exercise class note for complex analysis

Prof. H.Z.Li T.A. Yueheng Bao

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Abstract

In this note, we will roughly discuss some possible origins of the complex number, and point out some important points,in the limited view of the teaching assistant, relating to what was lectured by the porfessor. After all those preparations, we will give the reference key for some of our exercises. Some supplements might be given, according to the reality.

1 Some Understanding of complex number

Understanding 1 (A Pair of Real Number)**.** *A real number pair (a,b) which satisfy some computing rules*

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \times (c, d) = (ac - bd, bc + ad)$$

Remark 1. *Please check that the* $\mathbb{R}^2$ *with the computation metioned above forms a Field in Abstract Algebra*

this of course is the natural way to understand the complex number Field $\mathbb{C}$ if you are totally not familar with this "strange" structure while you are quite comfortable with the real number $\mathbb{R}$. I say this is natural since this was anyway the original rigulous definitiong given to complex number Field $\mathbb{C}$. However, a clear definition never equals to the true understanding.

I might exemplify this by another story which you must be very familiar with in the lecture Mathematical Analysis, namely the Real number $\mathbb{R}$. Looking back how ancient poeple got to reveal the existence of irrational number, $\sqrt{2}$, right? Despite the tragedy of the discoverer of this important number, people still made every effort to understand what this freak was. $x^2 = 2$, yes, it was the definition. Nevertheless, it never explained. Since rational number at that time was already completely mastered, and people believe that they can really densely describe any "length" in this world, and there must be some important connections between the freak with the known. Certainly, It takes centuries to find out the truth, i.e, Cauchy Sequence to portrait and connect.

$$1.4 \quad 1.41 \quad 1.414 \quad 1.4142 \quad 1.41421 \ ...$$

Very similarly, the moment the imaginary number firstly came out, people could not dirrectly understand this brand new object, which broke the exact connection between geometric "length" with the totally number.(note that the primiar definition or description of $\sqrt{2}$ is the length of the diagonal line of unit square.)

And since i was given by the necessaty representation of formula performed to solve the general algebraic equations, people feel ashamed that this kind of "number" was only something by imagination and never existed in the "real" world, and that was also why the name was given in the history. However, some great mathematicians would never gave in to the simple explanition of some thing only formally. Some great work,accordingly, was done in the late years, where the first step was only an trivial linear expansion (holds since i can't lies in $\mathbb{R}$).i.e.

$$z = a + bi$$

where $a, b \in \mathbb{R}$ That is exactly equivalent with the Understanding 1 given at the beginning.

Understanding 2 (by finite dimenisonal field extension)**.** *Algebraically, we can obtain the complex field by finite dimensional field expansion with the help of irreducible polynomial* $x^2 + 1$*, i.e.*

$$\mathbb{C} = \mathbb{R}[x]/x^2 + 1$$

Remark 2. *Please check the Ring isomorphism for the definition given by Understanding 1 and Understanding 2*

[Hint: $(a, b) \to \overline{a + bx}$*]*

Remark 3. *As for more details and tricks for finite dimenison field extension, just enjoy your Abstract Algebra and Galois Theory classes*

Understanding 3 (Complex Analysis)**.** *Understand the complex number z as a whole, which means you have already had a good idea of what complex number is, so that just understand it as a point on the complex plane with its own topology and category(holomorphism).*

Remark 4. *you see here how topology coincides with algebra! And actually, this is the typical example of TVS (over $\mathbb{R}$)*

Remark 5. *Only when people had a great progess in Complex Analysis, dare people asserted that they had already swallowed the new Field.*

Remark 6. *From the class, it is obvious that this understanding of complex number turn out to be the required understanding of complex number anyhow.*

2 The magic sqaure roots

It is remarkable that there are three important and extraordinary square roots in human's history!

The first is a natural geometric observation as I have already mentioned above, namely $\sqrt{2}$ is the length of the diagonal line of unit square.

The second is from the attempt to give the formula of solution to general algebraic equation. i.e.

$$ax^3 + bx^2 + cx + d = 0$$

And this is actually the origin of the complex number, which means people were eager to find some "imaginary number" I such that $i^2 = -1$. (though at that moment or even within a hundred years, people did not actually understand what that number actually meant.) this brand new idea had been puzzling people at that time until the complex analysis became mature. Afterwards, the complex number, mastered and understood by people, has become an important part of human's life. It's not an overstatement that people never described the very world without complex number.

While the last is in fact the origin of famous $\mathbb{H}$, by conducting square root of Klein-Gordan Equation.

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$$

(This indicated the existence of the positive electronics, which was exceedingly shocking and amazing when a few years later this prophecy was proved by experiment! See how magic the mathematics is!)

What turns out to be amazing is that, each magic square roots expand the number in an unbelievable degree. As the first is from countable to uncountable, whereas the second and the third is the extension of dimension.

3 Homework

1. 设 f 和 g 都在 z_0 处可微, 且 $f(z_0)=g(z_0)=0, g'(z_0)\neq 0$ 证明: $\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}$ 提示: $\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{g(z)-g(z_0)}$

$$=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\cdot\frac{z-z_0}{g(z)-g(z_0)}=\frac{f'(z_0)}{g'(z_0)}$$

2. 设域 G 和域 D 关于实轴对称, 证明: 如果 $f(z)$ 是 D 上的全纯都数, 那么 $\overline{\bar{f}(\bar{z})}$ 是 G 上 的全纯都数. 提示: $\lim_{\Delta z\to 0}\frac{\overline{f(\overline{z+\Delta z})-\overline{f(\bar{z})}}}{\Delta z}$
$=\lim_{\Delta z\to 0}\overline{\left[\frac{f(\bar{z}+\overline{\Delta z})-f(\bar{z})}{\overline{\Delta z}}\right]}\overline{f'(\overline{z})}, z\in G$

... It seems that my typing is kind of ugly... I then choose to cut some photos with some marvelous masterpiece among you.

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References

Strictly speaking, this is not something like bibloigrphy, nevertheless I want to mention this as to express my great gratitude towards Mr. H. Qin, who really does extremly well in his homework.

Exercise Class Note for Complex Analysis

Prof. H.Z.Li   T.A. Yueheng Bao

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Abstract

In this note, we will roughly discuss the basic ideas of universal covering space and the understanding of multivalue function. We will also review some important points,in the limited view of the teaching assistant, relating to what was lectured by the porfessor as we have done last time. After all those preparations, we will give the reference key for some of our exercises. Some supplements might be given, according to the reality.

1 Covering Space

As for this part, you can mainly refer to [1], and [3]. And you might alse see some examples from [2], Chapter 1.

Definition 1 (Covering Space)**.** *Let X be a topological space. A covering space of X is a topological space $\widetilde{X}$ together with a continuous map $p : \widetilde{X} \to X$ i.e. covering map, such that for any $x \in X$, there exists an open neighborhood U of x with the property*
(1) $p^{-1}(U) = \cup_\alpha V_\alpha$ is a disjoint union of open sets V_α in $\widetilde{X}$,
(2) For each α, the map $p_\alpha := p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Remark 1. *The space $\widetilde{X}$ is called the total space of the covering space, and X is called the base space, and for each $x \in X$, the pre-image $p^{-1}(x)$ is called the fiber over x.*

Remark 2. *We will always assume that both X and $\widetilde{X}$ are path-connected since*

(1) If $\widetilde{X}$ is a covering space of $X, X_0 \subset X$ is a subspace, then

$$\widetilde{X}_0 := p^{-1}(X_0)$$

is a covering space of $X_0 \rightsquigarrow$ We may (and will) assume X is path connected. In this case one can check - p is always surjective (provided $\widetilde{X} \neq \emptyset$). - for each x the fiber $p^{-1}(x)$ has the same cardinality (called the number of sheets of the covering). If $|p^{-1}(x)| = n$, we will call the covering an n-fold covering. (2) If X is path connected, then any path connected component of $\widetilde{X}$ is a covering space of $X \rightsquigarrow$ We may (and will) always assume $\widetilde{X}$ is path connected.

I assert this idea is quite essential in Complex Analysis, for there are some abundant origins you can pick up from the complex plane. See as follow,

Example 1. *$\mathbb{R}$ is a covering space of $S^1 \leftrightarrow$ with covering map $p : \mathbb{R} \to S^1, x \mapsto e^{2\pi i x}$. Similarly, S^1 is a covering space of S^1 in many different ways: For each $n \in \mathbb{Z}\backslash\{0\}$,*

$$p_n : S^1 \to S^1, z \to z^n$$

gives an $|n|$-fold covering of S^1.

Example 2. *The complex exponential map*

$$\exp : \mathbb{C} \to \mathbb{C}^* = \mathbb{C}\backslash\{0\}$$

is a covering map: for any $z = re^{i\theta} \in \mathbb{C}^*$, we have $\exp^{-1}(z) = \{\log r + (2k\pi + \theta)i \mid k \in \mathbb{Z}\}$, from which it is easy to check exp is a covering map. Similarly the map

$$p_n : \mathbb{C}^* \to \mathbb{C}^*, \quad z \mapsto z^n$$

is a $|n|$-fold covering map for any integer $n \in \mathbb{Z}\backslash\{0\}$.

exercise 1. *Please check what have mentioned in the example above is really covering maps*

exercise 2. *Show that the same map $p_n : \mathbb{C} \to \mathbb{C}, z \mapsto z^n$ is not a covering map.*

2 The Lifting Lemma

Now let $\widetilde{X}$ be a covering space of X and $p : \widetilde{X} \to X$ a covering map.

Definition 2. *Suppose $f : Y \to X$ is a continuous map. A lifting of f is a continuous map $\tilde{f} : Y \to \widetilde{X}$ such that the diagram commutes, i.e.*

$$\tilde{f} = p \circ f$$

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Lemma 1 (The Lifting Lemma)**.** *Let $p : \widetilde{X} \to X$ be a covering map. Given any continuous $F : P \times I \to X$ and any lifting $\widetilde{F}_0 : P \to \widetilde{X}$ of $F_0 = F|_{P\times\{0\}} : P \times \{0\} \to X$, there exists a unique lifting $\widetilde{F} : P \times I \to \widetilde{X}$ of F s.t. $\widetilde{F}\Big|_{P\times\{0\}} = \widetilde{F}_0$.*

By taking $P = \{pt\}$ and $P = [0,1]$ respectively, we get

Corollary 1 (Path lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering. Given any path $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$ and any $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique path $\tilde{\gamma} : [0,1] \to \widetilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ which is a lifting of γ, i.e. $p \circ \tilde{\gamma} = \gamma$.*

Corollary 2 (Homotopy lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering. Given any homotopy $F : [0,1] \times [0,1] \to X$ with $F(s,0) = x_0$ So that F is a homotopy fixing start points and any $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lifting $\widetilde{F} : [0,1] \times [0,1] \to \widetilde{X}$ with $\widetilde{F}(s,0) = \tilde{x}_0$ which is a lifting of F, i.e. $p \circ \widetilde{F} = F$.*

Uniqueness of lifting. Now we consider a general lifting. It turns out that the uniqueness of lifting always holds.

Proposition 1. *Let $p : \widetilde{X} \to X$ be a covering, $f : Y \to X$ be a continuous map, and let $\tilde{f}_1, \tilde{f}_2 : Y \to \widetilde{X}$ be two liftings of f. Suppose Y is connected, and suppose there exists $y_0 \in Y$ s.t. $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$. Then $\tilde{f}_1 = \tilde{f}_2$ on Y.*

Proof. Given any $y \in Y$, we let U be an open neighborhood of $f(y)$ in X such that

$$p^{-1}(U) = \bigcup_{\alpha} \widetilde{U}_\alpha,$$

(disjoint union)
and such that each

$$p_\alpha := p|_{\widetilde{U}_\alpha} : \widetilde{U}_\alpha \to U$$

is a homeomorphism. Take $\widetilde{U}_1$ and $\widetilde{U}_2$ such that

$$\tilde{f}_1(y) \in \widetilde{U}_1, \quad \tilde{f}_2(y) \in \widetilde{U}_2.$$

Now we use connectedness argument.

$$Y_0 = \left\{ y \in Y \mid |\tilde{f}_1(y) = \tilde{f}_2(y) \right\}$$

. Then $Y_0 \neq \emptyset$ since $y_0 \in Y_0$. In what follows we prove Y_0 is both open and closed. Suppose $y \notin Y_0$. Then we have $\widetilde{U}_1 \neq \widetilde{U}_2$, which implies $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$. By continuity, there exists an open neighborhood N of y in Y such that

$$\tilde{f}_1(N) \subset \widetilde{U}_1, \quad \tilde{f}_2(N) \subset \widetilde{U}_2.$$

It follows $N \cap Y_0 = \emptyset$. So Y_0^c is open, i.e. Y_0 is closed. - Suppose $y \in Y_0$. Then we have $\widetilde{U}_1 \cap \widetilde{U}_2 \neq \emptyset$ and thus $\widetilde{U}_1 = \widetilde{U}_2$. Again we will get an open neighborhood N of y as above. Since p is injective on $\widetilde{U}_1 = \widetilde{U}_2$, and since

$$p \circ \tilde{f}_1 = p \circ \tilde{f}_2,$$

we conclude that $\tilde{f}_1 = \tilde{f}_2$ on N.(Here you see how important the local homeomorphism can be) So $N \subset Y_0$, i.e. Y_0 is open. Finally since Y is connected, Y_0 is non-empty and is both open and closed, we conclude $Y_0 = Y$, i.e. $\tilde{f}_1 = \tilde{f}_2$ on Y. □

Remark 3. *The existence of a general lifting is more complicated. Suppose a lifting f of exists. Then by functoriality of π_1, we must have*

$$f_* \left(\pi_1 \left(Y, y_0\right)\right) = p_* \left(\tilde{f}_* \left(\pi_1 \left(Y, y_0\right)\right)\right) \subset p_* \left(\pi_1 \left(\widetilde{X}, \tilde{x}_0\right)\right).$$

It turns out that the condition above is also sutticient for the existence of a lifting, a long as we assume Y is path-connected and locally path-connected:

Theorem 1 (necessary and sufficient conditions for Existence of lifting). *Suppose* $p : \left(\widetilde{X}, \tilde{x}_0\right) \to (X, x_0)$ *is a covering, and* $f : (Y, y_0) \to (X, x_0)$ *is continuous. If* Y *is path-connected and locally path-connected, then a lifting* $\tilde{f}$ *of* f *exists if and only if* $(*)$

$$f_* \left(\pi_1 \left(Y, y_0\right)\right) \subset p_* \left(\pi_1 \left(\widetilde{X}, \tilde{x}_0\right)\right).$$

$$\begin{tikzcd} & (\widetilde{X}, \tilde{x_0}) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Remark 4. *This part is actually very difficult, since a new concept has come to you (to those who haven't perviouly learned the Topology),i.e. The fundamental group. Consequently, this part may not be detailedly given since Master Rocket will give you very marvelous courses over these objects.*

3 An application to complex analysis

We have seen above that the exponential map

$$\exp : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \backslash \{0\}$$

is a covering map. Now let's try to define the complex logarithm function. In complex analysis there are two different meanings of complex logarithm:
(1) Given $0 \neq z = re^{i\theta}, \log z$ is any complex number of the form $\ln r + i(\theta + 2k\pi)$, where $k \in \mathbb{Z}$. So this function log is a multi-valued function.
(2) Given subset $U \subset \mathbb{C}^*$, one would like to define a (single-valued) complex valued function $\log : U \to \mathbb{C}$ which is a "right inverse" of exp, i.e. $\exp \circ \log = \mathrm{Id}$.
Here we refer to the second one, i.e. the existence of a function $\log : U \to$ *mathbbC* satisfying $\exp \circ \log = \mathrm{Id}$. Using the language of covering $\exp : \mathbb{C} \to \mathbb{C}^*$,

Proposition 2. *The logarithm* $\log : U \to \mathbb{C}$ *is a lifting of the inclusion* map $\iota : U \hookrightarrow \mathbb{C}^*$.

According to the existence criterion of lifting,

Remark 5. *log can not be defined on the whole of $\mathbb{C}^*$, i.e. the map Id : $\mathbb{C}^* \to \mathbb{C}^*$ has no lifting, since* $\mathrm{Id}_* (\pi_1 (\mathbb{C}^*)) \not\subset \exp_* (\pi_1(\mathbb{C}))$.

Remark 6. *the logarithm* $\log : U \to \mathbb{C}$ *exists if and only if*

$$i_* (\pi_1(U)) \subset \exp_* (\pi_1(\mathbb{C})) = \{e\},$$

i.e. if and only if $i_* (\pi_1(U)) = \{e\}$ (or in other words, if and only if U contains no loop surrounding the origin).

Remark 7. *- log is well-defined if U is a simply connected region, but simply-connectedness is NOT a necessary condition. - the function*

$$z^t = e^{t \log z}$$

is a well-defined continuous function on U for all t if and only if $i_* (\pi_1(U)) = \{e\}$, i.e. if and only if U contains no loop surrounding the origin.

As a consequence, $F(t, z) := z^t$ is not a well-defined continuous function on S^1 and thus does not give a homotopy between the identity and the constant map on S^1. Similarly for any integer $d > 1$, the map

$$p_d : \mathbb{C}^* \to \mathbb{C}^*, \quad z \mapsto z^d$$

is a p-fold covering map. There does not exist a map $z^{1/d}$ on $\mathbb{C}^*$ since

$$(p_d)_* (\pi_1 (\mathbb{C}^*)) \simeq d\mathbb{Z} \not\supset \mathbb{Z} \simeq \pi_1 (\mathbb{C}^*) = \mathrm{Id}_* (\pi_1 (\mathbb{C}^*))$$

Remark 8. *In fact, by the same argument as above, it is easy to see that the map $z^{1/d}$ is welldefined on $U \subset \mathbb{C}^*$ if and only if U contains no loop surrounding the origin (since $i_* (\pi_1(U))$ is either $\mathbb{Z}$ or $\{e\}$).*

More generally, given any polynomial $f = f(z)$, one may ask: can we define $f^{1/d}$ on a region $U \subset \mathbb{C} \backslash Z_f$, where Z_f is the zero set of f ? The answer is: we can define $f^{1/d}$ on U if and only if

$$f_* (\pi_1(U)) \subset d\mathbb{Z} \simeq (p_d)_* (\pi_1 (\mathbb{C}^*)) \subset \mathbb{Z} \simeq \pi_1 (\mathbb{C}^*)$$

For example, if $a_1 < a_2 < \cdots < a_{2n}$ are real numbers, and

$$f(z) = (z - a_1) (z - a_2) \cdots (z - a_{2n})$$

then we can define $\sqrt{f(z)}$ on the set

$$U = \mathbb{C} \backslash \Cup_{1 \le k \le n} [a_{2k-1}, a_{2k}]$$

since each closed curve γ in U must surround even number of zeros of f, which implies that $[\gamma]_p$ (and thus $f_* ([\gamma]_p)$) is an "even" class.

Remark 9. *The fact mentioned above in fact plays a very important role in the theory of Riemann surfaces.*

4 Homework

(4.2.2)

(ii) $\sqrt[n]{\dfrac{1}{2^{n^2}}}=\dfrac{1}{2^n}$，$n\to\infty$时为0，由此收敛半径为无穷，

(iv) $\lim_{n\to\infty}\sqrt[n]{\dfrac{n^n}{n!}}=\mathrm{e}$，由此收敛半径为$\frac{1}{\mathrm{e}}$。

(4.2.4)

(i) $|z|<1$时$\sum_{n=0}^{\infty}|a_nz^n|<\sum_{n=0}^{\infty}a_0|z^n|=\frac{a_0}{1-|z|}$，由此绝对收敛，故收敛，从而$R\geq 1$。

(ii) *$R>1$时有反例。如令$a_n=\begin{cases}\frac{1}{(k+1)4^n} & n=4k\\ \frac{1}{(k+1)4^{n+1}} & n=4k+1,4k+2,4k+3\end{cases}$，可发现收敛半径为4，但在$z=4\mathrm{i}$不收敛。

当$R=1,z\neq 1$时，由于$|\sum_{n=0}^{A}z^n|=\left|\frac{1-z^{n+1}}{1-z}\right|\leq\frac{1}{|1-z|}$对$A$有界，$a_n$单调趋于0，由Dirichlet判别法知收敛。

(4.2.7)

由一致收敛，$\forall 0<r<1,\int_{|z|=r}f(z)\overline{f(z)}\mathrm{d}z=2\pi\sum_{n=0}^{\infty}a_nr^n\cdot\overline{a_n}r^n=2\pi\sum_{n=0}^{\infty}|a_n|^2r^{2n}$。由$f$有界$M$，此式对$0<r<1$有上界$2\pi M^2$。由此，$\sum_{n=0}^{\infty}|a_n|^2$的任意部分和由极限可知不超过$M^2$，从而根据单调有界知收敛，即得证。

(4.2.8)

(i) 由定义$\overline{\lim}_{n\to\infty}\sqrt[n]{a_n}<\infty$，而$\lim_{n\to\infty}\sqrt[n]{\frac{1}{n!}}=0$，由此知收敛半径为无穷，即为整函数。

(ii) *区域应为$|z|\leq r<R$，且将不等式中R换为r。

由$\sum_{n=0}^{\infty}a_nr^n$收敛，可知$|a_nr^n|$有上界$M$。

$$|\varphi^{(k)}(z)|=\left|\sum_{n=0}^{\infty}\frac{a_{n+k}}{n!}z^n\right|\leq\sum_{n=0}^{\infty}\left|\frac{a_{n+k}}{n!}\right||z|^n\leq\sum_{n=0}^{\infty}\frac{M}{r^k}\frac{|z|^n}{r^nn!}=\frac{M}{r^k}\mathrm{e}^{\frac{|z|}{r}}$$

(4.3.1) 令$g(z)=(z-a)f(z)$，定义$g(a)=0$。由f全纯可知g在$B\backslash\{a\}$全纯，又利用连续由Cauchy积分定理可知在B上全纯，因此a至少为1阶零点，从而由命题4.3.4知f在a点全纯。

(4.3.4)

(i)

$$\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=R} f(\zeta)\frac{\zeta^{n+1}-z^{n+1}}{(\zeta-z)\zeta^{n+1}}\mathrm{d}\zeta=\sum_{k=0}^{n}\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=R} f(\zeta)\frac{z^k}{\zeta^{z+1}}\mathrm{d}\zeta=\sum_{k=0}^{n}\frac{z^k}{k!}\cdot\frac{k!}{2\pi\mathrm{i}}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta^{z+1}}\mathrm{d}\zeta$$

由Cauchy积分公式知即为左式。
(ii) 由$f(z)=\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta$减去第一问即得结果。
(4.3.6)
(i) 由定义$a_n=\frac{f^{(n)}(0)}{n!}$，记$\operatorname{Re} f(z)=u(z)$，与习题3.4.9类似得结果。
(ii)

$$\frac{1}{\pi}\int_0^{2\pi} u(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta=\frac{1}{\pi}\int_0^{2\pi}\big(u(r\mathrm{e}^{\mathrm{i}\theta})-A(r)\big)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta$$

$$\leq\frac{1}{\pi}\int_0^{2\pi}\big|u(r\mathrm{e}^{\mathrm{i}\theta})-A(r)\big|\mathrm{d}\theta=\frac{1}{\pi}\int_0^{2\pi}\big(A(r)-u(r\mathrm{e}^{\mathrm{i}\theta})\big)\mathrm{d}\theta=2A(r)-2u(0)$$

最后一步利用Cauchy积分公式取实部。
(4.3.7)
(i) 记$\operatorname{Re} f(z)=u(z)$，由习题4.3.6(i)，$|a_n|\leq\frac{1}{\pi}\int_0^{2\pi}|u(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta=\frac{1}{\pi}\int_0^{2\pi}u(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta=2u(0)=2$。
(ii) 第一个不等号：取$|z|<r<1$，由习题3.4.8知

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}u\left(\frac{r\mathrm{e}^{\mathrm{i}\theta}+z}{r\mathrm{e}^{\mathrm{i}\theta}-z}\right)u(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta=\frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|z|^2}{|r\mathrm{e}^{\mathrm{i}\theta}-z|^2}u(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta$$

$$\geq\frac{r-|z|}{r+|z|}\frac{1}{2\pi}\int_0^{2\pi}u(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta=\frac{r-|z|}{r+|z|}$$

令$r\to1^-$可知成立。
第二个不等号：由模定义可知结果。
第三个不等号：$|f(z)|\leq1+\sum_{n=1}^{\infty}|a_nz^n|\leq1+\sum_{n=1}^{\infty}2|z|^n=\frac{1+|z|}{1-|z|}$。
(iii) 由(ii)知$g(z)=\frac{1}{f(z)}$也满足题设条件，考虑其二次、三次项利用(i)得结果。
(4.3.14)
(i) $\sum_{n=0}^{\infty}f^{(n)}(a)z^n$收敛半径至少为1，由习题4.2.8(i)知结论。
(ii) $\mathbb{C}$上的紧集不妨设包含在$B(a,R)$中。则

$$\left|\sum_{k=n+1}^{n+p}f^{(k)}(a)\right|=\left|\sum_{k=n+1}^{n+p}\sum_{m=0}^{\infty}\frac{(z-a)^m}{m!}f^{(k+m)}(a)\right|$$

$$= \left| \sum_{m=0}^{\infty} \frac{(z-a)^m}{m!} \sum_{k=0}^{p-1} f^{(k+m)}(a) \right| \le \sum_{m=0}^{\infty} \left| \frac{(z-a)^m}{m!} \right| \left| \sum_{k=m}^{m+p-1} f^{(k)}(a) \right|$$

由于收敛，可取n足够大使$\left| \sum_{k=m}^{m+p-1} f^{(k)}(a) \right| < \varepsilon$，此时原式不超过

$$\sum_{m=0}^{\infty} \left| \frac{(z-a)^m}{m!} \right| \varepsilon = \sum_{m=0}^{\infty} \frac{|z-a|^m}{m!} \varepsilon = \mathrm{e}^{|z-a|} \varepsilon \le \mathrm{e}^R \varepsilon$$

由此即有内闭一致收敛。
利用习题4.1.12(ii)中引理知$B(a,R)$中任何闭集必然包含于某$B(a,r), r < R$，可取到$r > 1$满足要求，再利用习题4.2.8(ii)知结论。
(5.1.2)
(i) $-\sum_{n=-1}^{\infty}(n+2)(1-z)^n$
(iii) 原式为$\mathrm{Log}(1-\frac{1}{z}) - \mathrm{Log}(1-\frac{2}{z})$，即$\sum_{n=0}^{\infty} \frac{2^n-1}{n} z^{-n}$。
(iv) 分别展开后相乘可知结果为$\pm \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n 2^k \binom{\frac{1}{2}}{n-k} \binom{\frac{1}{2}}{k} z^{-n+1}$
(5.2.6)
由有一列零点逼近z_0可知z_0不为极点，若其为可去奇点，由唯一性定理知f恒为0，矛盾，从而得证。
(5.2.7)
$A = \infty$直接取极点逼近即可。假设对所有有限的A，都有收敛于z_0的点列z_n满足$f(z_n) = A$，结论成立，否则设某A不满足此条件，即存在r使得$f(z) \ne A, \forall z \in B(z_0,r) \backslash z_0$，考虑$B(z_0,r)\backslash z_0$中的$\frac{1}{f(z)-A}$，由习题5.2.6可知$z_0$为$\frac{1}{f(z)-A}$的本性奇点，计算知$\frac{1}{f(z)-A}$收敛到$\mathbb{C}_\infty$中任何数可得$f$亦有此性质，从而得证。
(5.2.8)
由于$\mathrm{Re}\, f(z) > 0$，不可能存在子列收敛到实部小于0的数，从而不为本性奇点。由实部不为0可知$\frac{1}{f}$亦在此区域全纯，且计算得其非零处实部大于0。利用习题3.2.5可知$\frac{1}{f}$在零点处实部大于0，因此不为0，从而得证。 (5.3.5)
(i) 设$f(z) = \sum_{n=0}^{\infty} a_n z^n$考虑$g(z) = \frac{f(z)+f(-z)}{2} = \sum_{n=0}^{\infty} a_{2n} z^{2n}$，由于0为$\mathbb{R}, \mathrm{i}\mathbb{R}$的交可知$a_0 = 0$，从而$h(z) = \frac{g(z)}{z^2}$仍为整函数且满足$h(\mathbb{R}) \subset \mathbb{R}, h(\mathrm{i}\mathbb{R}) \subset \mathrm{i}\mathbb{R}$，由此归纳可知$g = 0$，即得证。
(ii) 记$f_0(z) = zf(z)$，满足上问条件，因此为奇函数，从而考虑展开式可知$f(z)$为偶函数。
(5.3.6)
由定理5.3.3知f为有理函数。因此其为$z^2 + z + \frac{1}{z-1} + \frac{1}{z-2} + \frac{1}{(z-2)^2} + \frac{5}{4}$。

(补充题)
定义$F_\varepsilon(z)=F(z)\mathrm{e}^{-\varepsilon z^\alpha},1<\alpha<2$，则其在$S$上全纯，$\overline{S}$上连续。当$\arg z=\pm\frac{\pi}{4}$时，考虑辐角可知$|F_\varepsilon(z)|=|f(z)|\mathrm{e}^{-\varepsilon|z|^2\cos\frac{\pi\alpha}{4}}\le 1$，且类似得$\lim_{z\to\infty}F_\varepsilon(z)=0$，因此将区域分为两部分后由最大模原理知$|F_\varepsilon(z)|\le 1$，令$\varepsilon\to 0$即得结果。
(4.5.4)
若否，不妨设$M(r_0)>M(r_1),r_0<r_1$，则$B(0,r_1)$上的最大模不在边界取到，矛盾。
(4.5.5)
若某不为常数的多项式$P(z)$无根，则考虑$\frac{1}{P(z)}$可发现其无穷远处趋于0，且无零点。但利用习题4.5.4可知$M(r)$在$[0,\infty)$上递增，与存在R使$|z|>R$时$\left|\frac{1}{P(z)}\right|<\left|\frac{1}{P(0)}\right|$矛盾。
(4.5.6)
设$g(z)=f\left(\frac{R^2}{z}\right)$，由$\lim_{z\to\infty}f(z)$存在知0是$g$的可去奇点，从而可使$g\in H(B(0,R))\cup C(\overline{B(0,R)})$，利用习题4.5.4知$\max_{z=r}|g(z)|$随$r$增加单调增，由非常数可知严格递增，从而$M(r)$严格减。
(4.5.9)
当$M(r_1)=0$或$M(r_2)=0$时，类似习题3.4.7使用Schwarz对称原理可知f恒为0，否则记$g(z)=M(r_1)^{\frac{\log r_2/z}{\log r_2/r_1}}M(r_2)^{\frac{\log z/r_1}{\log r_2/r_1}}$，有$|g(z)|=M(r_1)^{\frac{\log r_2/|z|}{\log r_2/r_1}}M(r_2)^{\frac{\log |z|/r_1}{\log r_2/r_1}}$，由此知边界上有$|f(z)|\le|g(z)|$，对$\frac{f}{g}$运用最大模原理可知$\overline{\Omega}$中$|f(z)|\le|g(z)|$，从而$M(r)\le M(r_1)^{\frac{\log r_2/r}{\log r_2/r_1}}M(r_2)^{\frac{\log r/r_1}{\log r_2/r_1}}$，两边取log即得结论。
(4.4.1)
在每个点附近作充分小圆盘，利用Cauchy积分定理知只需考虑一个零点处。设某零点z_0附近$f(z)=(z-z_0)^k h(z)$，$h(z_0)\neq 0$，则去掉全纯部分$\frac{h'(z)}{h(z)}$后积分即为$\frac{1}{2\pi\mathrm{i}}\int_{B(z_0,\varepsilon)}\frac{g(z)k}{z-z_0}=kg(z_0)$，因此得证。
(4.4.3)
由介值定理可知其有正实根，由于右半平面$|\mathrm{e}^{-z}|<1$，根一定落在$|z-\lambda|=1$内，而记$g(z)=z-\lambda$，利用Rouché定理可知$f(z)$在此内的根个数与$g(z)$相同，即得证。
(4.4.4)
先说明$P(z)=\sum_{k=0}^n a_k z^k$零点都在$B(0,1)$中。其显然无正实根，而若$z_0$为零点，考虑$(1-z_0)P(z_0)$可知$a_n z_0^{n+1}=a_0+\sum_{k=1}^n(a_k-a_{k-1})z_0^k$，若$|z_0|\ge 1$，利用无正实根可估算得左侧模大于右侧，矛盾。
利用其有n个零点，可知z绕$|z|=1$转一圈时$P(z)$转了n圈，从而与虚轴有$2n$个交点，即至少有$2n$个不同的θ使得$\operatorname{Re}P(\mathrm{e}^{\mathrm{i}\theta})$为0，即题目中的式子至少有$2n$个不同零点。

另一方面，记$z = \mathrm{e}^{\mathrm{i}\theta}$，则所求式子乘$z^n$后为$z$的$2n$次多项式，因此至多有$2n$个不同零点，即得证。 (4.4.6)
由于此级数在$B(0,1)$收敛于$\frac{1}{(1-z)^2}$，且幂级数的收敛满足内闭一致收敛，利用Hurwitz定理得证。
(4.4.7)
由于此级数在复平面上收敛于e^z，且幂级数的收敛满足内闭一致收敛，利用Hurwitz定理得证。
(4.4.11)
(ii) $|z| = 1$时$|2z^5 - z^3 + 3z^2 - z| \leq 2 + 1 + 3 + 1 < 8$，不存在零点。
(iv) $|z| = 1$时$|\mathrm{e}^z + 1| \leq |\mathrm{e} + 1| < 4$，因此其零点个数与$-4z^n$相同，为$n$个。

References

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Exercise Class Note for Complex Analysis

Prof. H.Z.Li T.A. Yueheng Bao

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Abstract

In this note, we will roughly discuss the basic knowledge of complex geometry. We will also review some important points,in the limited view of the teaching assistant, relating to what was lectured by the porfessor as we have done last time. After all those preparations, we will give the reference key for some of our exercises. Some supplements might be given, according to the reality.

1 Basic Simplectic Geometry

Let V be a (finite dimension) real vector space and $\Omega : V \times V \to \mathbb{R}$ a bilinear map. Ω is called *anti-symmetric* if for all u, $v \in V$,

$$\Omega(u, v) = -\Omega(v, u)$$

It is called *non-degenerate* if the associate map

$$\tilde{\Omega} : V \to V^*, \quad \tilde{\Omega}(u)(v) = \Omega(u, v)$$

is bijective. Obviously the non-degeneracy is equal to the condition

$$\Omega(u, v) = 0, \forall v \in \Omega \Longrightarrow u = 0$$

Note that we can regard Ω as a linear 2-form $\Omega \in \Lambda^2(V^*)$ via

$$\Omega(u, v) = \iota_v \iota_u \Omega$$

Definition 1. *A symplectic vector space is a pair V, Ω), where V is a real vector space, and Ω a non-degenerate anti-symmetric bilinear map. Ω is called a linear symplectic structure or a symplectic form on V.*

Example. Let $V = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and define

$$\Omega_0((X,\xi),(y,\eta)) := \langle x, \xi \rangle - \langle y, \eta \rangle,$$

then (V,Ω) is a symplectic vector space. Let $\{e_1 \dots e_n, f_1 \dots f_n\}$ be the standard basis of $\mathbb{R}^n \times \mathbb{R}^n$, then Ω is determined by the relations

$$\Omega_0(e_i, e_j) = \Omega_0(f_i, f_j) = 0, \quad \Omega_0(e_i, f_j) = \delta_{ij}, \qquad \forall i, j.$$

Donate by $\{e_1^* \dots e_n^*, f_1^* \dots f_n^*\}$ the dual basis of $(\mathbb{R}^n)^* \times (\mathbb{R}^n)^*$, than as a linear 2-form one has

$$\Omega_0 = \sum_{i=1}^{n} e_i^* \wedge f_i^*.$$

Example. More generally for any finitely dimension vector space U, the vector space $V = U \oplus U^*$ admits a canonical symplectic structure

$$\Omega((u,\alpha),(v,\beta)) = \beta(u) - \alpha(v).$$

Definition 2. *Let $(V_1,\Omega_1) and (V_2,\Omega_2)$ be two symplectic vector spaces. A linear map $F : V_1 \to V_2$ is called a linear symplectomorphism (or a linear canonical transformaton) if it is a linear isomorphism and satisfies*

$$F^*\Omega_2 = \Omega_1.$$

Example. Any linear isomorphism $L : U_1 \to U_2$ lifts to a linear symplectomorphism

$$F : U_1 \oplus U_1^* \to U_2 \oplus U_2^*, \quad F((u,\alpha)) = (L(u), (L^*)^{-1}(\alpha))$$

Proof. $\forall (u,\alpha),(v,\beta) \in U_1 \oplus U_1^*$, we have

$$\begin{aligned}
F^*\Omega_2((u,\alpha),(v,\beta)) &= \Omega_2(F((u,\alpha)),F((v,\beta))) \\
&= \Omega_2((L(u),(L^*)^{-1}(\alpha)),(L(v),(L^*)^{-1}(\beta))) \\
&= (L^*)^{-1}(\beta)(L(u)) - (L^*)^{-1}(\alpha)(L(v)) \\
&= \beta(u) - \alpha(v) \\
&= \Omega_1((u,\alpha),(v,\beta))
\end{aligned}$$

so $F^*\Omega_2 = \Omega_1$

□

Theorem 1. *For any linear vector space* (V,Ω), *there exists a basis* $\{e_1 \dots e_n, f_1 \dots f_n\}$ *of* V *so that*

$$\Omega_0(e_i, e_j) = \Omega_0(f_i, f_j) = 0, \quad \Omega_0(e_i, f_j) = \delta_{ij}, \qquad \forall i, j.$$

The basis is called the Darboux basis of (V,Ω).

Remarks.
(1)The Theorem is equivalent to saying that given any symplectic vector space(V,Ω), there exists a dual basis $\{e_1^* \dots e_n^*, f_1^* \dots f_n^*\}$ of V^* so that as a linear 2-form,

$$\Omega_0 = \sum_{i=1}^{n} e_i^* \wedge f_i^*.$$

This is also equivalent to saying that there exist a linear symplectomorphism

$$F : (V,\Omega) \to (\mathbb{R}^{2n}, \Omega_0)$$

in particular,

- Any symplectic vector space is even dimensional.
- Any even dimensional vector sapce admits a linear symplectic form.
- Up to linear symplectomorphism, there is a unique linear symplectic form on each even dimensional vector space.

proof of the linear Darboux Theorem. Apply the Gram-Schmidt process. Details will be added later. □

Let M be a smooth manifold, and

$$\omega \in \Omega^2(M) = \Gamma^\infty(\Lambda^2 T^* M)$$

a smooth 2-form on M. Recall that by definition this means that for any $p\ inM$,

$$\Omega_p : T_pM \times T_pM \to \mathbb{R}$$

is a skew-symplectic bilinear map (almost a symplectic form), and ω_p depends smoothly on p.

Definition 3. *We call* ω *a symplectic form on* M *if*

(1) *(closeness)* ω *is a closed 2-form, i.e.* $dw = 0$*.*

(2) *(non-degenerate) for each* $p \in M, \omega_p$ *is a linear symplectic form on* T_pM*.*

We will call the pair (M, ω) *a symplectic manifold.*

*Remarks.*According to the linear theory:

- dim $M = \dim T_pM$ must be even.
- if we denote dim$M$ = 2n, then

$$\omega - p^n \neq 0, \forall p \in M$$

  i.e. ω is a non-vanish 2n form, thus a $volumeform$, on M. As a consequence, M must be orientable. We will call $\frac{\omega^n}{n!}$ the *Liouville volume form* of(M, ω)

- If w is not only closed but also exact, i.e. there exacts a 1-form α on M so that $\omega = d\alpha$, then we say (M, ω) is an *exact symplectic manifold.*

Example 1. ⋄ (R^{2n}, Ω_0) *is the simplest symplectic manifold.*

⋄ *Let* S *be any oriented surface and* ω *any volume form on* S*. Then*

  - ω *is non-degenerate since it is a volume form;*
  - ω *is close since it is a top form.*

  So (S, ω) *is a symplectic manifold.*

Definition 4. *A complex structure on a vector space* V *is an automorphism* $J : V \to V$*such that* $J^2 = -Id$*. Such a pair*(V, J) *is called a complex vector space.*

The basic example is of course $\mathbb{C}^n = \mathbb{M}^{2n}$, with standard complex structure J_0 corresponding to the map "multiplication by $i = \sqrt{-1}$":

$$J_0 x_i = y_i, J_0 y_i = -x_i$$

. *Remarks.* Complex structure is very similar to symplectic structure:

(1) Since $\det(-Id) = \det(J^2) \geq 0$, dim V must be even.

(2) For any 2n dimensional vector space V with basis $x_1 \dots x_n, y_1 \dots y_n$, the linear map J defined by

$$J_0 x_i = y_i, J_0 y_i = -x_i$$

is a complex structure on V

Now, suppose (V, Ω) is a symplectic vector space which admits with a complex structure J.

Definition 5. *Let (V, Ω) be a symplectic vector space, and J is a complex structure on V.*

(1) We say J is tamed by Ω if the quadratic form $\Omega(v, Jv)$ is positive defined.

(2) We say J is compatible with Ω if it is tamed by Ω and J is a symplectomorphism, i.e.

$$\Omega(Ju, Jv) = \Omega(u, v)$$

An euqivalent condition for J compatible with Ω is that

$$G(u, v) = \Omega(u, Jv)$$

defines a inner product on V.

Proposition 1. *Every symplectic vector space admits a compatible complex structure. Moreover, given any inner product $g(\cdot, \cdot)$ on V, one can canonically construct such a J.*

Proof. □

Definition 6. *An almost complex structure J on a (real) manifold is an assignment of complex structure J_p on the tangent space T_pM which depends smoothly on p. The pair (M, J) is called an almost complex manifold.*

Remark. As in the symplectic case, an almost complex manifold must be even dimensional. Moreover, it is not hard to prove that an almost complex manifold must be orientable.
*Example.*As in the symplectic case, an orientable surface admits an almost complex structure. Let

$$\nu : \sigma\ toS^2$$

be the Gauss map with every point $x \in \sigma$ the outward unit normal vector $\nu(x)$. Define $J_x : T_pM \to T_pM$ by

$$J_x u = \nu x \times u$$

where $\times$ is the cross product between vectors in $\mathbb{R}^3$.
Example.S^2 andS^6 are the only spheres that admit almost complex structure.

Now let(M, Ω) be asymplectic manifold, and J the almost complex structure on M.

Definition 7. *We say an almost complex structure J on M is compatible with a symplectic structure ω on M if at each p, J_p is compatible with ω_p.*

As we mentioned before, this is equivalently to saying that the assignment

$$g_p : T_pM \times T_pM \to \mathbb{R}, g_p(u, v) := \omega(u, Jv)$$

defines a Riemann metric on M. So we get three structures on M: the symplectic structureω, an almost complex structure J and a Riemann structure g. They are related by

$$\begin{aligned} g(u, v) &= \omega(u, Jv) \\ \omega(u, v) &= g(Ju, v) \\ J(u) &= \tilde{g}^{-1}\tilde{(}\omega(u)) \end{aligned}$$

where $\tilde{g}$ and $\tilde{\omega}$ are the linear isomorphism form T_pM to T_p^*M induced by g and ω respectively. Such a triple (ω, g, J) is called a *compatible triple.*

Let (M, J) be an almost complex manifold. Denote by $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ the complexified tangent bundle. We extend J linear to $T_{\mathbb{C}}M$ by

$$J(v \otimes z) = J(u) \otimes z, \quad v \in TM, z \in \mathbb{C}.$$

Then again $J^2 = -Id$, but now on a complex vector space instead of a real vector space. So J has eigenvalues $\pm i$, and we have a eigenspace decomposition

$$TM \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1},$$

where

$$T_{1,0} = \{v \in TM \otimes \mathbb{C} | Ju = iu\} = \{v \otimes 1 - Jv \times i | v \in TM\}$$

is the $+i$-eigenspace of J and

$$T_{0,1} = \{v \in TM \otimes \mathbb{C} | Ju = -iu\} = \{v \otimes 1 + Jv \times i | v \in TM\}$$

is the $-i$-eigenspace of J. We call vectors in $T_{1,0}$ the *J-holomorphic vectors* and vectors in $T_{0,1}$ the *J-anti-holomorphic vectors*. They are both n dimensional (real) vector space. Moreover, Let

$$\pi_{1,0} : TM \to T_{1,0} \quad v \mapsto \frac{1}{2}(v \otimes 1 - Jv \otimes i)$$

and

$$\pi_{0,1} : TM \to T_{0,1} \quad v \mapsto \frac{1}{2}(v \otimes 1 + Jv \otimes i)$$

It is not hard to check They are (real) bundle isomorphism such that $\pi_{1,0} \circ J = i\pi_{1,0}$, and $\pi_{0,1} \circ J = -i\pi_{0,1}$.
Similarly one can split the complexified cotangent space $T^*M \otimes \mathbb{C}$ as

$$T^*M \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1},$$

where

$$\begin{aligned} T^{1,0} = (T_{1,0})^* &= \{\eta \in T^*M \otimes \mathbb{C} | J^*(\eta) = i\eta\} \\ &= \{\xi \otimes 1 - J^*(\xi) \otimes i | \xi \in T^*M\} \end{aligned}$$

is the dual space of $T_{1,0}$ and

$$\begin{aligned} T^{0,1} = (T_{0,1})^* &= \{\eta \in T * M \otimes \mathbb{C} | J^*(\eta) = -i\eta\} \\ &= \{\xi \otimes 1 + J^*(\xi) \otimes i | \xi \ inT^*M\} \end{aligned}$$

is the dual space of $T_{1,0}$. Moreover, any cotangent vector η has a splitting

$$\eta = \eta^{1,0} + \eta^{0,1},$$

where

$$\eta^{1,0} = \frac{1}{2}(\eta - i\eta \circ J), \quad \eta^{0,1} = \frac{1}{2}(\eta + i\eta \circ J).$$

Let $\Omega^k(M,\mathbb{C}) := \Gamma^\infty(\Lambda^k(T^*M \otimes \mathbb{C}))$ i.e. the complex-value k-forms on M. where

$$\begin{aligned}
\Gamma^\infty(\Lambda^k(T^*M \otimes \mathbb{C})) &= \Lambda^k(T^{1,0} \oplus T^{0,1}) \\
&= \oplus_{l+m=k}(\Lambda^l(T^{1,0})) \wedge (\Lambda^m(T^{0,1})) \\
&:= \oplus_{l+m=k}\Lambda^{l,m}
\end{aligned}$$

Definition 8. *The differential forms of type (l,m) on (M,J) are the sections of $\Lambda^{l,m}$:*

$$\Omega^{l,m} := \Gamma^\infty(\Lambda^{l,m})$$

Then

$$\Omega^k(M,\mathbb{C}) = \oplus_{l+m=k}\Omega^{l,m}$$

Let $\pi^{l,m} : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{l,m}$ be the projection map, where $l+m=k$. d is the usual exterior derivative We define

$$\begin{aligned}
\partial :=& \pi^{l+1,m} \circ d : \Omega^{l,m} \to \Omega^{l+1,m} \\
\bar{\partial} :=& \pi^{l,m+1} \circ d : \Omega^{l,m} \to \Omega^{l,m+1}
\end{aligned}$$

For $\beta \in \Omega^{l,m} \subset \Omega^k(M,\mathbb{C})$, we have $d\beta \in \Omega^{k+1}(M,\mathbb{C}$, and so

$$d\beta = \sum_{r+s=k+1} \pi^{r,s} d\beta = \pi^{k+1,0} d\beta + \cdots + \partial\beta + \bar{\partial}\beta + \cdots + \pi^{0,k+1} d\beta$$

Note that for functions we have $df = \partial f + \bar{\partial} f$, while for more general differential forms we don‘t have $d = \partial + \bar{\partial}$

2 complex manifold

Definition 9. *A complex manifold of complex dimension n is a manifold that locally homeomorphism to open subsets in $\mathbb{C}^n$, with biholomorphic transition function maps.*

Obviously any n-dimensional complex manifold is a real manifold of dimension 2n. and must be orientable if view as a real manifold (use the Cauchy-Riemann equation).

Proposition 2. *Any complex manifold has a canonical almost complex structure.*

Proof. Let M be a complex manifold and (U, V, φ) be a complex chart of M, where u is an open set in M and V is an open set in $\mathbb{C}$. We denote $\varphi = (z_1, \ldots, z_n)$, with $z_i = x_i + \sqrt{-1}y_i$. Then $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a coordinate system on U when we view M as a real manifold. So

$$T_pM = \mathbb{R}\text{-span of } \{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} | i = 1, \ldots, n\}.$$

We define J on U by the recipe

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$$

for $i = 1, \ldots, n$, and extends to T_pM by linearity. Obviously $J^2 = -1$. It remains to prove that J is global defined, i.e. it is independent of the choice of complex coordinate charts.
Suppose (U', V', φ') is another coordinate chart, with $\varphi' = (\omega_1 \ldots, \omega_n)$ and $\omega_i = u_i + \sqrt{-1}v_i$. Then on the overlap $U \cap U'$, the transition map

$$\psi : \varphi(U \cap U') \to \varphi(U \cap U'), \quad z \to \omega = \psi(z)$$

is a biholomorphism. If we write the map as

$$u_i = u_i(x, y), \quad v_i = v_i(x, y)$$

in real coordinates, then the real tangent vectors are related by

$$\frac{\partial}{\partial x_k} = \sum_j (\frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k} \frac{\partial}{\partial v_j})$$
$$\frac{\partial}{\partial y_k} = \sum_j (\frac{\partial u_j}{\partial y_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k} \frac{\partial}{\partial v_j})$$

while the Cauchy-Riemann equation gives

$$\frac{\partial u_i}{\partial x_k} = \frac{\partial v_j}{\partial y_k}, \quad \frac{\partial u_j}{\partial x_k} = -\frac{\partial v_j}{\partial y_k}$$

It follows that

$$J'(\frac{\partial}{\partial x_k}) = J'(\sum_j (\frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k} \frac{\partial}{\partial v_j})) = \sum_j \frac{\partial v_j}{\partial y_k} \frac{\partial}{\partial v_j} + \frac{\partial u_j}{\partial y_k} \frac{\partial}{\partial u_j} = \frac{\partial}{\partial y_k}.$$

Since $J'^2 = -Id$, we must have $J'(\frac{\partial}{\partial y_k}) = -\frac{\partial}{\partial x_k}$ $\square$

Remarks. One can find that for an almost complex manifold, we define the complex structure point by point. but for the complex manifold, we have an global definition on each coordinate chart. This will play a huge role later.

Now suppose M is a complex manifold and J is the canonical almost complex structure. Then in local coordinates

$$T_pM \otimes \mathbb{C} = \mathbb{C}\text{-span of } \{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} | i = 1, \ldots, n\}.$$

We define

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

Then the two eigenspace of J are

$$T_{1,0} = \mathbb{C}\text{-span of } \{\frac{\partial}{\partial z_j} | j = 1, \ldots, n\} \quad T_{0,1} = \mathbb{C}\text{-span of } \{\frac{\partial}{\partial \bar{z}_j} | j = 1, \ldots, n\},$$

Similarity if we put

$$dz_j = dx_j + idy_j \quad d\bar{z}_j = dx_j - idy_j$$

then

$$T^{1,0} = \mathbb{C}\text{-span of } \{dz_j | j = 1, \ldots, n\} \quad T^{0,1} = \mathbb{C}\text{-span of } \{d\bar{z}_j | j = 1, \ldots, n\},$$

If we use muti-index notation:$J = (j_1, \ldots, j_m), 1 \leq j_1 < \cdots < j_m \leq n, |J| = m, dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_m}$ then

$$\Omega^{l,m} = l, m - forms = \{ \sum_{|J|=l,|K|=k} b_{J,K} dz_J \wedge d\bar{z}_K | b_{J,K} \in \mathbb{C}^\infty(U, \mathbb{C})\}$$

Theorem 2. *On complex manifolds $d = \partial + \bar{\partial}$ for any (l, m)-forms.*

Proof. left as an exercise. □

3 Another view of Complex Geometry

Definition 10. *An δ-structure, S_M, on a K-manifold M is a family of K-valued continuous functions defined on the open sets M such that*

1. *For every $p \in M$, there exists an open neighborhood U of p and a homeomorphism $h : U \to U'$, where U' is open in K^n, such that for any open set $V \subset U$*
$$f : V \to K \in \mathcal{S}_M$$
if and only if $f \circ h^{-1} \in \mathcal{S}(h(V))$.

2. *If $f : U \to K$, where $U = \bigcup_i U_i$ and U_i is open in M, then $f \in \mathcal{S}_M$ if and only if $f|_{U_i} \in \mathcal{S}_M$ for each i.*

Definition 11. *- An $\mathcal{S}$-morphism $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ is a continuous map, $F : M \to N$, such that $f \in \mathcal{S}_N$ implies $f \circ F \in \mathcal{S}_M$. $F : M \to N$ is a homeomorphism, and*

$$F^{-1} : (N, \mathcal{S}_N) \to (M, \mathcal{S}_M)$$

is an $\mathcal{S}$-morphism.It follows from the above definitions that if on an $\mathcal{S}$-manifold $(M, \mathcal{S}_M)$ we have two coordinate systems $h_1 : U_1 \to K^n$ and $h_2 : U_2 \to K^n$ such that $U_1 \cap U_2 = \varnothing$, then

$$h_2 \circ h_1^{-1} : h_1 (U_1 \cap U_2) \to h_2 (U_1 \cap U_2)$$

is an $\mathcal{S}$-isomorphism on open subsets of $(K^n, \mathcal{S}_{K^n})$.

Definition 12. *Conversely, if we have an open covering $\{U_\alpha\}_{\alpha \in A}$ of M, a topological manifold, and a family of homeomorphisms $\{h_\alpha : U_\alpha \to U'_\alpha \subset K^n\}_{\alpha \in A}$ satisfying above, then this defines an $\mathcal{S}$-structure on M by setting $S_M = \{f : U \to K\}$ such that U is open in M and $f \circ h_\alpha^{-1} \in \mathcal{S}(h_\alpha (U \cap U_\alpha))$ for all $\alpha \in A$; i.e., the functions in S_M are pullbacks of functions in S by the homeomorphisms $\{h_\alpha\}_{\alpha \in A}$. The collection $\{(U_\alpha, h_\alpha)\}_{\alpha \in A}$ is called an atlas for (M, S_M).*

Definition 13. *Let N be a closed subset of an S-manifold M; then N is called an S-submanifold of M if for each point $x_0 \in N$, there is a coordinate system $h : U \to U' \subset K^n$, where $x_0 \in U$, with the property that $U \cap N$ is mapped onto $U' \cap K^k$, where $0 \leqslant k \leqslant n$. Here $K^k \subset K^n$ is the standard embedding of the linear subspace K^k into K^n, and k is called the K-dimension of N, and $n - k$ is called the K-codimension of N.*

Remark 1. *It is easy to see that an $\mathcal{S}$-submanifold of an S-manifold M is itself an δ-manifold with the $\mathcal{S}$-structure given by $\delta_M|_N$. Since the implicit* *function theorem is valid in each of our three categories, it is easy to verify that the above definition of submanifold coincides with the more common one that an δ-submanifold (of k dimensions) is a closed subset of an δ-manifold M which is locally the common set of zeros of $n-k$ S-functions whose Jacobian matrix has maximal rank.*

Example 2. $K^n, (\mathbb{R}^n, \mathcal{C}^n)$. *For every $p \in K^n, U = K^n$ and $h = i$ identity. Then $\mathbb{R}^n$ becomes a real-analytic (hence differentiable)manifold and* C^n *is a complex-analytic manifold.*

Example 3. *If (M, S_M) is an S-mantfold, then any open subset U of M has an S-structure, $S_U = \{f \mid U : f \in S_M\}$.*

Example 4 (Projective space). *If V is a finite dimensional vector space over K, then $\mathbf{P}(V) := \{$ the set of one-dimensional subspaces of $V\}$ is called the projective space of V. We shall study certain special projective spaces,*

$$\mathbf{P}_n(\mathbb{R}) := \mathbf{P}\left(\mathbb{R}^{n+1}\right) \quad \mathbf{P}_n(\mathbb{C}) := \mathbf{P}\left(\mathbb{C}^{n+1}\right)$$

We show how $\mathbf{P}_n(\mathbb{R})$ can be made into a differentiable manifold. There is a natural map $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbf{P}_n(\mathbb{R})$ given by $\pi(x) = \pi(x_0, \ldots, x_n) := \{$ subspace spanned by $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\}$

Theorem 3 (Whitney). *Let M be a differentiable n-manifoid. Then there exists a differentiable embedding f of M into R^{2n+1}. Moreover, the image of $M, f(M)$ can be realized as a real-analytic submanifold of $\mathbb{R}^{2n+1}$.*

Theorem 4. *Let X be a connnected compact complex manifold and let $f \in \mathcal{O}(X)$. Then f is constant; i.e., global holomorphic functions are necessarily constant.*

Proof. Suppose that $f \in \mathcal{O}(X)$. Then, since f is a continuous function on a compact space, $|f|$ assumes its maximum at some point $x_0 \in X$ and $S = \{x : f(x) = f(x_0)\}$ is closed. Let $z = (z_1, \ldots, z_n)$ be local coordinates at $x \in S$, with $z = 0$ corresponding to the point x. Consider a small ball B about $z = 0$ and let $z \in B$. Then the function $g(\lambda) = f(\lambda z)$ is a function of one complex variable (λ) which assumes its maximum absolute value at $\lambda = 0$ and is hence constant by the maximum principle. Therefore, $g(1) = g(0)$ and hence $f(z) = f(0)$, for all $z \in B$. By connectedness, $S = X$, and f is constant. $\square$

Corollary 1. *There are no compact complex submanifolds of $\mathbb{C}^n$ of positive dimension.*

Proof. Otherwise at least one of the coordinate functions $z_1, \ldots, z_n$ would be a nonconstant function when restricted to such a submanifold. $\square$

Definition 14 (vector bundle)**.** *A continuous map $\pi : E \to X$ of one Hausdorff space, E, onto another, X, is called a K-vector bundle of rank r if the following conditions are satisfied:*

1. *$E_p := \pi^{-1}(p)$, for $p \in X$, is a K-vector space of dimension r (E_p is called the fibre over p).*
2. *For every $p \in X$ there is a neighborhood U of p and a homeomorphism $h : \pi^{-1}(U) \to U \times K^r$ such that $h(E_p) \subset p \times K^r$, and h^p, defined by the composition*
$$h^p : E_p \xrightarrow{h} p \times K^r \xrightarrow{proj} K^r$$
is a K-vector space isomorphism

Remark 2. *the pair (U, h) is called a local trivialization*

Remark 3. *For a K-vector bundle $\pi : E \to X, E$ is called the total space and X is called the base space, and we often say that E is a vector bundle over X. Notice that for two local trivializations (U_α, h_α) and (U_β, h_β) the map $h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times K^r \to (U_\alpha \cap U_\beta) \times K^r$ induces a map*
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, K)$$
where $g_{\alpha\beta}(p) = h_\alpha^p \circ \left(h_\beta^p\right)^{-1} : K^r \to K^r$. The functions $g_{\alpha\beta}$ are called the transition functions of the K-vector bundle $\pi : E \to X$. The transition functions $g_{\alpha\beta}$ satisfy the following compatibility conditions:
$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = I_r \quad on \quad U_\alpha \cap U_\beta \cap U_\gamma$$
$$g_{\alpha\alpha} = I_r \text{ on } U_\alpha$$
where the product is a matrix product and I_r is the identity matrix of rank r.

Definition 15. *A K-vector bundle of rank $r, \pi : E \to X$, is said to be an $\mathcal{S}$-manifolds, π is an $\mathcal{S}$-morphism, and the local trivializations are $\mathcal{S}$-isomorphisms.*

Remark: Suppose that on an S-manifold we are given an open covering I $= \{U_a\}$ and that to each ordered nonempty intersection $U_a \cap U_b$ we have assigned an δ -function

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(r, K)$$

satisfying the compatibility conditions (2.2). Then one can construct a vector bundle $E \xrightarrow{a} X$ having these transition functions. An outline of the construction is as follows: Let

$$\tilde{E} = \bigcup_a U_a \times K^r \quad \text{(disjoint union)}$$

equipped with the natural product topology and s-structure. Define an equivalence relation in $\tilde{E}$ by setting

$$(x, v) \sim (y, w), \text{ for } (x, v) \in U_\beta \times K^r, (y, w) \in U_\alpha \times K^r$$

if and only if

$$y = x \quad \text{and} \quad w = g_{\alpha\beta}(x)v,$$

The fact that this is a well-defined equivalence relation is a consequence of the compatibility conditions (2.2). Let $E = \tilde{E}/\sim$ (the set of equivalence classes), equipped with the quotient topology, and let $\pi : E \to X$ be the mapping which sends a representative (x, v) of a point $p \in E$ into the first coordinate. One then shows that an E so constructed carries on S-structure and is an S-vector bundle. In the examples discussed below we shall see more details of such a construction.

Definition 16. *Let $\pi : E \to X$ be an S-bundle and U an open subset of X. Then the restriction of E to U, denoted by E_U is the $\mathcal{S}$-bundle $\pi_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$*

Definition 17. *Let E and F be S-bundles over X; i.e., $\pi_E : E \to X$ and $\pi_F : F \to X$. Then*

1. *a homomorphism of S-bundles, $f : E \to F$, is an S-morphism of the total spaces which preserves fibres and is K-linear on each fibre; i.e., f commutes with the projections and is a K-linear mapping when restricted to fibres.*

2. *An S-bundle isomorphism is an S-bundle homomorphism which is an S-isomorphism on the total spaces and a K-vector space isomorphism on the fibres.*

3. *Two S-bundles are equivalent if there is some $\mathcal{S}$-bundle isomorphism between them. This clearly defines an equivalence relation on the S-bundles over an S-manifold, X.*

Definition 18. *Let $E \to X$ be an S-bundle. An S-submanifold $F \subset E$ is said to be an S-subbundle of E if*

1. *$F \cap E_x$ is a vector subspace of E_x.*

2. *$\pi|_F : F \to X$ has the structure of an δ-bundle induced by the S-bundie structure of E, i.e., there exist local trivializations for E and F which are compatible as in the following diagram. where the map j is the natural inclusion mapping of K^s as a subspace of K^r and i is the inclusion of F in E.*

suppose that $f : E \to F$ is a vector bundle homomorphism of K-vector bundles over a space X. We define $Kerf = \bigcup_{x \in X} Kerf\ f_x, Imf = \bigcup_{x \in X} Imf_x$, where $f_x = f_{E_x}$. Moreover, we say that f has constant rank on X *if rankt (as a K-linear mapping) is constant for $x \in X$.*

$$
\begin{array}{ccc}
E|_U & \xrightarrow{\sim} & U \times K^r \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle id \times j} \\
F|_U & \xrightarrow{\sim} & U \times K^s
\end{array}
\qquad\qquad s \le r
$$

Definition 19. *An S-section of an S-bundle $E \to X$ is an S-morphism $s : X \to E$ such that*

$$\pi \circ s = 1_x$$

where 1_X is the identity on X; i.e., s maps a point in the base space into the fibre over that point.

$\delta(X, E)$ will denote the S-sections of E over X. $\mathrm{S}(U, E)$ will denote the δ-sections of $E|_U$ over $U \subset X$; i.e., $\mathrm{S}(U, E) = \mathrm{S}(U, E|_U)$

Remark 4. *we shall also occasionally use the common notation $\Gamma(X, E)$ for sections, provided that there is no confusion as to which category we are dealing with*

Definition 20. *An S-bundle morphism between two S-bundles $\pi_E : E \to X$ and $\pi_F : F \to Y$ is an s-morphism $f : E \to F$ which takes fibres of E isomorphically (as vector spaces) onto fibres in F. An S-bundle morphism $f : E \to F$ induces the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi_F} \\ X & \xrightarrow{\bar{f}} & Y \end{array}$$

Let V be a real vector space and suppose that J is an $\mathbb{R}$-linear isomorphism$J : V \to V$ such that $J^2 = -I$. Then J is called a complex structure on V . Suppose that V and a complex structure J are given. Then we can equip V with the structure of a complex vector space in the following manner:

$$(\alpha + i\beta)v := \alpha v + \beta Jv(\alpha, \beta \in \mathbb{R}, i = sqrt{-}1)$$

Thus scalar multiplication on V by complex numbers is defined, and it is easy to check that V becomes a complex vector space.
Conversely, if V is a complex vector space, then it can also be considered as a vector space over $\mathbb{R}$, and the operation of multiplication by i is an $\mathbb{R}$-linear endomorphism of V onto itself, which we can call J , and is a complex structure. Moreover, if $v_1, ..., v_n$ is a basis for V over $\mathbb{C}$, then $v_1, ..., v_n..., Jv_1, ..., Jv_n$ will be a basis for V over $\mathbb{R}$.

Let V be a real vector space with a complex structure J, and consider $V \otimes_{\mathbb{R}} \mathbb{C}$. The $\mathbb{R}$-linear mapping J extends to a $\mathbb{C}$-linear mapping on $V \otimes_{\mathbb{R}} \mathbb{C}$ by setting $J(v \otimes \alpha) = J(v) \otimes \alpha$ for $v \in V, \alpha \in \mathbb{C}$. Moreover, the extension has the property that $J^2 = -I$, and J has two eigenvalues $\{i, -i\}$. Let $V^{1,0}$ be the eigenspace corresponding to the eigenvalue i and $V^{0,1}$ to $-i$. Then $V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$.

Moreover, conjugation on $V \otimes_{\mathbb{R}} \mathbb{C}$ is defined by $\overline{v \otimes \alpha} = v \otimes \bar{\alpha}$ for $v \in V$ and $\alpha \in \mathbb{C}$. Thus $V^{1,0} \cong_{\mathbb{R}} V^{0,1}$ (conjugate-linear mapping). The complex vector space obtained from V by means of the complex structure J, is $\mathbb{C}$-linearly isomorphic to $V^{1,0}$, and we identify V_J with $V^{1,0}$. We now want to consider the exterior algebras of these complex vector spaces $V \otimes_{\mathbb{R}} \mathbb{C}$ by V_c and consider the exterior algebras $\wedge V_c, \wedge V^{1,0}$, and $\wedge V^{0,1}$ Then we have natural injections

exercise 1. *Specifically write down these isomorphisms and injections*

We let $\wedge^{p,q}V$ be the subspace of $\wedge V_c$ generated by elements of the form $\mu \wedge \omega$, where $\mu \in \wedge^p V^{1,0}$ and $w \in \wedge^q V^{0,1}$. Thus we have the direct sum (letting $n = \dim_{\mathbb{C}} V^{1,0}$)

$$\wedge V_c = \sum_{r=0}^{2n} \sum_{p+q=r} \wedge^{p,q} V$$

We now want to carry out the above algebraic construction on the tangent bundle to a manifold.

Definition 21. *Let X be a differentiable manifold of dimension $2n$. Suppose that J is a differentiable vector bundle isomorphism $J : T(X) \to T(X)$ such that $J_x : T_x(X) \to T_x(X)$ is a complex structure for $T_x(X)$; i.e., $J^2 = -I$, where I is the identity vector bundle isomorphism acting on $T(X)$. Then J is called an almost complex structure for the differentiable manifold X. If X is equipped with an almost complex structure J, then (X, J) is called an almost complex manifold.*

Remark 5. *We see that a differentiable manifold having an almost complex structure is equivalent to prescribing a $\mathbb{C}$-vector bundle structure on the $\mathbb{R}$-linear tangent bundle.*

Let X be a differentiable m-manifold, let $T(X)_c = T(X) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the tangent bundle, and let $T^*(X)_c$ be the complexification of the cotangent bundle. We can form the exterior algebra bundle $\wedge T^*(X)_c$, we let

$$\varepsilon^r(X)_c = \varepsilon\left(X, \wedge^r T^*(X)_c\right)$$

These are the complex-valued differential forms of total degree r on X. We shall usually drop the subscript c. In local coordinates we have $\varphi \in \mathcal{E}^r(X)$ if and only if φ can be expressed in a coordinate neighborhood by

$$\varphi(x) = \sum_{|I|=r} \varphi_I(x) dx_I$$

and $\varphi_I(x)$ is a C^∞ complex valued function on the neighborhood. The exterior derivative d is extended by complex linearity to act on complex-valued differential forms, and we have the sequence

$$\varepsilon^0(X) \xrightarrow{d} \varepsilon^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \varepsilon^m(X) \to 0$$

where $d^2 = 0$. Suppose that (X, J) is an almost complex manifold. Then we can apply the linear algebra to $T(X)_C$.J extends to a $\mathbb{C}$-linear bundle isomorphism on $T(X)_C$ and has (fibrewise) eigenvalues $\pm i$. Let $T(X)^{1,0}$ be the bundle of $(+i)$-eigenspaces for J and let $T(X)^{0,1}$ be the $(-i)$ for J [note that these are differentiable subbundles of $T(X)_c$]. We define a conjugation on $T(X)_c$,by fibrewise conjugation,

$$Q : T(X)^{1,0} \to T(X)^{0,1}$$

is a conjugate-linear isomorphism. Moreover, there is a $\mathbb{C}$-linear isomorphism

$$T(X)_J \cong T(X)^{1,0}$$

where $T(X)_J$ is the $\mathbb{C}$-linear bundle constructed from $T(X)$ by means of J. Let $T^*(X)^{1,0}, T^*(X)^{0,1}$ denote the $\mathbb{C}$-dual bundles. Consider the exterior algebra bundles, we have $T^*(X)_c = T^*(X)^{1,0} \oplus T^*(X)^{0,1}$. We let $\wedge p, qT^*(X)$ be the bundle whose fibre is $\wedge p, qT_x^*(X)$. The sections are the complex-valued differential forms of type (p, q) on X, which we denote by

$$\varepsilon^{p,q}(X) = \varepsilon\left(X, \wedge^{p,q}T^*(X)\right)$$

Moreover, we have that

$$\mathcal{E}^r(X) = \sum_{p+q=r} \mathcal{E}^{p,q}(X)$$

Definition 22. *Let $E \to X$ be an S-bundle of rank r and let U be an open subset of X. A frame for E over U is a set of r S-sections $\{s_1, , s_r\}$, $s_j \in (U, E)$, such that $\{s_1(x), , s_r(x)\}$ is a basis for Ex for any $x \in U$.*

Any base space. Let U be a trivializing neighborhood for E so that $h : E|_U \to U \times K^r$, and thus we have an isomorphism

$$h_* : \mathcal{S}\left(U, E|_U\right) \to \mathcal{S}\left(U, U \times K^r\right)$$

Consider the vector-valued functions $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots$, $e_r = (0, \ldots, 0, 1)$, which clearly form a (constant) frame for $U \times K^n$, and thus $\left\{(h_*)^{-1}(e_1), \ldots, (h_*)^{-1}(e_r)\right\}$ forms a frame for $E|_U$, since the bundle mapping h is an isomorphism on fibres, carrying a basis to a basis. Therefore we see that having a frame is equivalent to having a trivialization and that the existence of a global frame (defined over X) is equivalent to the bundle being trivial. Let now (X, J) be an almost complex manifold as before and let $\{w_1, \ldots, w_n\}$ be a local frame (defined over some open set U) for $T^*(X)^{1,0}$. It follows that $\left\{\bar{W}_1, \ldots, \bar{W}_n\right\}$ is a local frame for $T^*(X)^{0,1}$. Then a local frame for $\wedge^{p,q} T^*(X)$ is given by

$$\left\{w^I \wedge \bar{w}^J\right\}, |I| = p, \quad |J| = q, \quad (I, J \text{ strictly increasing })$$

Therefore any section $s \in \mathcal{E}^{p,q}(X)$ can be written (in U) as

$$s = \sum_{|I|=p,|J|=q} a_{IJ} w^I \wedge \bar{w}^J, a_{IJ} \in \varepsilon^0(U)$$

Note that

$$ds = \sum_{|I|=p,|J|=q} da_{IJ} \wedge w^I \wedge \bar{w}^J + a_{IJ} d\left(w^I \wedge \bar{w}^J\right)$$

where the second term is not necessarily zero, since $w_i(x)$ is not necessarily a constant function of the local coordinates in the base space. We now have, based on the almost complex structure, a direct sum decomposition of $\varepsilon^r(X)$ into subspaces $\varepsilon^{p,q}(X)$. Let $\pi_{p,q}$ denote the natural projection operators

$$\pi_{p,q} : \mathcal{E}^r(X) \to \mathcal{E}^{p,q}(X), p + q = r$$

We have in general

$$d : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+q+1}(X) = \sum_{r+s=p+q+1} \mathcal{E}^{r,s}(X)$$

by restricting d to $\mathcal{E}^{p,q}$. We define

$$\begin{aligned} \partial &: \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X) \\ \bar{\partial} &: \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X) \end{aligned}$$

by setting

$$\begin{aligned} \partial &= \pi_{p+1,q} \circ d \\ \bar{\partial} &= \pi_{p,q+1} \circ d \end{aligned}$$

We then extend ∂ and $\bar{\partial}$ to all

$$\mathcal{E}^*(X) = \sum_{r=0}^{\dim X} \mathcal{E}^r(X)$$

by complex linearity.

Proposition 3. *For $f \in \varepsilon^*(X)$,we have*

$$Q\overline{\partial}(Qf) = \partial f$$

Proof.

$$Q\pi_{p,q}f = \pi_{q,p}Qf \textit{ and } Qdf = dQf$$

□

It follows from Proposition that $\bar{\partial}^2 = 0$ if and only if $\partial^2 = 0$. In general

$$d : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+q+1}(X)$$

can be decomposed as

$$d = \sum_{r+s=p+q+1} \pi_{r,s} \circ d = \partial + \bar{\partial} + \cdots$$

If, however, $d = \partial + \bar{\partial}$, then

$$d^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$$

and since each operator projects to a different summand of $\varepsilon^{p+q+2}(X)$ (in which case the operators are said to be of different type), we obtain

$$\partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = \bar{\partial}^2 = 0$$

If $d = \partial + \bar{\partial}$ then we say that the almost complex structure is integrable.

Theorem 5. *The induced almost complex structure on a complex manifold is integrable.*

Theorem 6 (Newlander-Nirenberg)**.** *Let (X, J) be an integrable almost complex manifold. Then there exists a unique complex structure $\mathcal{O}_X$ on X which induces the almost complex structure J.*

4 Homework

不要迷信讲义！不要迷信讲义！！不要迷信讲义！！！学会独立判断，善于和同学交流。当然也欢迎来指出错误。

(4.4.12)
由于$|f(z)|<|z|$在边界成立，由Rouché定理知$z-f(z)$与z在$B(0,1)$内解个数相同，即得证。
(4.4.13)
(i) 由习题1.1.5知$|z|=1$时$|f(z)|=1$，从而由Rouché定理知$f(z)-b$与$f(z)$在$B(0,1)$内零点个数相同，可验证$f(z)$零点恰为$a_1,\ldots,a_n$，均在$B(0,1)$中，从而得证。
(ii) 类似(i)由Rouché定理知$b-f(z)$与b在$B(0,1)$内零点个数相同，即$B(0,1)$内无零点，而边界上$|f(z)|=1$因此无零点，从而只需说明$f(z)$有n个零点。$f(z)-b$的分子为关于z的n次多项式$\prod_{k=1}^n(a_k-z)-b\prod_{k=1}^n(1-\overline{a_k}z)$，当后半部分为0时$|z|>1$，因此前半部分不为0，由此此多项式的根不可能使后半部分为0，也即分母不为0，因此均为整个分式的根，从而得证。
(4.4.14)
利用辐角原理知$\frac{1}{2\pi\mathrm{i}}\int_{|z|=R}\frac{f'(z)}{f(z)}\mathrm{d}z=N$，令$z=Re^{\mathrm{i}\theta}$可得$\frac{1}{2\pi}\int_0^{2\pi}z\frac{f'(z)}{f(z)}\mathrm{d}\theta=N$，取实部即可知实部最大值$\geq N$。
(4.4.17)
由定理4.4.6与连续性可知$f(D)=G$，于是对任何$f(z_0),z_0\in D$，有$f(z_0)\notin\Gamma$。$f(z)-f(z_0)$在D中根的个数为(不妨设两曲线定向相同)$\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{f'(z)}{f(z)-f(z_0)}\mathrm{d}z=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{w}{w-f(z_0)}\mathrm{d}w$，而后者即为$z=f(z_0)$在$G$中根的个数，因此为1，从而得证。
(4.5.12)
当f为常数时，直接估算知成立。
当f不为常数且$f(0)=0$时，由习题4.5.11知$|f(Rz)|\leq\frac{2A(R)|z|}{1-|z|}$，再由最大模原理知结论(由于$\operatorname{Re}f(z)$为调和函数，其最大值在边界取到)。
当$f(0)\neq 0$时，令$g(z)=f(z)-f(0)$，则$|f(z)|\leq|g(z)|+|f(0)|$，再利用上一种情况可知

$$M(r)\leq\frac{2r}{R-r}\max_{|z|=R}g(z)+|f(0)|\leq\frac{2r}{R-r}A(R)+\frac{2r}{R-r}|f(0)|+|f(0)|$$

化简得结论。
(4.5.13)
(i) 令$\varphi(z)=\frac{z-1}{z+1}$，其将右半平面映射到$B(0,1)$，且1映射到0，因此对$w=\varphi\circ f$利用Schwarz引理知$|w(z)|\leq|z|$，此时$f(z)=\frac{1+w(z)}{1-w(z)}$。
第一个不等号：计算知$\operatorname{Re}f(z)=\operatorname{Re}\frac{1+w(z)}{1-w(z)}=\frac{1-|w(z)|^2}{|1-w(z)|^2}\geq\frac{1-|w(z)|}{1+|w(z)|}\geq\frac{1-|z|}{1+|z|}$。
第二个不等号：由实部与模定义知结论。
第三个不等号：计算知$|f(z)|\leq\frac{1+|w(z)|}{1-|w(z)|}\leq\frac{1+|z|}{1-|z|}$。

(ii) 由z_0处等号成立可推出$|w(z_0)|=|z_0|$，从而$w(z)=\mathrm{e}^{\mathrm{i}\theta}z$，代入即得证。
(4.5.15)
由于$\overline{B(0,1)}$为紧集，若其中有无穷多零点则存在聚点，因此f恒为0，矛盾。由其有有限多零点，类似习题4.5.17右侧g$g(z)$，令$h(z)=\frac{f(z)}{g(z)}$，其在$|z|=1$时模为1，且$h(B(0,1))\subset B(0,1)\backslash\{0\}$，考虑$h$与$\frac{1}{h}$可知$|h(z)|=1$，由习题2.2.2可知$h(z)$只能为常数，由模为1设其为$\mathrm{e}^{\mathrm{i}\theta}$，则$f(z)=\mathrm{e}^{\mathrm{i}\theta}g(z)$。由$f(z)$为整函数，若有非零根，会导致$g(z)$在某处趋于无穷，矛盾，因此只能$f(z)=\mathrm{e}^{\mathrm{i}\theta}z^n$。
(4.5.17)
当f零点总重数为1时，设$f(z_1)=0$，利用定理4.5.6直接知结论，利用归纳法，下假设f零点总重数为$k-1$时结论成立。
当f零点总重数为k时，设$f(z_1)$为k_1重零点，可设$f(z)=(z-z_1)^{k_1}g(z)$，$g(z)$其他零点与$f(z)$相同，但z_1不为零点，考虑$h(z)=f(z)\frac{1-\overline{z_1}z}{z_1-z}=(z-z_1)^{k_1-1}g(z)(1-\overline{z_1}z)$，由于$1-\overline{z_1}z$在$B(0,1)$中无零点，$h(z)$只有$z_1$的零点重数比$f(z)$少一重，从而零点总重数为$k-1$。利用归纳假设后两侧同乘$\left|\frac{z_1-z}{1-\overline{z_1}z}\right|$即得证。
(4.5.20)
记$h(z)=\frac{f(z_1)-f(z)}{1-\overline{f(z_1)}f(z)}\frac{1-\overline{z_1}z}{z_1-z}\frac{1-\overline{z_2}z}{z_2-z}$，由$z_1,z_2$均为$f(z_1)-f(z)$零点可知$h(z)\in H(B(0,1))$。$|z|=1$时$|h(z)|=\left|\frac{f(z_1)-f(z)}{1-\overline{f(z_1)}f(z)}\right|$，由$f(B(0,1))\subset B(0,1)$知模不超过1，从而由最大模原理$h(B(0,1))\subset\overline{B(0,1)}$，由此$|h(0)|\leq 1$，代入得证。
(4.5.24)
记$w(z)=\frac{z-\mathrm{i}}{z+\mathrm{i}}$，其为上半平面到$B(0,1)$的全纯同构，由此构造$\varphi:\mathrm{Aut}(B(0,1))\to\mathrm{Aut}(\mathbb{C}^+)$，$\varphi(f)=w^{-1}\circ f\circ w$，可知$\varphi$为群同构，由此可知$\mathrm{Aut}(\mathbb{C}^+)$即为所有$w^{-1}\circ f\circ w$，其中$f\in\mathrm{Aut}(B(0,1))$。
(4.5.22)
利用Schwarz-Pick定理可知$\left|\frac{f(0)-f(z)}{1-\overline{f(0)}f(z)}\right|\leq|z|$，记$g(z)=\frac{f(0)-f(z)}{1-\overline{f(0)}f(z)}$，利用$g(z)$替换$f(z)$知要证的式子可化为$|f(0)|z|^2-g(z)|\leq|z||1-\overline{f(0)}g(z)|$，同平方后可进一步化为$(|z|^2-|g(z)|^2)(1-|z|^2|f(0)|^2)\geq 0$，从而成立。
(4.5.29)
通过平移可不妨设$z_0=0$，在闭包在D中的某邻域$B(0,r)$展开为Taylor级数$z+\sum_{n=2}^{\infty}a_nz^n$。考虑使得$a_n\neq 0$的大于1的最小的$n$，记其为$m$。记$f_k(z)$为$f(z)$迭代$k$次的函数，可发现$f_k(z)$可在邻域中展开为$z+Na_mz^m+\dots$。由$D$有界可设$f_k(z)$有上界$M$，考虑$\overline{B(0,r)}$上的积分可知$|Na_mr^m|=\left|\frac{1}{2\pi}\int_0^{2\pi}f_N(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}m\theta}\mathrm{d}\theta\right|$，由长大不等式知$|Na_mr^m|<M$对任何$N$成立，与$a_m\neq 0$矛盾。
(4.5.30)

记$g(z)=\tan\frac{\pi f(z)}{4}$，可发现$g(z)\in B(0,1)$且$g(0)=0$，从而$|g(z)|\le|z|$。
$|\tan w|=\left|\frac{\mathrm{e}^{\mathrm{i}w}-\mathrm{e}^{-\mathrm{i}w}}{\mathrm{e}^{\mathrm{i}w}+\mathrm{e}^{-\mathrm{i}w}}\right|=\left|\frac{\mathrm{e}^{2\mathrm{i}w}-1}{\mathrm{e}^{2\mathrm{i}w}+1}\right|$，由于$\frac{|\mathrm{e}^{2\mathrm{i}w}|-1}{|\mathrm{e}^{2\mathrm{i}w}|+1}\le\left|\frac{\mathrm{e}^{2\mathrm{i}w}-1}{\mathrm{e}^{2\mathrm{i}w}+1}\right|$，代入$w=\frac{\pi}{4}f(z)$后化简可得第二问的式子。
另一方面，利用$|\tan w|=\left|\frac{\mathrm{e}^{2\mathrm{i}w}-1}{\mathrm{e}^{2\mathrm{i}w}+1}\right|$可知$\tan|\mathrm{Re}\,w|\le|\tan w|$，代入化简可得第一问的式子。
(补充题)
记$f(z)=\frac{\sin z}{z^7-1}$，在$z=\mathrm{e}^{\frac{2k\pi\mathrm{i}}{7}}$时利用命题5.4.5知$\mathrm{Res}(f,z)=7\mathrm{e}^{\frac{12k\pi\mathrm{i}}{7}}\sin\left(\mathrm{e}^{\frac{2k\pi\mathrm{i}}{7}}\right)$，从而所求积分为$14\pi\mathrm{i}\sum_{k=0}^{6}\mathrm{e}^{\frac{12k\pi\mathrm{i}}{7}}\sin\left(\mathrm{e}^{\frac{2k\pi\mathrm{i}}{7}}\right)$。
(5.5.1)
(1) $f(z)=\frac{z^2+1}{z^4+1}$为偶函数，可直接考虑$(-\infty,\infty)$上积分的值，利用推论5.5.2可知其为$2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}})+2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}})=2\pi\mathrm{i}(\frac{1}{2\sqrt{2}\mathrm{i}}+\frac{1}{2\sqrt{2}\mathrm{i}})=\sqrt{2}\pi$，从而所求积分为其一半，即$\frac{\sqrt{2}}{2}\pi$。
(7) 被积函数为偶函数，因此可考虑实轴上积分。记$f(z)=\frac{z\mathrm{e}^{\mathrm{i}az}}{z^2+b^2}$，利用正实轴上方充分大半圆围道，其上积分值为$2\pi\mathrm{i}\,\mathrm{Res}(f,b\mathrm{i})=\pi\mathrm{e}^{-ab}$，而由Jordan引理可知半圆部分在无穷远处积分趋于0，从而此即为实轴上积分，由此所求结果为$\frac{\pi}{2}\mathrm{e}^{-ab}$。
(5.5.1)
*f表示题目中的被积函数
(14) 考虑$\mathrm{Im}\,z\in(0,2\pi),|\mathrm{Re}\,z|<t$的矩形区域边界，区域中只有$\pi\mathrm{i}$处不全纯，且$t\to\infty$时左右边界积分趋于0，而上边界积分为下边界的$-\mathrm{e}^{2\pi\mathrm{i}p}$倍，由此设积分结果为$I$可知$(1-\mathrm{e}^{2\pi\mathrm{i}p})I=2\pi\mathrm{i}\,\mathrm{Res}(f,\pi\mathrm{i})$，因此$I=\frac{2\pi\mathrm{i}}{1-\mathrm{e}^{2\pi\mathrm{i}p}}(-\mathrm{e}^{\pi\mathrm{i}p})=\frac{\pi}{\sin p\pi}$。
(15) 可发现$\mathrm{Res}(f,\mathrm{i})=\frac{(-\mathrm{i}-1)^p}{2},\mathrm{Res}(f,-\mathrm{i})=\frac{(\mathrm{i}-1)^p}{2}$，由定理5.5.14取$r=1-p,s=p$可知结论。
(17) 可发现$\mathrm{Res}(f,\mathrm{i})=\frac{\sqrt[4]{-4\mathrm{i}}}{2\mathrm{i}},\mathrm{Res}(f,-\mathrm{i})=-\frac{\sqrt[4]{4\mathrm{i}}}{2\mathrm{i}}$，由定理5.5.14取$r=\frac{3}{4},s=\frac{1}{4}$可知结论。
(21) 图示曲线上积分为0，而类似例5.5.12可知弧线上取极限积分为0，从而实轴积分与虚轴积分相等，取实部知所求积分为$\mathrm{Re}\left(\int_0^\infty\frac{\log x+\mathrm{i}\frac{\pi}{2}}{-x^2-1}\mathrm{d}(x\mathrm{i})\right)=\frac{\pi}{2}\int_0^\infty\frac{1}{x^2+1}\mathrm{d}x=\frac{\pi^2}{4}$。
(29) 类似例5.5.12知$z=1$处先绕开再逼近结果不改变，因此$\int_{|z|=1}\frac{\log(z-1)}{z}\mathrm{d}z=\log(z-1)\big|_{z=0}=\pi\mathrm{i}$，令$z=\mathrm{e}^{\mathrm{i}\theta}$后取实部可知$\int_0^{2\pi}\log|1-\mathrm{e}^{\mathrm{i}\theta}|\mathrm{d}\theta=0$，由对称性可知$\int_0^{\pi}\log|1-\mathrm{e}^{\mathrm{i}\theta}|\mathrm{d}\theta=0$，而$|1-\mathrm{e}^{\mathrm{i}\theta}|=2\sin\frac{\theta}{2}$，代入换元即可知结论。
(6.1.2)
不妨设$z_0\in B(a,r)$，由于亚纯性，可取关于边界对称的域$D'\subset D$使得

其在$B(a,r)$内除了z_0外不包含其他$f(z)=A$的点或极点。在其中记$g(z)=\frac{z-w_0}{z-z_0}(f(z)-A)$，可发现$g(D'\cap\partial B(a,r))\subset\partial B(0,R)$且在其中全纯，从而利用Schwarz对称原理可延拓。由去掉极点后连续性可知在域中零点有极限点的亚纯函数亦只能为0，在D'在$B(a,r)$外的部分仍有$g(z)=\frac{z-w_0}{z-z_0}(f(z)-A)$，由$g(w_0)$与$g(z_0)$关于$\partial B(0,R)$对称可知$g(w_0)$为非零实数，因此只能$w_0$为$f$的一阶极点。由于$f(z)=A+\frac{z-z_0}{z-w_0}g(z)$，$g(z)$在$D'$上全纯，可知$f'(z_0)=\frac{g(z_0)}{z_0-w_0}$，$\mathrm{Res}(f,w_0)=(w_0-z_0)g(w_0)$，又由$g(z_0)$与$g(w_0)$关于$\partial B(0,R)$对称可知结论。

(6.1.3)

若f不恒为0，可取关于$\partial B(0,r)$对称的D使得f在$D\cap B(0,R)\backslash\overline{B(0,r)}$上恒不为0，由此利用Schwarz对称原理可将$f$延拓至$D$上，但此时利用唯一性定理可知$f$恒为0，矛盾。

(6.1.4)

与习题6.1.3证明相同。

(6.2.3)

不妨设$z_0=1$，否则考虑级数$\sum_{n=0}^{\infty}a_n\frac{z^n}{z_0^n}$即可。

类似定理6.2.3证明可将幂级数延拓为$B(0,\delta),\delta>1$上的亚纯函数$f(z)$，可设其在1处的Laurent展开为$\frac{b}{z-1}+\sum_{n=0}^{\infty}b_n(z-1)^n$，记$g(z)=f(z)-\frac{b}{z-1}$，可发现其在$B(0,\delta)$全纯。而其在0处的展开为$\sum_{n=0}^{\infty}(a_n+b)z^n$，由收敛半径大于1考虑1处可知$\lim_{n\to\infty}a_n+b=0$，从而$\lim_{n\to\infty}a_n=-b$，因此两项之比极限为1。

(6.2.9)

类似习题6.2.3知存在$b_1,\dots,b_m$使$\sum_{n=0}^{\infty}a_nz^n+\sum_{k=1}^{m}\frac{b_k}{z_k-z}$收敛，展开后取$z=1$可知$\lim_{n\to\infty}a_n-\sum_{k=1}^{m}b_kz_k^{-n-1}=0$，从而$\lim_{n\to\infty}|a_n|\le\sum_{k=1}^{m}|b_k|$，由此可知有界。(6.2.10) 参见丁袭明讲义（已发在群里）

(7.1.3)

由Montel定理知f_n有内闭一致收敛子列，设其收敛至f，记$g_n=f_n-f$，则$\lim_{n\to\infty}g_n(z_k)=0,\forall k$。在任何紧集$K$上，若$g_n$不一致收敛于0，由于其仍为正规族，存在一致收敛且收敛结果不为0的子列，假设收敛到h，由$h(z_k)=0,\forall k$即与唯一性定理矛盾，从而得证。

(7.1.4)

类似习题4.1.12，对D中任何紧集K，可扩张至紧集K'使得其包含z_0且其中任意两点存在长度不超过M的道路。取r使得K'中每点z作$\overline{B(z,r)}$取并后仍在D中，利用习题3.4.9可知$f'(z)=\frac{1}{\pi r}\int_0^{2\pi}\mathrm{Re}(z+re^{\mathrm{i}\theta})e^{-\mathrm{i}\theta}\mathrm{d}\theta$，取模可得$|f'(z)|\le\frac{2}{r}\mathrm{Re}\,f(z)\le\frac{2}{r}|f(z)|$。从而利用微分方程得$K'$中任何$f(z)$的模不超过$|f(z_0)|e^{2M/r}$，因此内闭一致有界，由Montel定理知为正规族。

第二条不成立的反例为$f_n(z)=n$。
(7.1.6)
由D有界可知取$M_0=\frac{M+m(D)}{2}$即有D上$|f(z)|\le\frac{|f(z)|^2+1}{2}$的积分不超过$M_0$。
对D中任何紧集K，类似习题4.1.12可取r使得K中每点z作$\overline{B(z,r)}$取并后仍在D中，利用平均值原理可知

$$|f(z)|=\frac{1}{\pi r^2}\left|\iint_{B(z,r)}f(w)\mathrm{d}x\mathrm{d}y\right|\le\frac{1}{\pi r^2}\iint_{B(z,r)}|f(w)|\mathrm{d}x\mathrm{d}y\le\frac{M_0}{\pi r^2}$$

从而内闭一致有界，由Montel定理知为正规族。
(7.2.1)
记φ将D双全纯映射至$B(0,1)$，则$\varphi\circ f$为有界整函数，从而为常值，由φ为单射知f为常值。
(7.2.2)
由平移不妨设$a=0$，记题中不等式左右分别为r,R。
考虑$\varphi:B(0,1)\to D,\varphi(z)=rz$，可发现$f\circ\varphi$为保持原点的$B(0,1)\to B(0,1)$映射，利用Schwarz引理可知$(f\circ\varphi)'(0)\le 1$，即$rf'(a)\le 1$，从而不等式左半边得证。
考虑$\psi:D\to B(0,1),\psi(z)=\frac{z}{R}$，可发现$\psi\circ f^{-1}$为保持原点的$B(0,1)\to B(0,1)$映射，利用Schwarz引理可知$(\psi\circ f^{-1})'(0)\le 1$，即$\frac{(f^{-1})'(0)}{R}\le 1$，由$(f^{-1})'(0)=\frac{1}{f'(0)}$可知得不等式右半边。
(7.2.3)
记$\varphi(z)=\frac{z-f(p)}{1-\overline{f(p)}z}$，考虑$\varphi\circ f\circ g^{-1}$，可发现其为0映射到0的$B(0,1)$自同构，从而其为$\mathrm{e}^{\mathrm{i}\theta}z$，从而代换$z$为$g(z)$可知$\varphi(f(z))=\mathrm{e}^{\mathrm{i}\theta}g(z)$。取$z=a$后两边求导得$f'(a)|f(p)|^2=\mathrm{e}^{\mathrm{i}\theta}g'(a)$，由$f'(a)>0$可知$\mathrm{e}^{\mathrm{i}\theta}$与$g'(a)$方向相反，从而$g(z)=\mathrm{e}^{-\mathrm{i}\theta}\varphi(f(z))=\frac{g'(a)}{|g'(a)|}\varphi(f(z))$，即为欲证的式子。

(Stein P127)
1.
Notice that

$$\begin{aligned} A(\xi) - B(\xi) &= \left(\int_{-\infty}^{t} + \int_{t}^{\infty} \right) f(x) \mathrm{e}^{-2\pi i \xi (x-t)} \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi i \xi (x-t)} \, \mathrm{d}x \\ &= \mathrm{e}^{2\pi i \xi t} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi i \xi x} \, \mathrm{d}x \\ &= \mathrm{e}^{2\pi i \xi t} \hat{f}(\xi) = 0. \end{aligned}$$

Then $A(\xi) = B(\xi), \quad \forall \epsilon \in \mathbb{R}$. Let

$$F(z) = \begin{cases} A(z), & \mathrm{Im}(z) \geqslant 0, \\ B(z), & \mathrm{Im}(z) < 0. \end{cases}$$

Then by (a) we have F continuous on $\left\{ \mathrm{Im}(z) = 0 \right\}$. Notice that for $\mathrm{Im}(z) = b > 0$,

$$\begin{aligned} \int_{-\infty}^{t} |\frac{\partial}{\partial z} f(x) \mathrm{e}^{-2\pi i z (x-t)}| \, \mathrm{d}x &= \int_{-\infty}^{t} |f(x) \cdot \left(-2\pi i (x-t) \mathrm{e}^{-2\pi i z (x-t)} \right)| \, \mathrm{d}x \\ &= 2\pi \int_{-\infty}^{t} |f(x)| \cdot (t-x) \mathrm{e}^{-2\pi b (t-x)} \, \mathrm{d}x \\ (y = t - x) \qquad &\leqslant 2\pi \int_{0}^{+\infty} \frac{A}{1 + (y-t)^2} \cdot y \mathrm{e}^{-2\pi b y} \, \mathrm{d}y \quad \text{which is bounded.} \end{aligned}$$

Also,

$$\int_{-\infty}^{t} |\frac{\partial}{\partial \overline{z}} f(x) \mathrm{e}^{-2\pi i z (x-t)}| \, \mathrm{d}x = \int_{-\infty}^{t} 0 \, \mathrm{d}x = 0.$$

Then by dominated convergence theorem, $F'(z)$ exists while $\partial F(z)/\partial \overline{z} = 0$ when $\mathrm{Im}(z) > 0$, which yields that F is holomorphic in the upper half-plane. The same is true for the lower half-plane similarly. With the continuity on $\left\{ \mathrm{Im}(z) = 0 \right\}$ and Morera theorem, we have F entire.

Notice that for $\mathrm{Im}(z) = b > 0$,

$$\begin{aligned}
|F(z)| &\leqslant \int_{-\infty}^{t} |f(x)| \cdot |e^{2\pi i z(t-x)}\,\mathrm{d}x| \\
&= \int_{-\infty}^{t} |f(x)| e^{-2\pi b(t-x)}\,\mathrm{d}x \\
&\leqslant \int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x \quad \text{which is bounded.}
\end{aligned}$$

The same is true for the lower half-plane similarly. With the continuity on $\left\{\mathrm{Im}(z) = 0\right\}$, we have F bounded in $\mathbb{C}$.
By Liouville theorem, F is constant. Let $z = ib$, $b \to +\infty$, by dominated convergence theorem, we have $F \equiv 0$.
By (b) we have $F(0) = 0$, thus

$$\int_{-\infty}^{t} f(x)\,\mathrm{d}x = 0.$$

Notice the equation above holds for all $t \in \mathbb{R}$. With the continuity of f, we have $f \equiv 0$.
3.
To prove

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{a}{a^2 + x^2} \mathrm{e}^{-2\pi i x\xi}\,\mathrm{d}x = \mathrm{e}^{-2\pi a|\xi|},$$

let

$$f(z) = \frac{a}{a^2 + z^2} \mathrm{e}^{-2\pi i z\xi}.$$

For $\xi \leqslant 0$, we choose the contour as
By residue theorem we have

$$\int_{\gamma \sqcup \gamma_R} f(z)\,\mathrm{d}z = 2\pi i \cdot \mathrm{Res}(f, ai) = \pi \mathrm{e}^{2\pi \xi a}.$$

Let $R \to \infty$, we have

$$\int_{\gamma} f(z)\,\mathrm{d}z \to \int_{-\infty}^{+\infty} \frac{a}{a^2+x^2}\mathrm{e}^{-2\pi i x\xi}\,\mathrm{d}x.$$

$$\begin{aligned} |\int_{\gamma_R} f(z)\,\mathrm{d}z| &= |\int_0^{\pi} \frac{a\mathrm{e}^{-2\pi i\xi R\mathrm{e}^{i\theta}}}{a^2+R^2\mathrm{e}^{2i\theta}} \cdot Ri\mathrm{e}^{i\theta}\,\mathrm{d}\theta| \\ &\leqslant \int_0^{\pi} \frac{a\mathrm{e}^{2\pi\xi R\sin\theta}}{R^2-a^2} \cdot R\,\mathrm{d}\theta \\ &\leqslant \int_0^{\pi} \frac{a}{R^2-a^2} \cdot R\,\mathrm{d}\theta \to 0. \end{aligned}$$

Then

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{a}{a^2+x^2}\mathrm{e}^{-2\pi i x\xi}\,\mathrm{d}x = \mathrm{e}^{2\pi a\xi}, \quad \forall \xi \leqslant 0.$$

For $\xi > 0$, choose the contour as the lower semi-circle instead(be aware of the direction of the path), we have

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{a}{a^2+x^2}\mathrm{e}^{-2\pi i x\xi}\,\mathrm{d}x = -2\pi i \cdot \mathrm{Res}(f, -ai) = \mathrm{e}^{-2\pi a\xi}, \quad \forall \xi > 0.$$

Then it holds that

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{a}{a^2+x^2}\mathrm{e}^{-2\pi i x\xi}\,\mathrm{d}x = \mathrm{e}^{-2\pi a|\xi|}.$$

Notice that $\frac{a}{a^2+z^2}\mathrm{e}^{-2\pi i z\xi}$ is holomorphic in $\left\{|\mathrm{Im}(z)| < b < a\right\}$, and it suffices that

$$|\frac{a}{a^2+z^2}\mathrm{e}^{-2\pi i z\xi}| \leqslant \frac{A}{1+x^2}, \quad \text{for} \quad \begin{cases} \xi \leqslant 0 \\ 0 \leqslant \mathrm{Im}(z) \leqslant b \end{cases} \quad \text{or} \quad \begin{cases} \xi > 0 \\ -b \leqslant \mathrm{Im}(z) < 0 \end{cases}.$$

Then by Fourier inverse transforms, we have

$$\int_{-\infty}^{+\infty} e^{-2\pi a|\xi|}\mathrm{e}^{2\pi i\xi x}\,\mathrm{d}x = \frac{1}{\pi} \cdot \frac{a}{a^2+x^2}.$$

6. Let

$$f(z) = \frac{1}{\pi}\frac{a}{a^2+z^2}.$$

It's obvious that $f \in \mathcal{F}_{\frac{a}{2}}$.
By Exercise 3 we have

$$\hat{f}(\xi) = \mathrm{e}^{-2\pi a|\xi|}.$$

By Possion summation formula we have

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \hat{f}(n).$$

Thus

$$\begin{aligned}
\frac{1}{\pi}\sum_{n=-\infty}^{+\infty} \frac{a}{a^2+n^2} &= \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-2\pi a|n|} \\
&= 2\sum_{n=0}^{+\infty} \mathrm{e}^{-2\pi an} - 1 \\
&= \frac{2}{1-\mathrm{e}^{-2\pi a}} - 1 \\
&= \coth \pi a.
\end{aligned}$$

8.
By inversion formula, we have

$$f(x) = \int_{-M}^{M} \hat{f}(\xi)\mathrm{e}^{-2\pi i x\xi}\,\mathrm{d}\xi = \int_{-M}^{M} \hat{f}(\xi)\left(\sum_{n=0}^{\infty} \frac{(-2\pi i)^n \cdot x^n \cdot \xi^n}{n!}\right)\mathrm{d}\xi = \sum_{n=0}^{\infty}\left(x^n \cdot \int_{-M}^{M} \hat{f}(\xi)\frac{(-2\pi i)^n \cdot \xi}{n!}\right.$$

By the uniqueness of Taylor expansion, we have

$$a_n = \int_{-M}^{M} \hat{f}(\xi)\frac{(-2\pi i)^n \cdot \xi^n}{n!}\,\mathrm{d}\xi = \frac{(-2\pi i)^n}{n!}\int_{-M}^{M} \hat{f}(\xi)\cdot \xi^n\,\mathrm{d}\xi.$$

Then

$$\begin{aligned}
|(n!a_n)^{\frac{1}{n}}| &= 2\pi\left|\int_{-M}^{M} \hat{f}(\xi)\cdot\xi^n\,\mathrm{d}\xi\right|^{\frac{1}{n}} \\
&\leqslant 2\pi M\left(\int_{-M}^{M} |\hat{f}(\xi)\,\mathrm{d}\xi|\right)^{\frac{1}{n}} \to 2\pi M.
\end{aligned}$$

In the converse direction, by Cauchy-Hadamard theorem and Stirling theorem, the convergence radius of $\sum_{n=0}^{\infty} a_n z^n$ is

$$\frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}} \geqslant \lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{2\pi M} = \lim_{n\to\infty} \frac{(2\pi n)^{\frac{1}{2n}} \cdot n}{2\pi M e} = \infty.$$

Then f is holomorphic in $\mathbb{C}$.
$\forall \epsilon > 0$, $\exists N$ subject to

$$|a_n| \leqslant \frac{(2\pi(M+\epsilon))^n}{n!}, \quad \forall n > N.$$

Then

$$\begin{aligned}
|f(z)| &\leqslant \sum_{n=0}^{N} |a_n z^n| + \sum_{n=N+1}^{\infty} \frac{(2\pi(M+\epsilon))^n}{n!}|z|^n \\
&= \mathrm{e}^{2\pi(M+\epsilon)|z|} + \sum_{n=0}^{N} \left(|a_n| - \frac{(2\pi(M+\epsilon))^n}{n!} \right) |z|^n \\
&\leqslant A_\epsilon \cdot \mathrm{e}^{2\pi(M+\epsilon)|z|}.
\end{aligned}$$

10.
Let $\zeta = \xi + \eta i$, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\frac{\partial}{\partial \zeta} f(x) \mathrm{e}^{-2\pi i x \zeta} \,\mathrm{d}x| \leqslant \int_{-\infty}^{+\infty} & 2\pi |x| c \mathrm{e}^{-ax^2} \mathrm{e}^{2\pi \eta x} \,\mathrm{d}x \\
&= 2\pi c \mathrm{e}^{\frac{\pi^2 \eta^2}{a}} \cdot \int_{-\infty}^{+\infty} |x| \mathrm{e}^{-a(x - \frac{\pi\eta}{a})^2} \,\mathrm{d}x \quad \text{finite.}
\end{aligned}$$

And

$$\int_{-\infty}^{+\infty} |\frac{\partial}{\partial \overline{\zeta}} f(x) \mathrm{e}^{-2\pi i x \zeta} \,\mathrm{d}x| = \int_{-\infty}^{+\infty} 0 \,\mathrm{d}x = 0.$$

By dominated convergence theorem, we have $\hat{f}(\zeta)$ derivable with respect to ζ in $\mathbb{C}$, while

$$\frac{\partial}{\partial \overline{\zeta}} \hat{f} = 0.$$

Then $\hat{f}(\zeta)$ is entire.

Notice that

$$
\begin{aligned}
|\hat{f}(\xi+\eta i)| &= |\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-2\pi i x(\xi+\eta i)}\,\mathrm{d}x| \\
\text{(rectangle contour)} &= |\int_{-\infty}^{+\infty} f(x-\frac{\pi}{b}\xi i)\mathrm{e}^{-2\pi i(x-\frac{\pi}{b}\xi i)(\xi+\eta i)}\,\mathrm{d}x| \\
&\leqslant \int_{-\infty}^{+\infty} |f(x-\frac{\pi}{b}\xi i)\mathrm{e}^{-2\pi i(x-\frac{\pi}{b}\xi i)(\xi+\eta i)}\,\mathrm{d}x| \\
&\leqslant \int_{-\infty}^{+\infty} c\mathrm{e}^{-ax^2+\frac{\pi^2\xi^2}{b}}\mathrm{e}^{2\pi(x\eta-\frac{\pi}{b}\xi^2)}\,\mathrm{d}x \\
&= c\mathrm{e}^{-\frac{\pi^2\xi^2}{b}+\frac{\pi^2\eta^2}{a}} \cdot \int_{-\infty}^{+\infty} \mathrm{e}^{-a(x-\frac{\pi\eta}{a})^2}\,\mathrm{d}x \\
&= c\mathrm{e}^{-\frac{\pi^2\xi^2}{b}+\frac{\pi^2\eta^2}{a}} \cdot \sqrt{\frac{\pi}{a}} \\
&= c'\mathrm{e}^{-a\xi^2+b'\eta^2}.
\end{aligned}
$$

5 Advertisement Moment

I strongly reccomend you considering about learning Geometry, which is really intriguing and fanscinating further!!
And if you are interested in Geometric Analysis, welcome to talk with me about this. And I suppose Prof. Li, who is really a master and an excellent expert in this field, would also be glad if you contact him to learn about Geometric Analysis!!
In addition, I am going to set up a reading seminar to learn about some Ricci flow theory. Anyone who is interested are welcomed to participate. And I am willing to offer help if anyone are determined to learn some basic Differential Manifold or Differential Geometry and want some guidance.
Anyway, feel free to chat with me for further learning plan.

References

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6 Special Thanks

I would like to give special thanks to my doommates and friends who have offered a hand for some part of the latex typing... It's really a tough work for me [humble]
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