \beta) \leq f(\alpha) + f(\beta), \qquad \forall \alpha, \beta \in [0, +\infty).$$

(b) Prove: a function $f : [0, +\infty) \to [0, +\infty)$ satisfying $f^{-1}(\{0\}) = \{0\}$ is metric-preserving if any one of the following conditions holds:

(i) f is non-decreasing and sub-additive.
(ii) f is concave.
(iii) There exists constant $c > 0$ so that for any $x > 0$, $f(x) \in [c, 2c]$.

(3) [Urysohn's lemma]

Let (X, d) be a metric space, For any subset $A \subset X$, define

$$d_A : X \to [0, +\infty), \quad x \mapsto d_A(x) = \inf_{a \in A} d(x, a).$$

Prove:

(a) d_A is a continuous function on X.
(b) A is closed if and only if $d_A(x) = 0$ implies $x \in A$.
(c) (*Urysohn's lemma for metric spaces*) If A and B are closed subsets in (X, d) and $A \cap B = \emptyset$. Then there exists a continuous function $f : X \to [0, 1]$ such that

$$f \equiv 0 \text{ on } A, \qquad \text{and} \qquad f \equiv 1 \text{ on } B.$$

(4) [Uniform convergence as a metric convergence]
Let $f_n : (X, d_X) \to (Y, d_Y)$ $(n \in \mathbb{N})$ and $f : (X, d_X) \to (Y, d_Y)$ be maps between metric spaces.
  (a) Define "uniform convergence": f_n converges uniformly to f on X if ...
  (b) Suppose f_n are continuous, and converges to f uniformly. Prove: f is continuous.
  (c) On the set $Y^X = \{f : X \to Y \mid f \text{ is any map}\}$, define

$$\bar{d}(f, g) := \sup_{x \in X} \frac{d_Y(f(x), g(x))}{1 + d_Y(f(x), g(x))}.$$

    (i) Prove: $\bar{d}$ is a metric on Y^X.
    (ii) Prove: f_n converges to f uniformly if and only if as elements in the metric space $(Y^X, \bar{d})$, f_n converges to f.

PROBLEM SET 2, PART 1: TOPOLOGY (H)
DUE: MARCH 07, 2022

(1) ["Uniform continuity" is not a topological conception]
Let (X, d_X) and (Y, d_Y) be metric spaces. We say a map $f : (X, d_X) \to (Y, d_Y)$ is *uniformly continuous* if

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

(a) Prove: $d_0(x, y) := |\arctan(x) - \arctan(y)|$ is a metric on $\mathbb{R}$.
(b) Prove: The metric d_0 and the absolute value metric $d(x, y) = |x - y|$ on $\mathbb{R}$ are topologically equivalent. Are they strongly equivalent?
(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the identity map, i.e. $f(x) = x$. Is $f : (\mathbb{R}, d) \to (\mathbb{R}, d_0)$ uniformly continuous? Is $f : (\mathbb{R}, d_0) \to (\mathbb{R}, d)$ uniformly continuous? Conclude that "Uniform continuity" is not a topological conception.
(d) Is "uniform continuity" preserved if we replace metrics d_X, d_Y by strongly e-quivalent ones? Prove your conclusion.

(More generally, there is a structure called "uniform structure", which is a generalization of metric structure, so that one can define uniform continuous maps between spaces with uniform structures.For details, c.f. J.L. Kelley, *General Topology*.)

(2) [The product topology and product metrics]
(a) Prove Proposition 1.44 (the product topology is a topology).
(b) Let (X, d_X) and (Y, d_Y) be metric spaces. Endow the product space $X \times Y$ with the metric

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Prove:
(i) If U is open in (X, d_X), V is open in (Y, d_Y), then $U \times V$ is open in $(X \times Y, d_{X\times Y})$.
(ii) W is an open set in $(X \times Y, d_{X\times Y})$ if and only if for any $(x, y) \in W$, there exists $r > 0$ such that $B(x, r) \times B(y, r) \subset W$. [So the metric topology induced by the product metric is the same as the product topology induced by metric topologies.]

(c) (**NOT REQUIRED**) Prove: The same conclusion holds if we replace the metric $d_{X\times Y}$ above by

$$d^p_{X\times Y}((x_1, y_1), (x_2, y_2)) := (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{1/p},$$

where $1 \le p \le +\infty$. Note: for $p = \infty$ we define

$$d^\infty_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

(3) [Equivalence of neighborhoods axioms and open sets axioms: Proposition 1.37]

(a) Given a neighborhood structure $\mathscr{N}$ on X, one can define a topology $\mathscr{T}$ via

$$\mathscr{T} = \{U \subset X \ : \ U \in \mathscr{N}(x) \text{ for any } x \in U.\}$$

Check: $\mathscr{T}$ is a topology on X, i.e. it satisfies (O1)-(O3)

(b) Given a topology $\mathscr{T}$ on X, one can define, for any $x \in X$,

$$\mathscr{N}(x) = \{N \subset X : \exists U \in \mathscr{T} \text{ s.t. } x \in U \text{ and } U \subset N\}.$$

Check: $\mathscr{N}$ is a neighborhood structure on X, i.e. it satisfies (N1)-(N4).

(c) You may have already noticed that in doing part (a), you used only (N1)-(N3). Can we conclude that the set of axioms (N1)-(N3) is equivalent to the set of axioms (O1)-(O3)?

(d) (**NOT REQUIRED**) Prove: the set of axioms (N1)-(N4) is equivalent to the set of axioms (O1)-(O3). Namely, the process $\mathscr{T} \rightsquigarrow \mathscr{N}$ and $\mathscr{N} \rightsquigarrow \mathscr{T}$ described above are inverse to each other.

(4) [Furstenberg's topological proof of the infinitude of primes]

For any $a, b \in \mathbb{Z}$ with $b > 0$ we define

$$N_{a,b} := \{a + nb \mid n \in \mathbb{Z}\}.$$

(a) (**NOT REQUIRED**) Define a topology on $\mathbb{Z}$ by

$$\mathscr{T}_{Furs} = \{U \subset \mathbb{Z} \mid \text{ either } U = \emptyset, \text{ or } \forall a \in U, \exists b \in \mathbb{Z}_{>0} \text{ s.t. } N_{a,b} \subset U\}.$$

(i) Prove: $\mathscr{T}_{Furs}$ is a topology on $\mathbb{Z}$.

(ii) Prove: Each $N_{a,b}$ is open.

(iii) Prove: Each $N_{a,b}$ is closed. [Hint: $N_{a,b} = \mathbb{Z} \setminus \cup_{i=1}^{b-1} N_{a+i,b}$.]

(iv) Let $\mathscr{P} = \{2, 3, \cdots\}$ be the set of all prime numbers. Prove:

$$\mathbb{Z} \setminus \{1, -1\} = \cup_{p \in \mathscr{P}} N_{0,p}.$$

(v) Conclude that $\mathscr{P}$ is not a finite set. [Hint: the set $\{1, -1\}$ can't be open.]

(b) Define a function $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$d(a, b) = \begin{cases} 0, & a = b \\ 2^{-\tau(a-b)}, & a \neq b, \end{cases}$$

where $\tau(a - b)$ is the smallest positive integer that does not divide $a - b$.

(i) Prove: d is a metric on $\mathbb{Z}$.

(ii) Describe the metric balls $B(a, r)$.

(iii) Show that the metric topology generated by d is the topology $\mathscr{T}_{Furs}$ above.

PROBLEM SET 2, PART 2: TOPOLOGY (H)
DUE: MARCH 07, 2022

(1) [The Sorgenfrey line]

Endow $\mathbb{R}$ with the Sorgenfrey topology

$$\mathscr{T}_{Sorgenfrey} = \{U \subset \mathbb{R} \mid \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } [x, x+\varepsilon) \subset U\}.$$

(a) Check: $\mathscr{T}_{Sorgenfrey}$ is a topology.
(b) Prove: Every left-closed-right-open interval $[a, b)$ is both open and closed.
(c) Prove: $\mathscr{T}_{Sorgenfrey}$ is strictly stronger than the usual topology $\mathscr{T}_{usual}$ on $\mathbb{R}$.
(d) Explore the meaning of convergence in $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$.
(e) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *right continuous* if $\lim_{x_n \to x_0+} f(x_n) = f(x_0)$. Prove: a function $f : \mathbb{R} \to \mathbb{R}$ is right continuous if and only if the map $f : (\mathbb{R}, \mathscr{T}_{Sorgenfrey}) \to (\mathbb{R}, \mathscr{T}_{usual})$ is continuous. [So people also call Sorgenfry topology *the right continuous topology*.]
(f) **[Upper semi-continuous topology]** Let $(X, \mathscr{T})$ be any topological space. We say a function $f : X \to \mathbb{R}$ is *upper semi-continuous* at a point $x_0 \in X$ if for any $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $f(x) \le f(x_0) + \varepsilon$ holds for all $x \in U$, and we say f is an *upper semi-continuous* function if it is upper semi-continuous everywhere. Construct a new topology $\mathscr{T}_{u.s.c}$ on $\mathbb{R}$ so that a function $f : X \to \mathbb{R}$ is upper semi-continuous if and only if the map $f : (X, \mathscr{T}) \to (\mathbb{R}, \mathscr{T}_{u.s.c})$ is continuous.

(2) [The pasting lemma]

Let X, Y be topological spaces. Consider a map $f : X \to Y$.

(a) Suppose $X = A \bigcup B$, where A, B are both closed subsets in X. Suppose $f|_A : A \to Y$ and $f|_B : B \to Y$ are continuous. Prove: $f : X \to Y$ is continuous.
(b) Show that the same result fails for $X = \bigcup_{n=1}^{\infty} A_n$, where each A_n is closed in X.
(c) Prove: If $X = \cup_\alpha U_\alpha$, where each U_α is open in X, and if $f|_{U_\alpha} : U_\alpha \to Y$ is continuous, then $f : X \to Y$ is continuous.

(3) [Homeomorphisms]

(a) Let $N = (0, \cdots, 0, 1)$ be the "north pole" of $S^n = \{(x^1, \cdots, x^{n+1}) | x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$. Show that $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$ by explicitly construct a homeomorphism. [Hint: stereographic projection.]
(b) Use Brouwer's invariance of domain theorem (see the end of Remark 1.58) to prove: If $n \neq m$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.
(c) Prove: If $f : X \to Y$ is a homeomorphism, then for any $A \subset X$, $f : X \setminus A \to Y \setminus f(A)$ is a homeomorphism.
(d) Let $\text{Homeo}(X)$ be the set of all homeomorphisms from X to X. Prove: $\text{Homeo}(X)$ is a group (with respect to the composition of maps). Moreover, if X and Y are homeomorphic, then the groups $\text{Homeo}(X)$ and $\text{Homeo}(Y)$ are isomorphic.

(4) (**NOT REQUIRED**) [Convergence in measure, almost everywhere convergence]
Let X be the set of all bounded measurable functions defined on $[0, 1]$. For any $f, g \in X$, we define

$$d(f, g) = \int_0^1 \min(|f(x) - g(x)|, 1)dx.$$

(a) Prove: d is a metric on X.
(b) Prove: $f_n \in X$ converges to f in measure if and only if f_n converges to f with respect to the metric d. (So in particular, "convergence in measure" is a topological convergence)
(c) Prove: almost everywhere convergence is not a topological convergence, i.e. there is no topology on X so that $f_n \to f$ a.e. if and only if $f_n \to f$ in that topology. [Hint: In real analysis, we learned that Riesz's theorem, which claims that if $f_n \to f$ in measure, then there is a subsequence $f_{n_k} \to f$ a.e.. Suppose there is such a topology. Find a sequence f_n in X that converges to f in measure, but fails to converge to f a.e.. Suppose such a topology exists. Since f_n fails to converge to f a.e., there is an open neighborhood U of f so that a subsequence sits outside U. But that subsequence still converges in measure, and thus has a sub-subsequence that converges a.e. to f, a contradiction.]

PROBLEM SET 3, PART 1: TOPOLOGY (H)
DUE: MARCH 14, 2022

(1) [Neighborhood basis]
Like a basis, we can define a *neighborhood basis* (or *neighborhood base*) as follows: A family $\mathcal{B}(x) \subset \mathcal{N}(x)$ of neighborhoods of x is called a *neighborhood basis at x* if for any $A \in \mathcal{N}(x)$, there exists $B \in \mathcal{B}(x)$ such that $B \subset A$.
  (a) Express $\mathcal{N}(x)$ in terms of a neighborhood basis $\mathcal{B}(x)$.
  (b) Define a conception of *neighborhood sub-basis*.
  (c) Write down a theorem that characterize the continuity of a map f at a point x via neighborhood basis and via neighborhood sub-basis, and prove your theorem.

(2) [Topologies on $\mathbb{R}^{\mathbb{N}}$]
Consider the space of sequences of real numbers,

$$X = \mathbb{R}^{\mathbb{N}} = \{(x_1, x_2, \cdots) \mid x_n \in \mathbb{R}\}.$$

On X we have defined three topologies: the box topology $\mathscr{T}_{box}$, the product topology $\mathscr{T}_{product}$, and the "uniform topology" $\mathscr{T}_{uniform}$ induced from the uniform metric

$$d_{uniform}((x_n), (y_n)) = \sup_{n \in \mathbb{N}} \min(|x_n - y_n|, 1).$$

  (a) Prove: $\mathscr{T}_{product} \subset \mathscr{T}_{uniform} \subset \mathscr{T}_{box}$.
  (b) One can also regard every element $(x_1, x_2, \cdots)$ in X as a map

$$f : \mathbb{N} \to \mathbb{R}, \quad n \mapsto x_n$$

  and thus identify X with the space of maps $\mathcal{M}(\mathbb{N}, \mathbb{R})$. Define the pointwise convergence topology $\mathscr{T}_{p.c.}$ on X, and prove $\mathscr{T}_{p.c.} = \mathscr{T}_{product}$.
  (c) Fix two elements $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ in X, and define a map

$$f : X \to X, \quad (x_1, x_2, \cdots,) \mapsto (a_1x_1 + b_1, a_2x_2 + b_2, \cdots).$$

  Prove that if we endow X with the product topology, then f is continuous. What if we endow X with the box topology?

(3) [Universality of the induced and co-induced topologies]
  (a) Prove Proposition 1.96.
  (b) Read page 38-39 on "co-induced topology" and prove Proposition 1.100.

(4) ["Product operation" for topologies is commutative and associative]
Let X_α $(\alpha \in \Lambda)$ be topological spaces. Prove: For any decomposition $\Lambda = \bigcup_\beta \Lambda_\beta$ of the set of indices Λ (where $\Lambda_\beta \cap \Lambda_{\beta'} = \emptyset$ for $\beta \neq \beta'$), the product topological space $\prod_{\alpha \in \Lambda} X_\alpha$ is homeomorphic to the product topological space $\prod_\beta \left(\prod_{\alpha \in \Lambda_\beta} X_\alpha\right)$, where each product appeared above is endowed with the product topology.

PROBLEM SET 3, PART 2: TOPOLOGY (H)
DUE: MARCH 14, 2022

(1) [Embedding $\mathbb{RP}^2$ into $\mathbb{R}^4$]
Consider the map

$$f: S^2 \to \mathbb{R}^4, (x, y, z) \mapsto (y^2 - x^2, xy, xz, yz).$$

Prove: the image is homeomorphic to $\mathbb{RP}^2$.

(2) [Cone and suspension of S^n]
Prove the following by constructing a homeomorphism for each pair of spaces.
  (a) $C(S^n) \simeq B^{n+1}$.
  (b) $S(S^n) \simeq S^{n+1}$.
  (c) $B^n/S^{n-1} \simeq S^n$.

(3) [Quotient map v.s. open/closed map]
  (a) Suppose $p : X \to Y$ is a surjective continuous map. Prove: If p is either open or closed, then it is a quotient map.
  (b) Construct a quotient map that is neither open nor closed.
  (c) (Not required) Let $\mathrm{SO}(n)$ be the special orthogonal group. Define a map

$$f : \mathrm{SO}(n) \to S^{n-1}, \qquad A \mapsto Ae_1,$$

  where $e_1 = (0, \cdots, 0, 1)$ is the "north pole vector" on S^{n-1}.
    (i) Prove: f is surjective, continuous and open, and thus is a quotient map.
    (ii) Consider the natural (right) action of $\mathrm{SO}(n-1)$ on $\mathrm{SO}(n)$ by

$$B \cdot A := A \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}, \qquad \forall B \in \mathrm{SO}(n-1), A \in \mathrm{SO}(n).$$

    Prove: the orbits of this action are the fibers of the quotient map f.
    (iii) Conclude that $\mathrm{SO}(n)/\mathrm{SO}(n-1) \simeq S^{n-1}$.

(4) [Covering space action]
Let G be a group acting on a topological space X. Let $Y = X/G$ be the orbit space, and $p : X \to Y$ be the quotient map. Let $U \subset X$ be an open set, such that

$$g \cdot U \cap U = \emptyset, \quad \forall g \neq e \in G.$$

Prove:
  (a) $V := p(U)$ is an open set in Y.
  (b) For any $g \in G$, the map $p_g = p \circ \tau_g : g^{-1} \cdot U \to V$ is a homeomorphism.

PROBLEM SET 4, PART 1: TOPOLOGY (H)
DUE: MARCH 21, 2022

(1) [“sequential conitnuous=continuous” for (A1) spaces]
Let X be an (A1) space, Y be any topological space. Prove: A map $f : X \to Y$ is continuous at x_0 if and only if it is *sequentially continuous* at x_0.

(2) [Locally finiteness]
Let $(X, \mathscr{T})$ be a topological space.
  (a) Let A, B be subsets in X. Prove: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
  (b) Let A_α be a family of subsets in X. Prove: $\cup_\alpha \overline{A_\alpha} \subset \overline{\cup_\alpha A_\alpha}$
  (c) Find an example so that $\cup_\alpha \overline{A_\alpha} \neq \overline{\cup_\alpha A_\alpha}$ for a family of subsets $A_\alpha \in \mathbb{R}$.
  (d) We say a family $\{A_\alpha\}$ of subsets in X is *locally finite* if for any $x \in X$, there exists an open neighborhood U_x of x so that $A_\alpha \cap U_x \neq \emptyset$ for only finitely many α's. Prove: If $\{A_\alpha\}$ is a *locally finite family*, then $\cup_\alpha \overline{A_\alpha} = \overline{\cup_\alpha A_\alpha}$.

(3) [Characterize continuity via interior]
In class we proved

> A map $f : X \to Y$ between two topological spaces is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ holds for any $A \subset X$.

Apply the idea of “open-closed” duality, write down the corresponding characterization of continuity of f via the interior operation, and then prove it.

(4) [Not required] [Convergence by net]
We call $(P, \preceq)$ a *directed set* if
- $(P, \preceq)$ is a partially ordered set (c.f. Def. 1.84),
- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

For a topological space X, a *net* is a map $f : (P, \preceq) \to X$ from a directed set $(P, \preceq)$ to X. We will use the notation (x_α) instead of a map “$f : \alpha \mapsto x_\alpha$” if there is no ambiguity. We say a net (x_α) *converges* to x_0, denoted by $x_\alpha \to x_0$, if for any neighborhood U of x, there is an $\alpha \in P$ such that $x_\beta \in U$ holds for any $\alpha \preceq \beta$.
  (a) Realize $\mathcal{N}(x)$ as a directed set. [You need to carefully choose the partial order relation so that it can be used in part (b) below.]
  (b) Prove: $x \in \overline{A}$ if and only if there exists a net (x_α) in A which converges to x_0.
  (c) Prove: A map $f : X \to Y$ is continuous if and only if for any net (x_α) in X which converges to a limit x_0, the net $(f(x_\alpha))$ in Y converges in Y to $f(x_0)$.

PROBLEM SET 4, PART 2: TOPOLOGY (H)
DUE: MARCH 21, 2022

(1) [Continuous maps from compact space to Hausdorff space]
Prove Lemma 2.1.20, Corollary 2.1.21 and Proposition 2.1.22.

(2) [Compactness for the "upper semi-continuous" topology]
In PSet2-2-1(f) you constructed the upper semi-continuous topology on $\mathbb{R}$,

$$\mathscr{T}_{u.s.c} = \{(-\infty, a) \mid a \in \mathbb{R}\}.$$

(a) Is $(\mathbb{R}, \mathscr{T}_{u.s.c})$ compact? sequentially compact? limit point compact?
(b) Describe all compact subsets in $(\mathbb{R}, \mathscr{T}_{u.s.c})$.
(c) State a theorem called "the extremal value theorem for upper semi-continuous functions" and prove it.

(3) [Countably compact]
A topological space X is called *countably compact* if every countable open covering of X has a finite subcovering.
(a) Prove: Closed subspace of a countably compact space is countably compact.
(b) Prove: Any countably compact space is limit point compact.
(c) Prove: X is countably compact if and only if it has the *nested sequence property*: for any nested sequence of non-empty closed sets $F_1 \supset F_2 \supset \cdots$, we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.
(d) Prove: Any sequentially compact space is countably compact.
(e) Prove: The continuous image of a countably compact space is countably compact.

(4) [One point compactification]
Given any topological space $(X, \mathscr{T})$, we say a compact topological space Y is a *compactification* of X if there exists a homeomorphism $f : X \to f(X) \subset Y$ such that $\overline{f(X)} = Y$.
(a) Prove: both S^1 and $[0, 1]$ are compactifications of $\mathbb{R}$.
(b) For any non-compact topological space $(X, \mathscr{T})$, define a topology $\mathscr{T}^*$ on the set $X^* = X \cup \{\infty\}$ by

$$\mathscr{T}^* = \mathscr{T} \bigcup \{X^*\} \bigcup \{K^c \cup \{\infty\} \mid K \subset X \text{ is closed and compact.}\}.$$

Prove: $\mathscr{T}^*$ is a topology on X^*, and $(X^*, \mathscr{T}^*)$ is a compactification of $(X, \mathscr{T})$. [This is called the *one-point compactification* of $(X, \mathscr{T})$.]
(c) Prove: the one-point compactification of $\mathbb{N}$ is homeomorphic to $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ (as a subset in $\mathbb{R}$).
(d) Construct a compact Hausdorff topology on any set X. [Hint: start with the discrete topology on $X \setminus \{x_0\}$]

PROBLEM SET 5, PART 1: TOPOLOGY (H)
DUE: MARCH 28, 2022

(1) [The topology of the Cantor set]
Recall that the Cantor set C is the following subset of $[0,1]$,

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right).$$

(a) Prove: Every point in the Cantor set is a limit point.
(b) Prove: As a subset of $[0,1]$, the Cantor set is nowhere dense.
(c) (Not required) For any closed subset $F \subset C$, prove: there exists a continuous map $f : C \to F$ so that $f(x) = x$ on F.
[Hints: F^c is the union of open intervals. Pick an element in each such interval that is not in C, and "push" points in the intervals to the "boundary points".]
(d) Define a map

$$g : \{0,1\}^{\mathbb{N}} \to [0,1], \quad a = (a_1, a_2, \cdots) \mapsto \sum_{k=1}^{\infty} \frac{2a_k}{3^k}.$$

Prove: g induces a homeomorphism between $(\{0,1\}^{\mathbb{N}}, \mathcal{T}_{product})$ and C.
(e) (Not required) Show that there is continuous surjective map from C to $[0,1]^2$, by showing that

$$h : \{0,1\}^{\mathbb{N}} \to [0,1]^2, \quad a = (a_1, a_2, \cdots) \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{2^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{2^k} \right)$$

is continuous and surjective. Is h injective?

(2) [Sequentially compactness for products]
(a) Let $X_1, \cdots, X_n$ be sequentially compact topological spaces. Prove: the product space $X = X_1 \times \cdots \times X_n$ is sequentially compact.
(b) Is $X = \{0,1\}^{\mathbb{N}}$ sequentially compact when equipped with the box topology $\mathscr{T}_{box}$? Prove you claim.
(c) Now suppose (X_n, d_n) are compact metric spaces. Define a product metric on $X = \prod_{n=1}^{\infty} X_n$ via

$$d((x_n),(y_n)) := \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{(1 + \mathrm{diam}(X_k)) \cdot 2^n}.$$

Prove: The metric topology on X induced by d coincides with the product topology on X.

(3) [Compactness in order topology]
Let $(X, \leq)$ be a totally ordered set. For any subset $A \subset X$, we say $x \in X$ is a *least upper bound* of A if x is an upper bound of A, and there is no $x' < x$ which is an upper bound of A. Now endowed X with the order topology introduced in Definition 1.85. Prove: X is compact if and only if every subset (including the empty set $\emptyset$) of X has a least upper bound.
[Hints: X has a least upper bound implies that X has a maximal element. $\emptyset$ has a least upper bound implies that X has a minimal element. Try to prove that for any sub-base covering $\mathcal{U}$, there are $a < b$ so that $\{x|x < b\}$ and $\{x|x > a\}$ are elements in $\mathcal{U}$, and then apply Alexander subbase theorem.]

(4) [The existence of Banach limit](Not required)
Consider the vector space of all bounded sequences of real numbers,

$$X = l^\infty = \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{R} \text{ and } \sup_n |a_n| < \infty\}.$$

On X there is a naturally defined *shift* operator

$$T : X \to X, \qquad \{a_1, a_2, \cdots\} \mapsto \{a_2, a_3, \cdots\}.$$

A *mean* on X is a linear map $L : X \to \mathbb{R}$ such that

$$\inf a_n \leq L(\{a_n\}) \leq \sup a_n$$

holds for all $\{a_n\} \in X$. A *Banach limit* is a mean that is invariant under the shift operator T, i.e. such that $L(\{a_n\}) = L(T(\{a_n\}))$ holds for all $\{a_n\} \in X$.

(a) Define $L_m : X \to \mathbb{R}$ by $L_m(\{a_n\}) = \frac{1}{m}\sum_{i=1}^m a_i$. Prove: L_m is a mean for each m, and $\lim_{m\to\infty} |L_m(T(\{a_n\})) - L_m(\{a_n\})| = 0$.
(b) Let $\mathcal{M}$ be the set of all means on X. One can regard $\mathcal{M}$ as a subset of $\mathcal{M}(X, \mathbb{R}) = \mathbb{R}^X$, equipped with the product topology. Prove: $\mathcal{M}$ is compact.
[Hint: $\mathcal{M}$ is contained in $\prod_{\{a_n\}\in X}[\inf a_n, \sup a_n]$.]
(c) Prove: There exists a Banach limit on X.
[Hint: Compact implies limit point compact. Use (a).]
(d) What is the Banach limit of a convergent sequence? What is the Banach limit of $\{0, 1, 0, 1, 0, \cdots\}$?

PROBLEM SET 5, PART 2: TOPOLOGY (H)
DUE: MARCH 28, 2022

(1) [Completion of metric spaces]
Let X be a set, and (Y, d_Y) be metric spaces. Consider the space of bounded maps,

$$\mathcal{B}(X, Y) = \{f : X \to Y \mid f(X) \text{ is bounded in } Y\}$$

(a) Prove: the supremum metric $d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x))$ is a metric on $\mathcal{B}(X, Y)$.
(b) Prove: If Y is complete, so is $(\mathcal{B}(X, Y), d_\infty)$.

In what follows, suppose (X, d_X) is a metric space, and take $Y = \mathbb{R}$.

(c) Fix a point $x_0 \in X$. For any $a \in X$, define a function $f_a : X \to \mathbb{R}$ via $f_a(x) := d_X(x, a) - d_X(x, x_0)$. Prove: $f_a \in \mathcal{B}(X, \mathbb{R})$.
(d) Prove: the map

$$\Phi : (X, d) \to (\mathcal{B}(X, \mathbb{R}), d_\infty), a \mapsto f_a$$

is an isometric embedding, i.e. $d_X(a, b) = d_\infty(f_a, f_b)$ for any $a, b \in X$.
(e) Prove: Any metric space (X, d_X) admits a completion.
(f) (Not required) Prove: If (Y_1, d_1) and (Y_2, d_2) are two completions of (X, d_X), then (Y_1, d_1) and (Y_2, d_2) are isometric.

(2) [From limit point compact to sequentially compact]
In the proof of Proposition 2.3.25, we only used the following two properties:

- Every $x \in X$ has a *descending* countable neighborhood basis $U_1^x \supset U_2^x \supset \cdots$.
- If x is a limit point of A, then every neighbourhood of x contains infinitely many points of A.

As a consequence, there are many other topological spaces in which limit point compact is equivalent to sequentially compact:

(a) Prove Proposition 2.3.26.
(b) Prove that in Proposition 2.3.26, one can weaken the Hausdorff condition to the following (T1) condition:

> (T1): For any $x \neq y$ in X, there exists open sets U and V in X so that $x \in U \setminus V$ and $y \in V \setminus U$.

(c) The (T1) condition is equivalent to a sentence on page 1 of today's notes. Find out it and prove the equivalence.

(3) [Closed unit ball in l^2]
Consider the metric space l^2 given in Example 1.6(3).

(a) Prove: l^2 is complete.
(b) Prove: The closed unit ball $\overline{B(0, 1)}$ and the unit sphere $S(0, 1)$ are non-compact.
(c) Prove: If $K \subset l^2$ is compact, then K has no interior point.

(4) [Lebesgue property]
We say a metric space (X, d) has the Lebesgue property if any open covering of X has a positive Lebesgue number.
  (a) Look at our proof of "sequentially compact $\Longrightarrow$ compact" in the proof of Theorem 2.3.28. What did we really proved? Your answer should be of the form ["condition A"+"condition B" implies compactness], and thus we have a new characterization of compactness in metric space.
  (b) Prove: If (X, d_X) has the Lebesgue property, then it is complete.
  (c) Prove: (X, d_X) has the Lebesgue property if and only if for any metric space (Y, d_Y), any continuous map $f : X \to Y$ is uniformly continuous.
  (d) Suppose (X, d_X) has the Lebesgue property. Prove: If A, B are non-empty disjoint closed subsets in (X, d), then $dist(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\} > 0$.

PROBLEM SET 6, PART 1: TOPOLOGY (H)
DUE: APRIL 6, 2022

(1) [More on LCH]
  (a) [Structure of noncompact LCH]
    (i) Let K be a compact Hausdorff space, $p \in K$ and $X = K \setminus \{p\}$ is non-compact. Prove: X is LCH.
    (ii) Conversely, suppose X be a non-compact LCH. Let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. Prove: X^* is compact and Hausdorff.
  (b) [The evaluation map could fail to be continuous without local compactness] Consider the evaluation map

$$e : \mathbb{Q} \times \mathcal{C}(\mathbb{Q}, [0,1]) \to [0,1], \quad (x, f) \mapsto e(x, f) = f(x).$$

    (i) Prove: $\mathbb{Q}$ is not locally compact.
    (ii) Prove: for any $q_1 \in \mathbb{Q}$ and any closed subset $A \subset \mathbb{Q}$ with $q_1 \notin A$, there is a continuous function $f_1 \in \mathcal{C}(\mathbb{Q}, [0,1])$ such that $f(q_1) = 1, f(A) = 0$.
    (iii) (Not required) Now let $f_0 \in \mathcal{C}(\mathbb{Q}, [0,1])$ be the zero map $f_0(\mathbb{Q}) = 0$, and take any $q_0 \in \mathbb{Q}$. Prove: e is not continuous at (q_0, f_0) (where we endow $\mathcal{C}(\mathbb{Q}, [0,1])$ with the compact convergence topology).
    [Hint: For any open neighborhood U of q_0 and any compact set K in $\mathbb{Q}$, there exists $q_1 \in U \setminus K$. Construct a continuous function using (b).]

(2) [More on compact-open topology]
  (a) Prove Proposition 2.4.22, i.e. (Y, d) is a metric space, then $\mathscr{T}_{c.c.} = \mathscr{T}_{c.o.}$.
  (b) Prove Proposition 2.4.23, i.e. if Y s LCH, then the composition map is continuous with respect to $\mathscr{T}_{c.o.}$.
  (c) Prove: If X is locally compact and Hausdorff, then

$$S(\{x\}, U) = \bigcup_{\text{compact neighborhood } K \text{ of } x} S(K, U).$$

  [Hint for (b) and (c): Use Proposition 2.4.16]

(3) [Compactly generated spaces]
  (a) Read the materials on compactly generated spaces (page 99), and prove: any locally compact space is compactly generated.
  (b) Prove: Any first countable space is compactly generated.
  (c) Find a compactly generated space that is not locally compact. [Hint: PSet5-2]
  (d) Let $(X, \mathscr{T})$ be any topological space. Prove: there exists a topology $\mathscr{T}' \supset \mathscr{T}$ such that $(X, \mathscr{T}')$ is compactly generated, and a set is compact with respect to $\mathscr{T}'$ if and only if it is compact with respect to $\mathscr{T}$.
  [Hint: Construct topology by needs!]

(4) [Applications of Arzela-Ascoli]

(a) Suppose $k = k(x, y) \in \mathcal{C}([0, 1] \times [0, 1], \mathbb{R})$. For any $f \in \mathcal{C}([0, 1], \mathbb{R})$, define

$$Kf(x) = \int_0^1 k(x, y)f(y)dy.$$

Prove: K is a *compact operator*, i.e. it maps any bounded subset in $(\mathcal{C}([0, 1], \mathbb{R}), d_\infty)$ into a compact subset in the same space.

(b) (Not required) We want to minimize the functional $\Phi[f] := \int_{-1}^1 f(t)dt$. Consider the set

$$\mathcal{F} = \{f \in \mathcal{C}([-1, 1], [0, 1]) \mid f(-1) = f(1) = 1\}.$$

(i) What is $\inf_{f\in\mathcal{F}} \Phi[f]$? Is the infimum attained?

(ii) For any constant $C > 0$, let

$$\mathcal{F}_C = \{f \in \mathcal{F} \mid |f(x) - f(y)| \le C|x - y|\}.$$

Prove: The infimum $\inf_{f\in\mathcal{F}_C} \Phi[f]$ is attained. Can you find the function?

(c) (Not required) Prove Theorem 2.5.12.

PROBLEM SET 6, PART 2: TOPOLOGY (H)
DUE: APRIL 06, 2022

(1) [Topological algebra]
Let X be a topological space. Endow $\mathcal{C}(X,\mathbb{R})$ with the compact convergence topology.
  (a) Prove: The addition, multiplication and the scalar multiplication

$$a : \mathcal{C}(X,\mathbb{R}) \times \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R}), \quad (f,g) \mapsto a(f,g) = f+g,$$
$$m : \mathcal{C}(X,\mathbb{R}) \times \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R}), \quad (f,g) \mapsto m(f,g) = fg,$$
$$s : \mathbb{R} \times \mathcal{C}(X,\mathbb{R}) \to \mathcal{C}(X,\mathbb{R}), \quad (\lambda,g) \mapsto s(\lambda,f) = \lambda g$$

  are continuous. (As a consequence, $\mathcal{C}(X,\mathbb{R})$ is a topological algebra.)
  (b) Prove Proposition 2.6.4 (the closure of a subalgebra of topological algebra is a closed subalgebra).

(2) [Applications of Stone-Weierstrass]
  (a) Prove: Any continuous function on $[0,1]$ can be approximated uniformly by functions of the form

$$a_0 + a_1 e^x + a_2 e^{2x} + \cdots + a_n e^{nx}, \quad n \in \mathbb{N}.$$

  - As a consequence, prove if f is a continuous function on $[0,1]$ satisfying

(*) $$\int_0^1 f(x) e^{nx} dx = 0, \quad n = 0,1,2\cdots,$$

  then $f = 0$.
  - What if (*) holds only for even n?

  (b) Let X, Y be compact Hausdorff spaces. Prove: any $f \in \mathcal{C}(X \times Y,\mathbb{R})$ can be approximated uniformly by functions of the form

$$f_1(x)g_1(y) + f_2(x)g_2(y) + \cdots + f_n(x)g_n(y), \quad n \in \mathbb{N},$$

  where $f_k \in \mathcal{C}(X,\mathbb{R}), g_k \in \mathcal{C}(Y,\mathbb{R})$.

(3) [Stone-Weierstrass for complex-valued functions]
  (a) Prove Theorem 2.6.14 (Stone-Weierstrass for complex-valued functions).
  (b) Prove: Any complex-valued continuous function on $S^1 = \mathbb{R}/\mathbb{Z}$ can be approximated uniformly by functions of the form

$$\sum_{k=-n}^{n} a_k e^{-2\pi i k x}, \quad n \in \mathbb{N}.$$

(4) [Stone-Weierstrass on LCH] (Not requried)
   (a) Let X be LCH. Prove: $\mathcal{C}_0(X, \mathbb{R})$ is an algebra.
   (b) Prove Theorem 2.6.15 (Stone-Weierstrass theorem on LCH).
   (c) Prove: Any $f \in C_0([0, +\infty), \mathbb{R})$ can be approximated uniformly by functions of the form

$$\sum_{k=-n}^{n} a_k e^{-kx}, \quad n \in \mathbb{N}.$$

PROBLEM SET 7, PART 1: TOPOLOGY (H)
DUE: APRIL 11, 2022

(1) [Lindelöff Property]
  (a) Prove Proposition 2.7.13.
  (b) Prove Proposition 2.7.14.
  (c) Check: $(\mathbb{R}, \mathscr{T}_{cocountable})$ is Lindelöf but not σ-compact.
  (d) Check: The Sorgenfrey line $(\mathbb{R}, \mathscr{T}_{Sorgenfrey})$ is Lindelöf.

(2) [The Sorgenfrey plane]
Consider the product of two Sorgenfrey lines,

$$(\mathbb{R}^2, \mathscr{T}_{Sorgenfrey}) := (\mathbb{R}, \mathscr{T}_{Sorgenfrey}) \times (\mathbb{R}, \mathscr{T}_{Sorgenfrey}),$$

which is known as the *Sorgenfrey plane*.
  (a) Prove: It is first countable, separable but not second countable.
  (b) Prove: Is it Hausdorff?
  (c) Consider the subspace $A = \{(x, -x) \mid x \in \mathbb{R}\}$. Is it closed? What is the induced subspace topology on A?
  (d) Prove: It is not Lindelöf.

(3) [Closedness of graph]
Let X, Y be topological spaces, define the *graph* of a map $f : X \to Y$ to be the set

$$G_f := \{(x, f(x) \mid x \in X\} \subset X \times Y.$$

  (a) Prove: Y is Hausdorff $\Longleftrightarrow$ for any X and $f \in \mathcal{C}(X, Y)$, G_f is closed in $X \times Y$.
  (b) Construct a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed.
  (c) Prove: If Y is compact Hausdorff, then f is continuous $\Longleftrightarrow$ G_f is closed.

(4) [Hereditary properties]
A topological property P is called *hereditary* if

$$\boxed{(X, \mathscr{T}) \text{ satisfies } P \Longrightarrow \text{ Any subspace } Y \text{ of } X \text{ satisfies } P.}$$

  (a) Prove: (A1) and (A2) are hereditary, but (T4) is not hereditary.
  [Hint: Given any $(X, \mathscr{T})$, consider $(X \cup \{\infty\}, \mathscr{T} \cup \{X \cup \{\infty\}\})$]
  (b) Which of the following properties are hereditary:
  compact/sequentially compact/locally compact/separable/Lindelöf/(T1)/(T2)/(T3)
  (c) A topological property P is called *closed hereditary* if

$$\boxed{(X, \mathscr{T}) \text{ satisfies } P \Longrightarrow \text{ Any closed subspace } Y \text{ of } X \text{ satisfies } P.}$$

  For those non-hereditary properties above, determine whether they are closed hereditary.

PROBLEM SET 7, PART 2: TOPOLOGY (H)
DUE: APRIL 11, 2022

(1) [Productive properties]

A topological property P is called *productive* if

$$\boxed{\text{Each } \ (X_\alpha, \mathscr{T}_\alpha) \text{ satisfies } P \Longrightarrow (\prod_\alpha X_\alpha, \mathscr{T}_{prodcut}) \text{ satisfies } P.}$$

(a) Prove: (T1), (T2) and (T3) are productive.
(b) Conversely, if $(\prod_\alpha X_\alpha, \mathscr{T}_{prodcut})$ is (T1), (T2) or (T3), can we conclude that each $(X_\alpha, \mathscr{T}_\alpha)$ is (T1), (T2) or (T3)?
(c) Is (T4) productive? Is Lindelöf productive?
(d) Prove: *separable* and *metrizable* are not productive. What about (A1), (A2)?
(e) Can you introduce a weaker version of productivity, so that those non-productive properties in part (d) satisfy the weaker one?

(2) [Baire space]

A topological space is called a *Baire space* if every intersection of a countable collection of open dense sets in the space is also dense.

(a) Use "open-closed" duality to give an equivalent characterization of Baire space.
(b) Prove: Any complete metric space is a Baire space.
(c) Prove: Any compact Hausdorff space is a Baire space.
(d) Prove: Any locally compact Hausdorff space is a Baire space.

(3) [Applications of Urysohn lemma]

(a) Let X be a compact Hausdorff space, $x_0 \in X$, and U is an open neighborhood of x_0. Prove: For any $\varepsilon > 0$ and any continuous function $f : X \to \mathbb{R}$, there exists a continuous function $g : X \to \mathbb{R}$ satisfying all of the following three conditions:
- $\sup_{x \in X} |g(x) - f(x)| < \varepsilon$.
- $g = f$ on U^c.
- there exists a neighborhood V of x_0 such that $g(x) \equiv f(x_0)$ on V.

(b) Let X be LCH. Recall
- $\mathcal{C}_b(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and bounded}\}$.
- $\mathcal{C}_c(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and compactly supported}\}$.
- $\mathcal{C}_0(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and vanishes at infinity}\}$.

On $\mathcal{C}_b(X, \mathbb{R})$ we have a metric $d_\infty(f, g) := \sup_{x \in X} |f(x) - g(x)|$. Prove: The closure of $\mathcal{C}_c(X, \mathbb{R})$ in $\mathcal{C}_b(X, \mathbb{R})$ is $\mathcal{C}_0(X, \mathbb{R})$.

(4) [Locally metrizable] (Not required)

A topological space X is said to be *locally metrizable* if for any $x \in X$, there is a neighborhood U of x that is metrizable. Prove: If X is compact Hausdorff, then X is metrizable if and only if it is locally metrizable.

[Hint: Cover X by finitely many compact metrizable neighborhood.]

PROBLEM SET 8, PART 1: TOPOLOGY (H)
DUE: APRIL 18, 2022

(1) [Uniqueness of extension]
Let X, Y be topological spaces, $A \subset X$ be a *dense subset*, and $f : A \to Y$ be a continuous map.
 (a) Prove: If Y is a (T2) space, then f admits at most one continuous extension.
 (b) Does the same conclusion hold if Y is a (T1) space? If yes, prove it; if no, give a counterexample.

(2) [Tietze extensions with restrictions]
Let $(X, \mathscr{T})$ be a (T4) space, $A \subset X$ be closed.
 (a) Let $f : A \to \mathbb{C}$ be a continuous complex-valued function with

$$|f(x)| \leq 1, \quad \forall x \in A.$$

 Prove: f can be extended to a continuous function $\tilde{f} : X \to \mathbb{C}$ so that

$$|\tilde{f}(x)| \leq 1, \quad \forall x \in X.$$

 (b) Let $f : A \to \mathbb{R}$ and $g_1, g_2 : X \to \mathbb{R}$ be continuous functions, and suppose

$$g_1(x) \leq f(x) \leq g_2(x), \ \forall x \in A \qquad \underline{\text{and}} \qquad g_1(x) \leq g_2(x), \ \forall x \in X.$$

 Prove: f can be extended to a continuous function $\tilde{f} : X \to \mathbb{R}$ such that

$$g_1(x) \leq \tilde{f}(x) \leq g_2(x), \quad \forall x \in X.$$

(3) [Retraction]
Let X be a topological space, $A \subset X$ be a subspace. We say A is a *retract* of X if there exists a continuous map $r : X \to A$ such that

$$r(x) = x, \quad \forall x \in A.$$

Such a map r is called a *retraction*.
 (a) Prove: If X is Hausdorff, A is a retract of X, then A is closed.
 (b) Prove: A is a retract of X if and only if for any topological space Y, any continuous map $f : A \to Y$ has an extension $\tilde{f} : X \to Y$.
 (c) Suppose X is normal and A is closed. Prove: If Y is a retract of $\mathbb{R}^J$ (with product topology, where J is any set), then any continuous map $f : A \to Y$ admits a continuous extension $\tilde{f} : X \to Y$.

(4) [Different compactifications][Not required]
Let X, Y, Z be LCH spaces.
 (a) Construct at least three different compactifications of the interval $\mathbb{R}^2$.

(b) Prove that the Cech-Stone compactification βX is the largest compactification of X: For any compact Hausdorff compatification K of X (with an embedding $\varphi : X \to K$), there is a surjective continuous closed map $F : \beta X \to K$ which extends the embedding $\varphi : X \to K$.

(c) Similarly, prove that the one point compactification X^* is the smallest compactification of X.

(d) Given any continuous map $\varphi : X \to Y$, we constructed a continuous map $\beta\varphi : \beta X \to \beta Y$. Prove that the "lifting" $\varphi \rightsquigarrow \beta\varphi$ is "functorial" in the following sense:

  (i) If $\mathrm{Id_X}$ is the identity map, then $\beta\mathrm{Id_X} = \mathrm{Id}_{\beta X}$.

  (ii) If $\varphi : X \to Y$, $\psi : Y \to Z$ be continuous maps, then $\beta(\psi \circ \varphi) = \beta\psi \circ \beta\varphi$.

PROBLEM SET 8, PART 2: TOPOLOGY (H)
DUE: APRIL 18, 2022

(1) [Products of paracompact spaces]
   (a) Prove: The Sorgenfrey line is paracompact, while the Sorgenfrey plane is not. [Hint: The Sorgenfrey plane is not normal.]
   (b) Is paracompactness productive? Is it preserved under continuous maps?
   (c) Prove: If X is compact, Y is paracompact, then $X \times Y$ is paracompact.

(2) [LCH version of P.O.U.]
Let X be a locally compact, σ-compact, Hausdorff space, and $\mathscr{U} = \{U_\alpha\}$ is an open cover of X. Prove:
   (a) There exits two locally finite open coverings $\mathscr{V} = \{V_n\}$ and $\mathscr{W} = \{W_n\}$ such that
      - $W_n \subset \overline{W_n} \subset V_n \subset \overline{V_n}$, and $\overline{V_n}$ is compact,
      - For each n, there exists $U_\alpha \in \mathscr{U}$ such that $\overline{V_n} \subset U_\alpha$.
   (b) Prove Theorem 2.10.15(LCH version of P.O.U.).

(3) [Examples and non-Examples of topological manifolds]
   (a) Prove: Every topological manifold is σ-compact.
   (b) Prove: $\mathbb{RP}^n$ is a topological manifold.
   (c) (line with doubled point) Let $X = (\mathbb{R} \times \{0,1\})/\sim$, where $(x,0) \sim (x,1)$ for all $x \neq 0$. Prove: X is (A2) and locally Euclidian, but not (T2).
   (d) [NOT Required] (long line) Let Ω be the smallest uncountable well-ordered set. That is, Ω is an uncountable set, and there is a well-order $<$ on Ω such that for any $a \in \Omega$, the set $\{b \in \Omega \mid b < a\}$ is countable. Let $L = \Omega \times [0,1)$. Define an order on L via
   $$(a,t) \prec (b,s) \text{ if and only if "} a < b \text{" or "} a = b \text{ and } t < s\text{".}$$
   For any $x \prec y$ in L, we define $(x,y) = \{z \in L \mid x \prec z \prec y\}$.
      (i) Prove: These "intervals" (x,y) form a basis of a topology on L.
      (ii) Prove: With respect to this topology, L is (T2), locally Euclidean but not (A2). It is called the *long line*.
      [Hint: By the definition of well-order, for any $a \in \Omega$, the set $\{b \in \Omega \mid a < b\}$ has a minimal element, called the successor of a. Define charts on L by
      $$\varphi : \{a\} \times (0,1] \cup \{a'\} \times (0,1) \to (-1,1), \quad \varphi(a,t) = t - 1 \text{ and } \varphi(a',t) = t.$$
      where a' is the successor of a.]

(4) [An application of P.O.U.]
Let X be Hausdorff and paracompact, $f : X \to \mathbb{R}$ be lower semi-continuous and $g : X \to \mathbb{R}$ be upper semi-continuous. Moreover, assume $f(x) > g(x), \forall x \in X$. Prove: there exists a continuous function $h : X \to \mathbb{R}$ such that $f(x) > h(x) > g(x), \forall x \in X$.

PROBLEM SET 9, PART 1: TOPOLOGY (H)
DUE: MAY 07, 2022

(1) [Connectedness of subspace]
Let $(X, \mathscr{T})$ be a topological space, and $Y \subset X$ be a subspace. Which of the following statements are equivalent to the fact

$$\boxed{Y \text{ is disconnected}} ?$$

Prove the equivalence for the correct ones and give counterexamples for the wrong ones:

(a) There exists non-empty sets $A, B \subset X$ with $A \cap \overline{B} = \overline{A} \cap B = \emptyset$, such that $Y = A \cup B$, where the closure is taken in to be the closure in X.
(b) There exists open sets A, B in X with $A \cap B \cap Y = \emptyset$, such that $Y \subset A \cup B$ and $A \cap Y \neq \emptyset, B \cap Y \neq \emptyset$.
(c) There exists disjoint open sets A, B in X with $A \cap Y \neq \emptyset, B \cap Y \neq \emptyset$, such that $Y \subset A \cup B$.
(d) There exists disjoint closed sets A, B in X with $A \cap Y \neq \emptyset, B \cap Y \neq \emptyset$, such that $Y \subset A \cup B$.
(e) There exists a set A which is both open and closed in X such that $A \cap Y \neq \emptyset$ and $A \cap Y \neq Y$.
(f) There is a surjective continuous map $f : Y \to \{0, 1\}$.

(2) [Connected components]
Let X be a topological space. The *connected component* containing $x \in X$ is defined to be the maximal connected subsets of X containing x.

(a) Prove: The connected component containing x is the union of all connected subsets of X that contains x.
(b) Prove: Each connected component is a closed subset.
(c) Give an example showing that the connected component need not be open.
(d) (Generalization of Proposition 3.1.8) Prove: If $f : X \to Y$ is continuous, then for any subset A of X, the cardinality of connected components of $f(A)$ is no more than the cardinality of connected components of A.
(e) (Generalization of Proposition 3.1.18) Denote the connected component of X_α containing x_α to be $C(x_\alpha)$. Prove: the connected component of $\prod_\alpha X_\alpha$ containing the point (x_α) is $\prod_\alpha C(x_\alpha)$.

(3) [Non-homeomorphic spaces]

(a) Show that the following spaces are pairwise non-homeomorphic:

$$\mathbb{R}, \quad \mathbb{Z}, \quad S^1, \quad \mathbb{R}^2, \quad [0,1], \quad [0,1)$$

(b) Consider

$$A = (0,1) \cup \{2\} \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \cdots,$$
$$B = (0,1] \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \cdots.$$

Prove: There exists continuous bijection $f : A \to B$ and continuous bijection $g : B \to A$, however, A and B are not homeomorphic.

[You may compare this with Cantor–Schröder–Bernstein theorem in set theory]

(4) [Connected + suitable separation axioms v.s. countability]

(a) Prove: If $(X, \mathscr{T})$ is (T1), (T4) and connected, and X contains at least two elements, then X contains uncountably many elements.

(b) Can we replace (T4) by (T3)?

(c) [Not required][The Golomb space] Define a topology on $\mathbb{N}_{>0}$ as follows: For any coprime positive integers a and b, let $D_{a,b} = \mathbb{N}_{>0} \cap \{a + bk \mid k \in \mathbb{N}_{\geq 0}\}$. Consider the topology $\mathscr{T}_{Golomb}$ generated by these $D_{a,b}$'s. It turns out that $(\mathbb{N}_{>0}, \mathscr{T}_{Golomb})$ is (T2), connected but contains countably elements:

(i) Prove: $\mathcal{B} = \{D_{a,b} \mid a, b \text{ are coprime positive integers}\}$ is a basis of $\mathscr{T}_{Golomb}$.

(ii) Prove: $(\mathbb{N}_{>0}, \mathscr{T}_{Golomb})$ is (T2).

(iii) Prove: $(\mathbb{N}_{>0}, \mathscr{T}_{Golomb})$ is connected. Is it compact or (T3) or metrizable? [In proving connectedness, you may need the following consequence of *Chinese remainder theorem* from number theory: If b_1 and b_2 are coprime, then $D_{a_1,b_1} \cap D_{a_2,b_2} \neq \emptyset$.]

(iv) The *Dirichlet Theorem* in number theory asserts that every $D_{a,b}$ (with a, b coprime) contains infinitely many prime numbers. Explain this using the language of topology.

PROBLEM SET 9, PART 2: TOPOLOGY (H)
DUE: MAY 04, 2022

(1) [Path connectedness: examples]

(a) Although looks quite non-obvious, the set $\mathbb{R}^2 - \mathbb{Q}^2$ is path-connected. We give two proofs here:

First proof. Since $\mathbb{Q}^2$ is a countable set, for any $x \in \mathbb{R}^2 - \mathbb{Q}^2$, there exist uncountably many lines l s.t.

$$x \in l \subset \mathbb{R}^2 - \mathbb{Q}^2.$$

Now for $x \neq y \in \mathbb{R}^2 - \mathbb{Q}^2$, pick two such lines, one contains x and the other contains y, such that they are not parallel. Now you can connect x to the intersection point through the first line, then to y through the second line. □

Second proof. Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 - \mathbb{Q}^2$. If $x_1, x_2 \in \mathbb{R} - \mathbb{Q}$, then we pick $y_0 \in \mathbb{R} - \mathbb{Q}$, and connect (x_1, y_1) to (x_1, y_0) through the line $x = x_1$, and connect (x_1, y_0) to (x_2, y_0) through the line $y = y_0$, and finally connect (x_2, y_0) to (x_2, y_2) through the line $x = x_2$. Similar arguments holds if $x_1, y_2 \in \mathbb{R} - \mathbb{Q}$ or $y_1, y_2 \in \mathbb{R} - \mathbb{Q}$ or $x_2, y_1 \in \mathbb{R} - \mathbb{Q}$. □

It turns out that each proof can be extended to prove a more general result on path-connectedness:

Proposition 0.1. *Let $S \subset \mathbb{R}^n$ be ... then $\mathbb{R}^n - S$ is path connected.*

Proposition 0.2. *Let X, Y are path-connected, and ...*

Complete the full statements.

(b) Show that the topological space

$$(X = \{v, s\}, \mathscr{T} = \{\emptyset, \{s\}, \{v, s\}\}$$

is path-connected.

(2) [Locally connectedness]

(a) Define the conception:

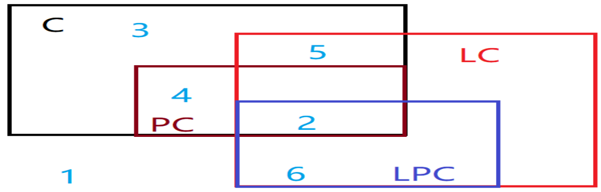
Definition 0.3. We say a topological space X is <u>locally connected</u> if ......

(b) Consider $(\mathbb{R}, \mathscr{T}_{cocountable})$. Is it connected? locally connected? path connected? locally path connected?

(c) For simplicity, let's denote

C=connected, LC=locally connected,
PC=path connected, LPC=locally path connected.

Give examples in region 1-6 for the following picture: (the remaining two parts are more complicated. You can try if you want to challenge yourself..)



(d) Prove: If X is compact and locally connected, then X has finitely many connected components. Can we remove the locally connectedness condition?

(e) Prove: X is locally connected if and only if for any open set U in X, any connected component of U is open.
(In particular, any connected component of a locally connected space is open.)

(f) (Not required) Suppose X is locally connected, $f : X \to Y$ is continuous. Prove: if f is either open or closed, then $f(X)$ is locally connected.
Can we remove the assumption "f is either open or closed"?

(3) [Components and path components]

(a) Find the components and path component for the following spaces:
  (i) The Sorgenfrey line.
  (ii) $(\mathbb{R}, \mathscr{T}_{cocountable})$.
  (iii) $(\mathbb{R}^{\mathbb{N}}, \mathscr{T}_{uniform})$.

(b) Prove Proposition 3.2.22 and Proposition 3.2.23, namely, π_0 and π_c are functors.

(4) (Not required) [Components of topological groups]
Let G be a topological group.

(a) Prove: For any normal subgroup N of G, the quotient group G/N is a topological group.

(b) Prove: $\pi_0(G)$, $\pi_c(G)$ are both topological groups. What's the relation between these two groups?

(c) Are $\pi_0(G)$ and $\pi_c(G)$ Hausdorff spaces?

(d) Find the relations between $\pi_0(G_1 \times G_2)$ and $\pi_0(G_1), \pi_0(G_2)$, where G_1, G_2 are topological groups.

PROBLEM SET 10, PART 1: TOPOLOGY (H)
DUE: MAY 9, 2022

(1) [Constructing homotopies]
   (a) Prove Proposition 3.3.3 (composition, pull-back and push-forward)
   (b) Prove that "homotopy equivalence between topological spaces" is an equivalence relation (Remark 3.3.9(3)).
   (c) Prove Proposition 3.3.17(1) and (3).

(2) [Maps to S^n]
   (a) Prove: Any non-surjective continuous map $f : X \to S^n$ is null-homotopic.
   (b) Let $f, g : X \to S^n$ be continuous maps. Suppose they are never anti-podal, i.e. $g(x) \neq -f(x)$ holds for all x. Prove: f is homotopic to g.
   (c) Let $\overline{B^{n+1}}$ be the closed unit ball in $\mathbb{R}^{n+1}$. Prove: There exists a retraction $f \in \mathcal{C}(\overline{B^{n+1}}, S^n)$ if and only if Id_{S^n} is null-homotopic.
   [Hint: For "only if" part, use the fact $\overline{B^{n+1}}$ is convex; for "if" part, use the fact "$\overline{B^{n+1}}$ is the cone over S^n".]

(3) [Deformation retract]
We say A is a *weak deformation retract* of X if there exists a retraction $r : X \to A$ so that Id_X is homotopic to $\iota \circ r : X \to X$, where $\iota : A \hookrightarrow X$ is the inclusion map. In other words, A is a *weak deformation retract* of X if there exists a continuous map (called a *weak deformation retraction*) $F : [0,1] \times X \to X$ such that

$$F(0, x) = x, F(1, x) \in A,\ \forall x \in X \quad \text{and} \quad F(1, a) = a, \forall a \in A.$$

A weak deformation retraction F is called a *strong deformation retraction* if

$$F(t, a) = a, \quad \forall a \in A, \forall t \in [0, 1].$$

[In some books, people call weak deformation retract defined above a deformation retract, while in some other books (includes Munkres's book and Hatcher's book) people call strong deformation retract defined above a deformation retract.]
   (a) Construct a strong deformation retraction $\mathbb{R}^{n+1} \setminus \{0\}$ to S^n.
   (b) Construct a strong deformation retraction from $\mathbb{T}^2 - \{pt\}$ (i.e. the torus with one point removed) $S^1 \vee S^1$ (i.e. "figure 8").
   (c) Prove: If $A \subset X$ is a weak deformation retract, then $A \sim X$.
   (d) (Not required)[Compare with Exercise for Section 2.9] Prove: $A \subset X$ is a weak deformation retract of X if and only if it satisfies the following two properties:
      - For any topological space Y, any continuous map $f : A \to Y$ has a continuous extension $\tilde{f} : X \to Y$.
      - For any topological space Y and any continuous maps $f, g : X \to Y$, if $f|_A$ is homotopic to $g|_A$, then f is homotopic to g.

(4) [Contractible spaces]
  (a) Prove that the following are equivalent:
    (i) X is contractible.
    (ii) X is homotopy equivalent to a point.
    (iii) X weak deformation retracts to a point. [However, there are examples of topological spaces that are contractible but do not strong deformation retract to any point(c.f. Hatcher, Algebraic Topology, Exercise 1.6).]
  (b) Recall that the (topological) cone $C(X)$ of any space topological space X is

$$C(X) = X \times [0,1]/X \times \{0\}.$$

    (i) Prove: For any X, the topological cone $C(X)$ is contractible.
    (ii) (Not required) Let Y be any topological space, and $f \in \mathcal{C}(X,Y)$ be a continuous map. Prove: f is null-homotopic if and only if f has a continuous extension $\hat{f} : C(X) \to Y$.
  (c) Suppose Brouwer's fixed point theorem holds, i.e. any continuous map $f : \overline{B^n} \to \overline{B^n}$ has a fixed point (that is, a point p with $f(p) = p$). Prove: S^{n-1} is not contractible.
  (d) (Not required) Find "(Bing's) house with two rooms" from literature/internet and show that it is contractible.

PROBLEM SET 10, PART 2: TOPOLOGY (H)
DUE: MAY 9 , 2022

(1) [Simply connected]
  (a) Let X be path connected. Prove that the following statements are equivalent:
    (i) X is simply connected, i.e. $\pi_1(X) = \{e\}$.
    (ii) Any loop in X can be continuously deformed to a point in X.
    (iii) For any $x_0, x_1 \in X$, any two paths $\gamma_1, \gamma_2 \in \Omega(X; x_0, x_1)$ are path homotopic.
  (b) Show that "simply connectedness" is a topological property. Is it multiplicative? preserved under continuous maps? hereditary?

(2) [The fundamental group of the product space]
  (a) Prove: $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.
  (b) (Not required) Write down a formula for the fundamental group of an arbitrary product, $\pi_1(\prod_\alpha X_\alpha, (x_\alpha))$, and prove your formula.
  [Warning: for infinitely many groups G_α, there are two ways to "multiply" them together: the direct sum $\bigoplus_\alpha G_\alpha$ and the direct product $\bigotimes_\alpha G_\alpha$.]

(3) [Base point change isomorphism]
Let X be path connected, $x_0, x_1 \in X$. We have seen in Proposition 3.4.9 that any path λ from x_0 to x_1 induces a group isomorphism $\Gamma_\lambda : \pi_1(X, x_1) \to \pi_1(X, x_2)$.
  (a) Suppose λ_1 is a path from x_0 to x_1, and λ_2 is a path from x_1 to x_2.
  Prove: $\Gamma_{\lambda_1 * \lambda_2} = \Gamma_{\lambda_2} \circ \Gamma_{\lambda_1}$.
  (b) Prove: $\pi_1(X, x_0)$ is abelian <u>if and only if</u> for any two paths λ_1, λ_2 from x_0 to x_1, we have $\Gamma_{\lambda_1} = \Gamma_{\lambda_2}$.
  (c) Suppose X, Y are path connected, and $f \in \mathcal{C}(X, Y)$. I have a vague idea that the group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is independent of the choice of x_0. Please write down an explicit formula/rigorous statement and prove it.

(4) [The fundamental group of topological groups]
Let G be a topological group. We want to prove $\pi_1(G, e)$ is an abelian group. There is a one-sentence proof:

> Topological groups are group objects in the category $\mathcal{TOP}$, so under the functor π_1 (which preserves products), they become group objects in the category $\mathcal{GROUP}$, which are abelian groups.

Unfortunately, I don't understand that fancy proof. So I want more elementary proofs. In what follows we give two proofs.
We let γ_1, γ_2 be two loops in G based at e.

(a) (First proof) Denote by γ_e the constant loop at e. Check:

$$F(s,t) = (\gamma_1 * \gamma_e)(\max(0, t - \frac{s}{2})) \bullet (\gamma_e * \gamma_2)(\min(1, t + \frac{s}{2}))$$

is a path homotopy between $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$, where $\bullet$ is the group multiplication.

(b) (Not required)(Second proof) Construct explicit path homotopies to verify
  (i) $\gamma_1(t) \bullet \gamma_2(t) \sim \gamma_2(t) \bullet \gamma_1(t)$;
  (ii) $(\gamma_1 * \gamma_2)(t) \sim \gamma_1(t) \bullet \gamma_2(t)$.
  (Hint: $\gamma_1 * \gamma_2 = (\gamma_1 * \gamma_e) \bullet (\gamma_e * \gamma_2)$)

PROBLEM SET 11, PART 1: TOPOLOGY (H)
DUE: MAY 16, 2022

(1) [More fundamental groups]
Find the fundamental groups of the following spaces:
(a) $\mathbb{R}^{n+k} \setminus (\mathbb{R}^n \times \{(0, \cdots, 0)\})$ $(k \geq 2)$
(b) $\mathbb{R}^3 \setminus \mathbb{Z}^3$
(c) $S^2 \vee S^2$ (See section 1.4 for the definition of the wedge product)
(d) $S^1 \vee S^2$
(e) $\{(x, y, 0) \mid x, y \in \mathbb{R}\} \cup \{(0, y, z) \mid y^2 + z^2 = 1, z \geq 0\}$
(f) $\mathbb{R}^3 \setminus (\{(0, 0, z) \mid z \in \mathbb{R}\} \cup \{(x, y, 0) \mid x^2 + y^2 = 1\})$
(g) (Not required) $\mathbb{R}^3 \setminus \{(x, y, 0) | x^2 + y^2 = 1\}$
(h) (Not required) $\mathbb{R}^3 \setminus (\{(0, 0, 0)\} \cup \{(1, 1, z) \mid z \in \mathbb{R}\})$

(2) [Maps with trivial induced homomorphism]
(a) Suppose $h : S^1 \to X$ is a continuous map. Prove: The following are equivalent
  (i) The induced homomorphism $h_* : \pi_1(S^1, 1) \to \pi_1(X, h(1))$ is the trivial homomorphism (i.e. $h_*([\gamma]_p) = e$ holds for all $[\gamma]_p \in \pi_1(S^1, 1)$).
  (ii) h is null homotopic.
  (iii) h can be extended to a smooth map $H : \overline{D} \to S^1$.
(b) Now suppose $X = S^1$. Prove: (i)-(iii) are equivalent to
  (iv) h can be lifted to a continuous map $\tilde{h} : S^1 \to \mathbb{R}$ so that $p \circ \tilde{h} = h$.
(c) Read the proof of Borsuk-Ulam theorem (in which (i)$\Longrightarrow$(iv) is used) and the proof of pancake theorem on page 220-221.

(3) [The degree for maps between the circle]
For any continuous map $f : S^1 \to S^1$, there exists $n \in \mathbb{Z}$ such that $f_*([\gamma_1]_p) = [\gamma_n]_p$. The integer n is called the *degree* of the map f, and is denoted by $\deg(f)$.
(a) Prove: If $f \in \mathcal{C}(S^1, S^1)$ is not surjective, then $\deg(f) = 0$.
(b) Prove: If $f, g \in \mathcal{C}(S^1, S^1)$, then $\deg(f \circ g) = \deg(f)\deg(g)$.
(c) Prove: f is homotopic to g if and only if $\deg(f) = \deg(g)$.
(d) Read the following paragraph which gives a descriptive definition of the winding number:

> Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a continuous map and $p \notin \mathrm{Im}(\gamma)$. The *winding number* $W(\gamma, p)$ of the closed curve γ around the point p is defined to be the integer representing the total number of times that curve travels counterclockwise around the point.

Use the language of mapping degree to give a rigorous definition of winding number $W(\gamma, p)$.

(4) (Not required) [Not-so-fundamental group]
Let X be a path connected topological space, and $x_0 \in X$ be a base point. Given any two loops γ_0, γ_1 based at x_0, we define a *pseudo-homotopy* between γ_0 and γ_1 to be a map [NOT NECESSARY CONTINUOUS] $F : [0,1] \times [0,1] \to X$ s.t.

- For any fixed t, the map $\gamma_t(s) := F(t,s)$ is continuous in s.
- For any fixed s, the map $\lambda_s(t) := F(t,s)$ is continuous in t.
- For any s, $F(0,s) = \gamma_0(s),\ F(1,s) = \gamma_1(s)$.
- For any t, $F(t,0) = F(t,1) = x_0$.

We define the "*NOT-SO-Fundamental group*" of X at x_0 to be the pseudo-homotopy classes.

(a) Show that the "*NOT-SO-Fundamental group*" of S^1 is the trivial group $\{e\}$.
(b) Show that the "*NOT-SO-Fundamental group*" is not so interesting, since it is always the trivial group $\{e\}$.
(c) In proving $\pi_1(S^1) \simeq \mathbb{Z}$, where did we use the continuity of the homotopy?

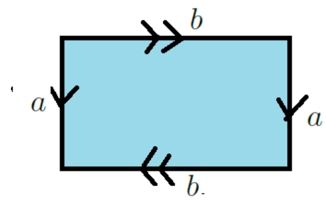
PROBLEM SET 11, PART 2: TOPOLOGY (H)
DUE: MAY 16, 2022

(1) (NOT required). [Abelianization]
Let G be a group.
(a) Let $[G, G]$ be the subgroup of G that is generated by all elements of the form $xyx^{-1}y^{-1}$ for all $x, y \in G$. Prove: $[G, G]$ is a normal subgroup of G.
(b) Prove: The group $Ab(G) := G/[G, G]$ is abelian (called the *abelianization* of G).
(c) Prove: The abelianization defines a functor from $\mathcal{GROUP}$ to $\mathcal{ABELGROUP}$.
(d) What is the abelianization of $\mathbb{Z} * \cdots * \mathbb{Z}$?
(e) Prove: $Ab(\langle a_1, b_1, \cdots, a_n, b_n | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1\rangle) = \mathbb{Z}^{2n}$.
(f) Prove: $Ab(\langle a_1, \cdots, a_n | a_1^2 \cdots a_n^2 = 1\rangle) = \mathbb{Z}^{n-1} \times \mathbb{Z}_2$.

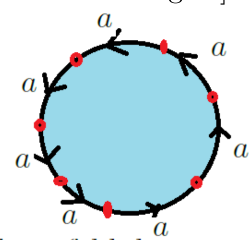
(2) [The wedge sum of circles]
(a) Finite wedge sum and applications.
(i) Prove: $\pi_1(S^1 \vee S^1 \vee \cdots \vee S^1) \simeq \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$.
(ii) What is the fundamental group of $\mathbb{R}^2 - \{\text{finitely many points}\}$?
(iii) (NOT required) What is the fundamental group of $\mathbb{R}^2 - \mathbb{Z}^2$?
(iv) (Not required) What is the fundamental group of the set $\mathbb{R}^3 - \{\text{finitely many lines passing } 0\}$?
(v) (NOT required). A group is called *finitely presented* if it has a presentation $G = \langle S | R\rangle$ where both S and R are finite sets. Prove: any finitely presented group is the fundamental group of some compact Hausdorff space.
[Hint: First construct a wedge sum of circles with fundamental group $\langle S\rangle$, then for each element in R attach a disk to kill the relation.]
(b) Infinite wedge sum.
(i) Let $X = \cup_{n\geq 1} C_n$, where C_n is the circle in $\mathbb{R}^2$ of radius n centered at $(n, 0)$. Compute $\pi_1(X)$.
(ii) Let $Y = \{(x, 0) \mid x \in \mathbb{R}\} \cup \bigcup_{n\geq 1} \widetilde{C}_n$, where $\widetilde{C}_n$ is the circle in $\mathbb{R}^2$ of radius $1/3$ centered at $(n, 1/3)$. Compute $\pi_1(Y)$. Are X and Y homeomorphic? homotopic equivalent?
(iii) (NOT required). Let $Z = \cup_{n\geq 1} C_{1/n}$, where $C_{1/n}$ is the circle of radius $1/n$ centered at $(1/n, 0)$. Prove: There is a surjective homeomorphism from $\pi_1(Z)$ to the *direct product* $\prod_{n\geq 1} \mathbb{Z}$. As a consequence, $\pi_1(Z)$ contains uncountably many elements [So Z is not homotopy equivalent to X or Y].
(iv) (NOT required).Use (iii) to prove: $\pi_1(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ contains uncountably many elements.

(3) [Application of van Kampen]
Use van Kampen theorem to compute the fundamental group of

(a) $\mathbb{RP}^2$
(b) The Klein bottle.
(c) (Not required) The n-fold dunce cap. [Split the boundary circle of a closed disk into n parts (by n red dots), and identify the boundary segments according to the picture below (but keep the interior of the disk unchanged.]



The Klein bottle



The n-fold dunce cap

(d) Prove: The fundamental group of $\Sigma_g = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$ is given by

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$

(e) Remove k small disjoint discs from Σ_g and denote the resulting space by $\Sigma_{g,m}$. Compute $\pi_1(\Sigma_{g,m})$
(f) (Not required) Compute the fundamental group of $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$

(4) [The fundamental group of topological manifolds]
Let X, Y be connected topological manifolds.
(a) Suppose $\dim X > 2$. Prove: For any point $x \in X$, $\pi_1(X) \simeq \pi_1(X \setminus x)$.
(b) Prove: $\pi_1(X \vee Y) \simeq \pi_1(X) * \pi_1(Y)$.
(c) Suppose $\dim X = \dim Y > 2$. Prove: $\pi_1(X \# Y) \simeq \pi_1(X) * \pi_1(Y)$.
(d) (NOT required) Prove: The fundamental group of any topological manifold is countable (i.e. contains only countably many elements).
[Hint: cover X by countably many open sets U_i that are homeomorphic to Euclidean balls. Pick a point from each U_i and from each component of all possible $U_i \cap U_j$. Try to show that each loop is path homotopic to loops consisting of segments connecting the chosen points.]

PROBLEM SET 12, PART 1: TOPOLOGY (H)
DUE: MAY 23, 2022

(1) [Products of coverings]
  (a) Prove: If X is connected, $\widetilde{X} \neq \emptyset$, then p is surjective, and the cardinality of $p^{-1}(x)$ is independent of x.
  (b) Prove: If $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X'$ are covering maps, so is their product $p \times p' : \widetilde{X} \times \widetilde{X}' \to X \times X'$.
  (c) Let $p : \mathbb{R} \to S^1$ be the standard covering map. Prove: The infinite product $\prod_{n\in\mathbb{N}} p : \prod_{n\in\mathbb{N}} \mathbb{R} \to \prod_{n\in\mathbb{N}} S^1$ is NOT a covering map.

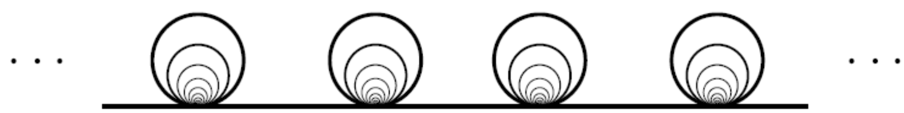
(2) [Fundamental groups of covering spaces]
Suppose $X, \widetilde{X}$ are path-connected, $p : \widetilde{X} \to X$ is a covering map, and $p(\tilde{x}_0) = x_0$.
  (a) Suppose γ is a loop in X based at x_0. Prove: γ can be lifted to a *loop* in $\widetilde{X}$ based at $\tilde{x}_0$ if and only if $[\gamma] \in p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.
  (b) Prove: the index of the subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ is the cardinality of $p^{-1}(x_0)$.
  (c) Prove: If the base space X is simply connected, then p is a homeomorphism.
  (d) Suppose $\tilde{x}_1 \in p^{-1}(x_0)$. Prove: as subgroups of $\pi_1(X, x_0)$, the two groups $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ and $p_*(\pi_1(\widetilde{X}, \tilde{x}_1))$ are conjugate to each other.

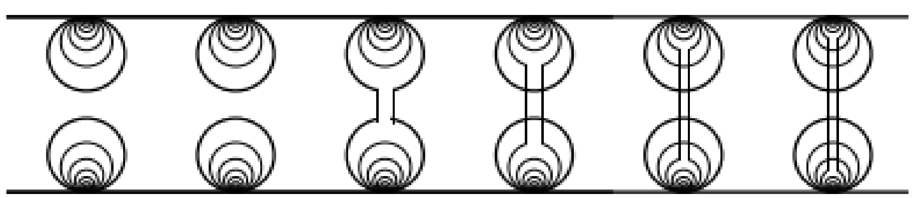
(3) [properly discontinuous actions]
  (a) Let $G = \langle a, b | a^{-1}bab = 1\rangle$. Consider the action of G on $\mathbb{R}^2$ generated by
$$a \cdot (x, y) := (-x, y - 1), \qquad b \cdot (x, y) = (x + 1, y).$$
    (i) Show that this action is a properly discontinuous action. What is the fundamental group of the Klein bottle?
    (ii) Also check that the quotient space in Example 3.7.6 is Klein bottle, and thus $\mathbb{T}^2$ is a double covering of the Klein bottle.
  (b) (Not required) Suppose group G acts on $\widetilde{X}$. We say the action is *free* if
$$\text{for any } g \neq e \text{ and any } x \in \widetilde{X},\ g \cdot x \neq x.$$
  Prove: If $\widetilde{X}$ is Hausdorff, G is a finite group, and the G-action on $\widetilde{X}$ is free, then the action is properly discontinuous.
  (c) (Not required) More generally, Let $\widetilde{X}$ be a LCH space. Suppose the G-action on $\widetilde{X}$ is free, and satisfies the following condition (known as *proper action*):
$$\text{for any compact subset } C \subset \widetilde{X}, \text{ the set } \{g \mid g \cdot C \cap C \neq \emptyset\} \text{ is finite,}$$
  Prove: the G-action is properly discontinuous, and $\widetilde{X}/G$ is a LCH space.
  [Hint: By locally finiteness, for any compact C, $\cup_g g \cdot C$ is closed.]

(4) [$SU(2)$ and $SO(3)$](Not required)
Let $SU(2)$ be the special unitary group, i.e. the group of 2×2 unitary matrices with determinant 1, and $SO(3)$ the special orthogonal group, i.e. the group of 3×3 orthogonal matrices with determinant 1.
(a) Prove: $SU(2)$ is homeomorphic to S^3 (and thus is simply connected).
(b) Prove: $SU(2)$ is a double covering of $SO(3)$ (and thus $SO(3) \simeq \mathbb{RP}^3$).
(c) What is the fundamental group of $SO(3)$?

PROBLEM SET 12, PART 2: TOPOLOGY (H)
DUE: MAY 23, 2022

(1) [Covering of covering space]
Let X, Y, Z be path-connected and locally path-connected spaces, and $f : X \to Y$, $g : Y \to Z$ be continuous maps.
  (a) Suppose both g and $g \circ f$ are covering maps. Prove: f is a covering map.
  (b) Suppose both f and $g \circ f$ are covering maps. Prove: g is a covering map.
  (c) Suppose f is a covering, and g is finite covering. Prove: $g \circ f$ is a covering.
  (d) Suppose f and g are covering, and suppose Z is semi-locally simply connected. Prove: $g \circ f$ is a covering.
  (e) (Not required) Let X be the second space below, Y be the first space below, and Z be the Hawaii earing. Construct a natural covering map $g : Y \to Z$, and a natural double covering map $f : X \to Y$ (as a double covering), so that the composition $g \circ f$ is NOT a covering map. [So in general the composition of covering maps may fail to be a covering map.]





(2) [Classify covering spaces]
  (a) Find all path connected covering spaces of $S^1 \vee S^2$.
  (b) Find all path connected covering spaces of $\mathbb{T}^2 = S^1 \times S^1$.
  [You may use the fact that the subgroups of $\mathbb{Z}^2$ are
  - $\{(0, 0)\}$,
  - $\{k(p, q) \mid k \in \mathbb{Z}\}$ (where $(p, q) \in \mathbb{Z}^2$)
  - $\{k_1(p, q) + k_2(r, s) \mid k_1, k_2 \in \mathbb{Z}\}$ (where $(p, q), (r, s) \in \mathbb{Z}$, and $ps - qr \neq 0$.]

(3) [Covering of topological groups and manifolds]

(a) (Not required) Let G be a topological group which is path-connected and locally path-connected.

(i) Suppose $\widetilde{G}$ is path-connected, and let $p : \widetilde{G} \to G$ be a covering map. Fix an element $\tilde{e} \in p^{-1}(e)$. Define a map $m : \widetilde{G} \times \widetilde{G} \to G$ by

$$m(\tilde{a}, \tilde{b}) := p(\tilde{a}) \cdot p(\tilde{b})$$

Prove: m can be lifted to a map $\tilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ with $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$.

(ii) Prove: Any covering space of a topological group is a topological group.

(b) Let M be a topological manifold.

(i) Prove: Any topological manifold admits a universal covering.

(ii) Prove: Any covering space of a topological manifold is still a topological manifold. [It follows that any Lie group admits a universal covering which is still a Lie group. This fact plays an important role in classifying Lie groups.][Hint:what do we know about the fundamental group of a topological manifold?]

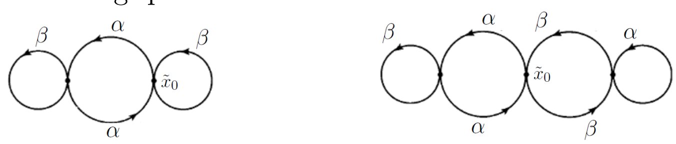
(4) [Deck transformation] (Not required)

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Its *Deck transformation group* is

$$\mathrm{Aut}(p) := \{h : \widetilde{X} \to \widetilde{X} \mid h \text{ is a covering space isomorphism}\}.$$

(a) For each path connected covering space of S^1, find its Deck transformation group.

(b) Below are two covering spaces of $S^1 \vee S^1$. Find their Deck transformation groups.



(c) Suppose G acts on $\widetilde{X}$ which is path-connected, and suppose the action is properly discontinuous. Prove: G is the deck transformation group of the covering $p : \widetilde{X} \to X = \widetilde{X}/G$.

(d) Let $p:(\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal covering. Define an action of $G = \pi_1(X, x_0)$ on $\widetilde{X}$, and prove that the action you defined is properly discontinuous..

[Thus the deck transformation group of the universal covering is $\pi_1(X, x_0)$.]

PROBLEM SET 13, PART 1: TOPOLOGY (H)
DUE: MAY 30, 2022

(1) [Applications of Brouwer's Fixed Point Theorem]
   (a) [A special case of Poincare-Hopf Theorem, proved by Hadamard] Let $f : \overline{B^n} \to \mathbb{R}^n$ be a continuous map (i.e. f is a vector field on $\overline{B^n}$) such that $x \cdot f(x) > 0$ for all $x \in S^{n-1} = \partial \overline{B^n}$. Prove: there exists $x \in B^n$ such that $f(x) = 0$.
   (b) [Poincare-Bohl] Let $f : \overline{B^n} \to \mathbb{R}^n$ be a continuous map such that $f(x) \notin \{\alpha x \mid \alpha > 0\}$ for any $x \in S^{n-1}$. Prove: there exists $x \in \overline{B^n}$ such that $f(x) = 0$.
   (c) [Perron-Frobenius] Any $n \times n$ real matrix with positive entries has a positive eigenvalue, and the corresponding eigenvector can be chosen to have strictly positive entries.
   (d) [Kuratowski-Steinhuas] Let $f : \overline{B^n} \to \overline{B^n}$ be a continuous map such that $f(S^{n-1}) \subset S^{n-1}$, and suppose for any $x \in S^{n-1}$, $f(x) \neq x$. Prove: $f(\overline{B^n}) = \overline{B^n}$.

(2) [Brouwer's Fixed Point Theorem, 2^{nd} version](Not required)
Let $K \subset \mathbb{R}^n$ be any non-empty compact convex subset.
   (a) Suppose K has non-empty interior. Prove: K is homeomorphic to $\overline{B^n}$.
   (b) Prove: K has non-empty interior if and only if K is not contained in a proper hyperplane (i.e. a set of the form $x_0 + V$, where $V \subset \mathbb{R}^n$ is a linear subspace).
   (c) Prove Theorem 4.1.5.

(3) [Poincaré-Miranda theorem]
The following theorem was first announced by H. Poincaré in 1883, which can be viewed at first glance as a higher dimension generalization of intermediate value theoremz. Miranda showed in 1940 that the theorem was equivalent to the Brouwer's fixed point theorem.

Poincaré-Miranda Theorem. Let $f = (f_1, \cdots, f_n) : [0,1]^n \to \mathbb{R}^n$ be continuous. Suppose for any $1 \leq i \leq n$, we have

$$f_i \leq 0 \quad \text{on} \quad \{x \in [0,1]^n \mid x_i = 0\},$$
$$f_i \geq 0 \quad \text{on} \quad \{x \in [0,1]^n \mid x_i = 1\}.$$

Then there exists $p \in [0,1]^n$ such that $f(p) = 0$.

   (a) Prove Poincaré-Miranda theorem via Brouwer's fixed point theorem.
   [Hint: Let $r : \mathbb{R} \to [0,1]$ be the retraction with $r((-\infty,0)) = 0, r((1,+\infty)) = 1$ and let $r(x) = (r(x_1), \cdots, r(x_n))$. Consider $h(x) = r(x) - f(r(x))$. Then h maps into a large ball into itself. Show that the fixed point of h lies in $[0,1]^n$.]
   (b) Prove Brouwer's fixed point theorem via Poincaré-Miranda theorem.

(4) [Applications of Brouwer's invariance of domain theorem]
  (a) Prove: there is no injective continuous map $f : S^n \to \mathbb{R}^n$.
    - Then show that there is no proper subset of S^n that is homeomorphic to S^n itself.
  (b) Show that conception of the boundary point is well-defined in the definition of "topological manifold with boundary".
    - Then show that if X is a topological manifold with boundary of dimension n, then its boundary ∂X is a topological manifold of dimension $n - 1$.

PROBLEM SET 13, PART 2: TOPOLOGY (H)
DUE: MAY 30, 2022

(1) [A story about love and hates]
In a certain country there are two towns, A and B, and two disjoint roads, α and β, connecting them. Two lovers in town A must travel to town B, one by road α and one by road β. So great is the force of their love that if at any instant they are separated by ten kilometers or more, they will surely die. There are also two enemies, one lives in town A and must travel to town B by road α, while the other lives in town B and must travel to town A by road β. So great is the force of their hatred that if at any instant they are separated by ten kilometers or less, they will surely die.

(a) Show that at least two people will end up dead by converting the previous problem to the following one:

Let $\gamma_1 : [0,1] \to [0,1]^2$ be a path from the point $(0,0)$ to the point $(1,1)$, and $\gamma_2 : [0,1] \to [0,1]^2$ be a path from the point $(0,1)$ to the point $(1,0)$.

Claim: γ_1 and γ_2 must intersect.

(b) Here is a fake proof the claim above:

Since γ_1 is a path in the square $[0,1]^2$ and since paths are continuous, we may find a continuous function $f : [0,1] \to [0,1]$ so that the image of the path γ_1 is the graph of f. Similarly we may find a continuous function g whose graph is the path γ_2. By assumption, we have $f(0) = 0$, $f(1) = 1$ and $g(0) = 1$, $g(1) = 0$. Consider the function $h(x) := f(x) - g(x)$. Then h is a continuous function with $h(0) = -1$, $h(1) = 1$, so there is $x_0 \in [0,1]$ so that $h(x_0) = 0$, i.e. $f(x_0) = g(x_0)$. So the paths γ_1 and γ_2 intersect at the point $(x_0, f(x_0))$.

Find the mistake in this proof.

(2) [Brouwer's Invariance of domain theorem revisited]

(a) (Higher dimensional analogue of "arc non-separation" theorem) Prove: If $K \subset \mathbb{R}^n$ is compact and is a retract of $\mathbb{R}^n$, then $\mathbb{R}^n \setminus K$ is connected.

(b) Let $D = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ be the open unit disc. Use Jordan curve theorem to prove: If $f : \overline{D} \to \mathbb{R}^2$ is continuous and injective, then $f(D)$ is the interior (=the bounded component) of the Jordan curve $f(S^1)$. [Hint: $f(\overline{D})$ is a retract of $\mathbb{R}^2$.]

(c) (Not required) Assume Jordan-Brouwer Theorem holds. State a higher dimensional analogue of (b) and prove it.

(3) [Application to the square peg problem]
Let $J \subset \mathbb{R}^2$ be a Jordan curve that is symmetric about the origin (i.e. $P \in J$ if and only if $-P \in J$). Moreover, assume the origin O lies in the bounded connected component of $\mathbb{R}^2 \setminus J$. Prove: J has an inscribed square, i.e. there exists four points

on J that are the vertices of a square.
[Hint: rotate the curve C by $\pi/2$ and try to find an intersection point.]

(4) [Applications to graph theory](Not required)
We say a graph $G = (V, E)$ is a *planar graph*, if it can be embedded into $\mathbb{R}^2$, i.e. can be drawn in $\mathbb{R}^2$ so that no edge cross.
  (a) Prove: The graph K_5 (=the graph with vertices $V = \{a_i \mid 1 \le i \le 5\}$ and edges $\{a_i a_j \mid 1 \le i < j \le 5\}$) is not a planar graph.
  (b) We say a space $X \subset \mathbb{R}^2$ is a θ-*space* if X is the union of three arcs A, B, C, so that they intersect and only intersect each other at their end points (so that the space looks like the letter "θ"). Prove: If $X \subset \mathbb{R}^2$ is a θ-space with arcs A, B, C, then $\mathbb{R}^2 \setminus X$ has three connected components, whose boundaries are $A \cup B$, $B \cup C$ and $C \cup A$ respectively.
  (c) Prove: The graph $K_{3,3}$ (=the graph with vertices $V = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and edges $\{a_i b_j \mid 1 \le i, j \le 3\}$) is not a planar graph.

PROBLEM SET 14, PART 1: TOPOLOGY (H)
DUE: JUNE 8, 2022

(1) [Maps on intervals]
  (a) Prove Lemma 4.3.5, Lemma 4.3.6 and Lemma 4.3.7.
  (b) Construct two coordinate charts on the "line with two doubled point" (see PSet8-2-3) that violates lemma 4.3.4.

(2) [Classification of 1-manifold with boundary]
  (a) Write down an analogue of Proposition 4.3.8 (and of Lemma 4.3.4 if you want) that can help you to prove the classification theorem of 1-manifold with boundary. [You don't need to prove your proposition.]
  (b) [NOT Required] Prove Theorem 4.3.3 (Classification of 1-manifold with boundary) using the proposition you wrote above.

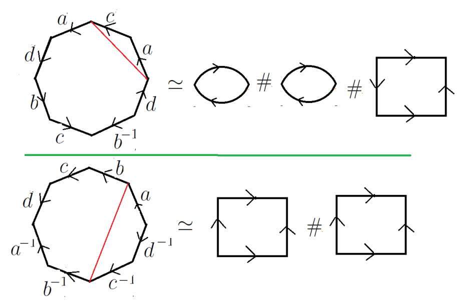
(3) [Knot groups]
  (a) For any knot K, show that the abelianization of the knot group $\pi_1(\mathbb{R}^3 \setminus K)$ is $\mathbb{Z}$.
  (b) Write down the knot groups of the knots 4_1 and 7_1 (See figure 4.1).
  (c) [Not required] Show that the unknot, the knot 3_1 and the knot 4_1 are pairwise non-equivalent knots.

(4) [Knot in $\mathbb{R}^4$?] [Not required]
Let K be a polygonal knot in $\mathbb{R}^4$, that is, the image of an embedding of S^1 into $\mathbb{R}^4$ that consists of finitely many line segments.
  (a) Prove: There exists a direction $v \in S^3$ in $\mathbb{R}^4$ such that for any $x, y \in K$, $x - y$ is not parallel to v.
  (b) Use the projection $pr_v : \mathbb{R}^4 \to v^\perp$ to construct an ambient isotopy in $\mathbb{R}^4$ that converts the knot K to a polygonal knot in $v^\perp \simeq \mathbb{R}^3$.
  (c) Prove: Any knot in $\mathbb{R}^4$ is a trivial knot.

PROBLEM SET 14, PART 2: TOPOLOGY (H)
DUE: JUNE 8, 2022

(1) [Cut the Möbius band]
Prove your conclusion via polygonal presentation. (You may use scissor to find out the answers.)
  (a) Cut the Möbius band along the center circle, what do you get?
  (b) Cut the Möbius band along a circle that is close to the boundary circle, what do you get?
  (c) What if you cut the Möbius band along both circles mentioned above? Does the order of cutting matter?
  (d) What if you cut the Möbius band along k circles that are parallel to the center circle?

(2) [Cut and Paste Polygons]
Prove the following identities by doing "cutting and pasting" on the polygons.
[Hint: The first cutting is given. The first pasting is to eliminating a. You will need a second cutting and pasting.]



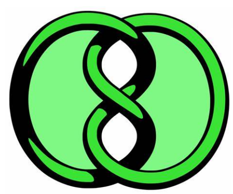
(3) [Triangulated surface]
Let S be a compact surface which is connected and without boundary.
  (a) Prove: If a finite simplicial complex K is a triangulation of S, then
    (i) Any 1-simplex in K is the intersection of exactly two 2-simplexes in K.
    [What if three triangles meeting at one edge? Use a theorem that we learned in this chapter.]
    (ii) For any 0-simplex v (i.e. vertex) in K, we can arrange the 2-simplexes containing v "cyclicly" as $\sigma_1, \sigma_2, \cdots, \sigma_k, \sigma_1$, so that $\sigma_i \cap \sigma_{i+1}$ is a 1-simplex (where we denote $\sigma_{k+1} = \sigma_1$).
    [What if these 2-simplexes can be arranged into more than two such "cycles"?]
  (b) (Not required) Conversely, suppose K is a simplicial complex consisting of finitely many 2-simplexes and their faces, so that the conditions (i) and (ii) are satisfied.

Show that $|K|$ is a surface.
[You need to show that any point has an Euclidean neighborhood.]

(4) (Not required) [Polygon presentation is a surface]
   (a) Complete the proof of Theorem 4.4.11 (the existence of polygonal presentation).
   (b) Prove: Any polygon presentation is a surface.
   [Again you need to show that any point has an Euclidean neighborhood. What if many vertices get glued into one point?]

PROBLEM SET 14, PART 3: TOPOLOGY (H)
DUE: JUNE 8, 2022

(1) [Symbolic presentation of surfaces]
Find out the surfaces in our list (i.e. $S^2, \Sigma_k, \widetilde{\Sigma}_l, S^2_m, \Sigma_{k,m}, \widetilde{\Sigma}_{l,m}$ with specified k, l, m) which are homeomorphic to the ones given below:
  (a) $\langle a, b, c, d \mid acadbcb^{-1}d \rangle$.
  (b) $\langle a, b, c, d, e \mid abcb^{-1}adede^{-1} \rangle$.
  (c) See picture below.
  (d) See picture below.


(c)


(d)

(2) [Orientability]
  (a) Prove Proposition 4.4.21.
  (b) (Not required) For each orientable compact surface (without boundary) $\Sigma_{k,m}$, prove: there exists an orientation-reversing self-homomorphism (that is, a homeomorphism $f : \Sigma_{k,m} \to \Sigma_{k,m}$ so that for some oriented triangulation K of σ, f maps simplexes of K to simplexes of K, such that for each triangle ABC in K, the orientation on $\langle f(A), f(B), f(C) \rangle$ is $-[f(A)f(B)f(C)]$)
  [Hint: Just put the surface at a nice position, and consider the map that maps one coordinate to its inverse. You need to handle even/odd number of boundary circles separately.]

(3) [Euler characteristic v.s. covering]
We know that S^2 is a double covering of $\mathbb{RP}^2$. In Section 3.7 we have seen that Σ_{11} is a 5-fold covering space of Σ_3.
  (a) Compare $\chi(S^2)$ and $\chi(\mathbb{RP}^2)$. Compare $\chi(\Sigma_{11})$ and $\chi(\Sigma_3)$.
  (b) In general, suppose S_1, S_2 are compact connected surfaces without boundary, and $p : S_1 \to S_2$ is a k-fold covering. Find the relations between $\chi(S_1)$ and $\chi(S_2)$, and prove it.
  (c) In general, if Σ_m is a covering space of Σ_n, Find the relation between m and n.
  (d) [Not required] For each non-orientable connected surface without boundary, i.e. $\widetilde{\Sigma}_{l,m}$, there exists an orientable connected surface which is a double covering of $\widetilde{\Sigma}_{l,m}$. Which surface is it?

(4) [Triangulation of surface] [Not required]
Let K be a triangulation of a compact surface S without boundary, and let $|V|, |E|, |F|$ be the number of vertices, edges and triangles in K. Prove:
(a) $3|F| = 2|E|$.
(b) $|E| = 3(|V| - \chi(S))$.
(c) $|V| \geq \frac{7+\sqrt{49-24\chi(S)}}{2}$. [So we have seen the triangulation of $\mathbb{T}^2$ and $\mathbb{RP}^2$ with least vertices.][This is also related to the following question: how many color do you need to color a map on surface S?]

[**Last Problem**]
We learned many beautiful theorems in this course. Write down at least two of them, one from part one of this course, and the other from part two of this course.