1.2.2 由定义.

1.3.1. 见答案. $P(A)+P(B)=P(A\cup B)+P(A\cap B)$

1.3.4. 见答案 . 归纳.

1.3.5. $P(\bigcap_{r=1}^{+\infty} A_r) = \lim_{n\to+\infty} P(\bigcap_{r=1}^{n} A_r)$. 根据概率连续性.

$\forall\ n \in N^*$, $P(\bigcap_{r=1}^{n} A_r) = 1 - P((\bigcap_{r=1}^{n} A_r)^c) = 1 - P(\bigcup_{r=1}^{n} A_r^c)$

$\geq 1 - \sum_{r=1}^{n} P(A_r^c) = 1$.

$\therefore P(\bigcap_{r=1}^{+\infty} A_r) = 1$.

.4.7 记红色为R, 品红为M C_i 为第i次的颜色 $i=1,2$.

则 $P(C_1=M) = \sum_{r=1}^{n} P(C_1=M \mid r\text{th urn})\, P(r\text{th urn})$. 由全概率公式.

$= \sum_{r=1}^{n} \frac{n-r}{n-1} \times \frac{1}{n} = \frac{1}{n(n-1)} \sum_{r=1}^{n} (n-r) = \frac{1}{2}$. each urn contains exactly $n-1$ balls.

$P(C_2=M) = \sum_{r=1}^{n} P(C_2=M \mid r\text{-th urn})\, P(r\text{-th urn})$.

$= \sum_{r=1}^{n} \left(\frac{r-1}{n-1} \times \frac{n-r}{n-2} + \frac{n-r}{n-1} \cdot \frac{n-r-1}{n-2}\right) \times \frac{1}{n}$. 由概率链式.

$= \frac{1}{n(n-1)(n-2)} \sum_{r=1}^{n} \left[(r-1)(n-r) + (n-r)(n-r-1)\right]$

$= \frac{1}{n(n-1)(n-2)} \sum_{r=1}^{n} (n-r)(n-2) = \frac{1}{2}$.

(a). $P(C_2=M) = \frac{1}{2}$.

(b). $P(C_2=M \mid C_1=M) = \frac{P(C_2=M, C_1=M)}{P(C_1=M)} = 2\sum_{r=1}^{n} P(C_2=M, C_1=M \mid r\text{-th urn})\, P(r\text{th urn})$

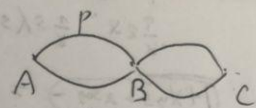
$\sum_{r=1}^{n} \frac{(n-r)(n-r-1)}{(n-1)(n-2)} = \frac{2}{3}$. by $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$.

. 由独立性的定义, 见答案.

. 见答案

见答案.

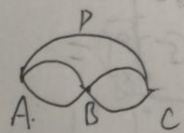
1.7.1.



(a) $P(A\leftrightarrow B \mid A\leftrightarrow C) = \dfrac{P(A\leftrightarrow B \,\&\, A\leftrightarrow C)}{P(A\leftrightarrow C)}$

$P(A\leftrightarrow C)=(1-p^2)^2$,

$= \dfrac{[1-p^2]\, p^2}{1-(1-p^2)^2}$.



(b) $P(A\leftrightarrow B \mid A\leftrightarrow C) = \dfrac{P(A\leftrightarrow B \,\&\, A\leftrightarrow C)}{P(A\leftrightarrow C)}$

$= \dfrac{p[1-p^2]p^2}{p[1-(1-p^2)^2]} = \dfrac{(1-p^2)p^2}{1-(1-p^2)^2}$.

1.7.3. 设 $P_k = P$(point starting at k finally absorbed).

以第一步为条件. $0<k<N$. then.

$P_k=\frac{1}{2}(P_{k-1}+P_{k+1})$. 边界: $P_0=P_N=1$. 等差数列.

$\Rightarrow P_k=1$. $k=0,1,\cdots,N$.

1.8.20. 显然 $P_0=1$, ~~根据第n-1投为条件~~

~~$P(A_n)=pP(A_n \mid n-1投为H)+(1-p)P(A_n \mid n-1投为T)$~~

根据第一投为条件.

$P(A_n)=pP(A_n \mid 1投为H)+P(A_n \mid 1投为T)\cdot(1-p)$

$=p\underbrace{[1-P(A_{n-1})]}_{n-1投:奇}+(1-p)\underbrace{P(A_{n-1})}_{n-1投:偶}$. 对应的剩下n-1投.

i.e. $P_n=p(1-P_{n-1})+(1-p)P_{n-1}$.

剩下见答案.

补充: 设 $H'_甲$ 为前n投 H数. 以甲的第n+1投为条件.

$P(H_甲>H_乙)=P(H_甲>H_乙 \mid n+1投为T)\frac{1}{2}+P(H_甲>H_乙 \mid n+1投为H)\frac{1}{2}$

$=\frac{1}{2}\underbrace{P(H'_甲>H_乙)}_{\substack{\shortparallel \\ P(H'_甲<H_乙) \\ 对称性}}+\frac{1}{2}P(H'_甲\ge H_乙)=\frac{1}{2}$.

$X \sim U(0,1)$，F 严格增分布函数，则 $Y = F^{-1}(X)$ 有分布函数 F.

$\therefore \quad P(Y \le x) = P(F^{-1}(X) \le x) = P(X \le F(x)) = F(x)$

更一般、对分布函数 $F(x)$，定义

$$F^{-1}(y) = \sup\{x \mid F(x) < y\}. \quad 0<y<1$$

则 $F^{-1}(y)$ 单调增. 一个基本事实：

Lemma: $X \sim U(0,1)$ 则 $Y = F^{-1}(X)$ 的分布函数为 $F(y)$.

Pf: $P(Y \le y) = \underline{P(F^{-1}(X) \le y) \overset{?}{=} P(X \le F(y))} = F(y)$

$F^{-1}(y) \le x \iff y \le F(x)$. Fact.

Suffice to show: $F^{-1}(y) > x \iff y > F(x)$.

$(\Leftarrow)$ $y > F(x)$ 由 F 右连续性知. $\exists\ \delta > 0$. s.t.

$y > F(x+\delta) \Rightarrow x+\delta \le F^{-1}(y) \Rightarrow x < F^{-1}(y)$

$(\Rightarrow)$: Let $x < F^{-1}(y)$ 由定义，$\exists\ x_*$ s.t.

$x < x_*$ 且 $x_* \in \{x \mid F(x) < y\}$.

从而 $F(x) \le F(x_*) < y$.

$\square$

此结论表明. 均匀分布可以生成其它分布，在随机模拟中.

相当重要！

矩阵积分基础，

有两个基本点，

1°. 积分元. 2°. 积分换元时的伸缩因子.（Jacobians）

Chp 1. Real case.

$X = [x_{ij}]_{\substack{1\le i\le m,\\ 1\le j\le n}} \in M_{m\times n}(\mathbb{R})$, without any symmetry, that is.

X has mn independent real entries x_{ij}'s.

matrix of differentials of X, named dX,

$dX = [dx_{ij}]_{\substack{1\le i\le m,\\ 1\le j\le n}}$, then.

volume element : $[dX] := \prod_{i=1}^{m}\prod_{j=1}^{n} dx_{ij}$.

For a real square symmetric matrix X, $m=n$, $X = X^t$.

$[dX] := \cancel{\prod_{i\le j} dx_{ij}}\ \prod_{i\ge j} dx_{ij}$. principle: independent elements or freedom

basics of elementary calculus:

$Y = AX$, $X, Y \in \mathbb{R}^n$, then $[dY] = \det(A)[dX]$,

△ $X, Y \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$ invertible, If $Y = AXB$.

then $[dY] = (\det A)^n (\det B)^m [dX]$

△ $\underset{\text{变量}}{X}$. 下三角 $\underset{\text{常量}}{A = [a_{ij}]}$. $B = [b_{ij}]$ 下三角 with $a_{jj} > 0$, $b_{jj} > 0$ $j = 1, \dots, n$.

then $Y = AX + X^t A^t \Rightarrow [dY] = 2^n \left(\prod_{j=1}^{n} a_{jj}^{j}\right)[dX]$.

$Y = XB + B^t X^t \Rightarrow [dY] = 2^n \left(\prod_{j=1}^{n} b_{jj}^{n-j+1}\right)[dX]$

X, Y $n\times n$ symmetric matrices. 变量. $A \in \mathbb{R}^{n\times n}$ invertible. 常量

$Y = AXA^t$, then $[dY] = (\det A)^{n+1}[dX]$. Hint: Write A as products of elementary matrices.

Δ $X, Y \in R^{n\times n}$ skew symmetric. 变 $A \in R^{n\times n}$ invertible 常

$$Y = AXA^t, \quad \Rightarrow [dY] = (\det A)^{n-1}[dX].$$

Freedom of a skew symmetric matrices is $\frac{n(n-1)}{2}$.

Δ $X \in R^{n\times n}$ symmetric positive definite. $(X > 0)$.

$T = [t_{ij}]$. 下三角. s.t. $t_{jj} > 0$, $j = 1, \dots, n$. then:

$$X = T^tT \Rightarrow [dX] = 2^n \left(\prod_{j=1}^{n} t_{jj}^{j}\right)[dT].$$

$$X = TT^t \Rightarrow [dX] = 2^n \left(\prod_{j=1}^{n} t_{jj}^{n-j+1}\right)[dT].$$

Hint: $X = TT^t \Rightarrow dX = dT \cdot T^t + T \cdot dT^t$. use previous result.

e.g. Let $Re(\alpha) > \frac{n-1}{2}$, then we can compute that:

$$\Gamma_n(\alpha) := \int_{\substack{X \in R^{n\times n} \\ X > 0}} [dX](\det X)^{\alpha - \frac{n+1}{2}} e^{-Tr(X)}$$

$$= \pi^{\frac{n(n-1)}{4}} \Gamma(\alpha)\Gamma(\alpha - \tfrac{1}{2}) \cdots \Gamma(\alpha - \tfrac{n-1}{2}).$$

Hint: $A_{m\times m} > 0$. $\exists$ unique $m\times m$ 下三角 T, 对角元为正 s.t.

$A = TT^t$.

$$\det(B)^{-\alpha} = \frac{1}{\Gamma_n(\alpha)} \int_{X>0} [dX] \det(X)^{\alpha - \frac{n+1}{2}} e^{-Tr(BX)}.$$

$$Re(\alpha) > \frac{n-1}{2}$$

$$B = B^t > 0.$$

概率论习题课

1. 习题

2.1.(2, 3, 4, 5(b)(d)) 2.7.9

2.3(3.5) 2.4.2 2.5(2,5).

3.2(3)(5) 补充 骰子 + ~~硬~~币

3.3.3(1). 3.3.3

~~概率方法~~

一些要点:

- 什么是随机变量?
- 概率空间上的 $X:\Omega\to\mathbb{R}$ 的可测函数.
- 什么是分布函数?

~~P和X诱导的~~R.

$P(X(\omega)\le x)=\mu\{\omega \mid X(\omega)\le x\}$ 这一次测度空间的测度.

P和X诱导了一个R上的测度.

- Real analysis fact:

$\cancel{\int f(X(\omega))P(d\omega)=\int}$

$$\int f(X(\omega))\mu(d\omega)=\int f(x)P(dx)=E(f(X)).\ \text{or}\ =\int f(x)\,dP(x)$$

这使得我们~~什么是~~可以将~~X看成~~单个随机变量概率空间看成是R.(更便于思考)

n个随机变量便可视为是 $\mathbb{R}^n$.

- 什么是连续分布函数?

$p(dx)=f(x)dx$, 即 $P(X\le x)$ 可导, 导数为 f.

一种便于理解, 计算和常用的类型.

- 一些常见误区及解释:

~~概率论~~ 随机变量一定是现实某物的抽象模型!

- 概率论实际上研究的是一类先验的事物,我们预先知道某一随机变量的分布函数再去探究这种分布函数具有的性质,不过大数定理让我们能够从多次实验得到对应随机变量的分布函数.

• 一种看待独立随机变量的方式:

$X_1, X_2, \cdots, X_n$ 是一列独立随机变量, μ_i 是其在 $\mathbb{R}$ 上对应的概率测度

n 有限之时. $(X_1, \cdots, X_n)$ 可以看成是 $\mathbb{R}^n$ 上 $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n$ 构成的乘积概率测度, ~~赋~~对应的一个取值在 $\mathbb{R}^n$ 上的随机变量. $X: \Omega \to \mathbb{R}^n$ (我们可能只定义过 $\Omega \to \mathbb{R}$ 的情况, 但这应当可以理解!)

这里是 $\mathbb{R}^n$

这里 X_i 为 X 第 i 个分量. 易见在这种乘积测度下 $\{X_i\}$ 是独立的.

而当 n 为变为无穷时, 我们可以视其为 $\prod_{i \in I} \mathbb{R}$ 上的乘积测度 (由前面有限的个乘积测度生成), ~~这时任取之个出~~

• $\mathbb{P}(X \le x)$ 与 $\lim_{y \to x^-} \mathbb{P}(X \le y)$ 的异同.

相差一个 $\mathbb{P}(X = x)$. 作业中很多同学没注意这个.

• 样本空间:

我们考虑的随机变量都是在同一个概率空间中考虑具体构造可以其从上面得到. 如果不在同一个概率空间, 我们几乎没什么东西可干.

• ~~由~~期望, 条件期望与条件概率.

以我们先前提到的乘积测度空间来看

$\mathbb{E}(f(X,Y)|X)$ 就是 $\int f(x,y)\,dy$, 为 x 的一个可测函数

而条件概率则为 ~~丑~~ $\mathbb{P}(Y \le x | X) = \mathbb{E}(\mathbb{I}_{\{Y \le x\}} | X)$

$= \int \mathbb{I}_{\{Y \le x\}}\, dY$ (X 和 Y 可能有关).

• 一些具体例子.

$\{B_n\}$ B_n i.i.d ~~伯~~伯努利分布. $\overset{\text{同构}}{\sim} L[0,1]$

几个概率方法例题

1. splitting graphs.

$G=(V,E)$ 有n个顶点e个边，则G存在一个二划分有至少 $\frac{e}{2}$ 条边.

2. Ramsey 问题.

若 $\binom{n}{k}\cdot 2^{1-\binom{k}{2}}<1$，则 $R(k,k)>n$. 从而 $R(k,k)>[2^{\frac{k}{2}}]$

对 $k\geq 3$.

3. 组合数论.

(Erdös 1965). 所有集合 $B=\{b_1,\cdots,b_n\}$. $b_i\neq 0$. $\exists$一个 sum-free 子集 A

使 $|A|>\frac{1}{3}n$. sum-free 指 $A+A\cap A=\phi$.

proof 取 $p=3k+2$ 为一素数. $p>2\max|b_i|$.

则 $C=\{k+1,\cdots,2k+1\}$ sum-free.

而 $\mathbb{E}\, xC\cap B=\frac{k+1}{3k+1}>\frac{1}{3}$.

4. Balancing vectors.

(a) $v_1,\cdots v_n\in\mathbb{R}^n$, $|v_i|=1$, $\varepsilon_i=\pm 1$, 则 $\exists\ \varepsilon_i$ 使

$$|\sum \varepsilon_i v_i|\geq\sqrt{n}$$

也 $\exists\ \varepsilon_i$.

$$|\sum\varepsilon_i v_i|\leq\sqrt{n}.$$

(b). $v_1,v_2,\cdots v_n\in\mathbb{R}^n$, $|v_i|\leq 1$, $p_1,\cdots p_n\in[0,1]$,

$w=\sum p_i v_i$, 则存在 $\varepsilon_i\in\{0,1\}$ 使.

$v=\varepsilon_1 v_1+\cdots+\varepsilon_n v_n$. $|w-v|\leq\frac{\sqrt{n}}{2}$

5. UnBalanceing light.

设 $a_{ij}=\pm 1$. $1\le i,j\le n$. 则 $\exists\ x_i, y_j=\pm 1$. 使得

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i y_j \ge \left(\sqrt{\frac{2}{\pi}}+o(1)\right)n^{\frac{3}{2}}.$$

（要用中心极限定理）.

($S_i=\sum_j a_{ij}y_j$, 则 $\mathbb{E}(|S_j|)\approx\mathbb{E}(|N(0,1)|\sqrt{n}) = \sqrt{n}\sqrt{\frac{2}{\pi}}$).

6. $\nu(n)$ 渐近分布.

令 $\nu(n)$ 为整除 n 的素因子个数. $\omega(n)\to\infty$, 则

$\{1,\dots,n\}$ 中使得 $|\nu(x)-\ln\ln n|>\omega(n)\sqrt{\ln\ln n}$ 的 x 是 $o(n)$ 的.

（事实上是说 $\mathbb{P}(|\nu(x)-\ln\ln n|>\omega(n)\sqrt{\ln\ln n})\to 0$ ）.

Erdös-Kac.

$$\frac{\nu(x)-\ln\ln n}{\sqrt{\ln\ln n}}\longrightarrow N(0,1).$$

ref: Probability and example by Rick Durrett 3.4.3

Or Arithmetic Randonnée An introduction to probabilistic number theory by E. Kowalski. 2.2.3

1. 条件概率与概率空间的子空间 A是$(\Omega, \mathcal{F}, P)$的可测子集并且$P(A) > 0$

想在A上构造概率子空间$(A, \mathcal{F}_A, P_A)$

A上的σ-代数: $\mathcal{F}_A = \{B \subset A | \exists \widetilde{B} \in \mathcal{F} \text{ s.t. } \widetilde{B} \cap A = B\}$

A上的概率: $P_A(B) = P(\widetilde{B}|A)$

要验证: 若 $\widetilde{B}_1 \cap A = \widetilde{B}_2 \cap A$, 则 $P(\widetilde{B}_1|A) = P(\widetilde{B}_2 \cap A)$

注: 条件概率其实是缩小了总的概率空间(从Ω变成了A)

2. 事件的关系:attract/repel (1.8.29)

A, B是两个事件

A, B independent: $P(B|A) = P(B)$ or $P(AB) = P(A)P(B)$

A repels B: $P(B|A) < P(B)$ or $P(AB) < P(A)P(B)$

A attracts B: $P(B|A) > P(B)$ or $P(AB) > P(A)P(B)$

对称性: A repels $B \Leftrightarrow B$ repels A

A repels $B \Leftrightarrow A$ attracts $B^c \Leftrightarrow A^c$ attracts B

3. 事件的惊奇程度

$P(A) = p \to S(A) = \log \frac{1}{p}$

$P(A|B) = p \to S(A|B) = \log \frac{1}{p}$

事件发生的概率越小,则事件发生产生的惊奇程度就越大

$S(AB) = S(A) + S(B|A)$

若A, B独立, 则$S(AB) = S(A) + S(B)$

4. Shannon信息熵: $H(X) = \mathbb{E}[S(X)] = \mathbb{E}[\log \frac{1}{f(X)}] = \sum_i f(x_i) \log \frac{1}{f(x_i)}$

熵表示随机变量的混乱程度,即该随机变量包含的信息量

条件熵/相对熵: $H(X|Y) = \mathbb{E}[\log \frac{1}{f(X|Y)}] = \sum_{i,j} f(x_i, y_j) \log \frac{1}{f(x_i|y_j)}$

联合熵: $H(X, Y) = \mathbb{E}[\log \frac{1}{f(X,Y)}] = \sum_{i,j} f(x_i, y_j) \log \frac{1}{f(x_i, y_j)}$

性质:$H(X) \geq 0$

$H(X, Y) = H(X) + H(Y|X)$

若X是离散型随机变量且有n个取值,则当$\mathbb{P}(x_i) = \frac{1}{n}$时信息熵最大

若X是连续型随机变量且在$[a, b]$中取值,则当X为均匀分布时信息熵最大

若给定X的期望μ和方程σ^2, 则当X为正态分布时信息熵最大

一个封闭系统有熵增的趋势,如果没有外力的介入,总是倾向于熵增的方向发展

5. 互信息: $I(X, Y) = H(X) + H(Y) - H(X, Y) = H(X) - H(X|Y)$

$H(X)$ 表示X包含的信息

$H(X|Y)$ 表示在已知Y的情况下X增加的信息

$H(X, Y)$ 表示X, Y共同包含的信息(X, Y所含信息的并集)

$I(X, Y)$ 表示X, Y公共的信息(X, Y所含信息的交集)

正定性: $I(X, Y) \geq 0$, 取等$\Leftrightarrow X, Y$独立

对称性: $I(X, Y) = I(Y, X)$

6. 概率空间的距离

欧式距离: $d(X, Y) = \sqrt{\mathbb{E}[X - Y]^2}$ (实际上这是一个内积空间)

$d_p(X,Y)=(\mathbb{E}[X-Y]^p)^{\frac{1}{p}}$

$d(X,Y)=H(X,Y)-I(X,Y)=H(X|Y)+H(Y|X)$

Jaccard距离: $d(X,Y)=\frac{H(X,Y)-I(X,Y)}{H(X,Y)}$

7. Poisson分布的性质

$X\sim$Poisson$(\lambda):\mathbb{P}(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$

$\mathbb{E}[X]=\lambda,\ \mathbb{E}[X(X-1)\dots(X-t)]=\lambda^{t+1}$

$Var(X)=\lambda$

母函数: $G_X(s)=e^{\lambda(1-s)}$

λ的意义:intensity, 单位时间内随机事件的平均发生次数

若$X\sim$Poisson(λ), $Y\sim$Poisson(μ), 且X,Y 独立, 则$X+Y\sim$Poisson$(\lambda+\mu)$

(3.5.2) 掷N枚硬币, $N\sim$Poisson(λ), 每次硬币朝上的概率为p, 则朝上的硬币数$\sim$Poisson$(p\lambda)$

8. 组合数

$C_n^k=C_{n-1}^k+C_{n-1}^{k-1}$

$C_n^k=\frac{n}{k}C_{n-1}^{k-1}$

$\sum\limits_{k=0}^{n}C_n^k=2^n$

范德蒙恒等式: $C_{m+n}^k=\sum\limits_{i=0}^{k}C_m^iC_n^{k-i}$

9. 随机游走

$S_n=S_0+X_1+\cdots+X_n$

$\{S_n\}$是一列随机变量满足马氏性,时齐性,空齐性

增量独立性: S_m,S_n-S_m独立 $(m<n)$

鞅性(需要对称随机游走): $\mathbb{E}[S_n|S_m]=S_m\ (m<n)$

10. 轨道

$N_n(a,b)=\#\{(0,a)\to(n,b)\}$

$N_n^P(a,b)=\#\{(0,a)\to(n,b)$且满足性质$P\}$

$N_n(a,b)=C_n^{\frac{n+a-b}{2}}$

反射原理: 若a,b在r的同侧, 则$N_n^{pass\ y=r}(a,b)=N_n(a,2r-b)$

投票定理: $N_n^{not\ return\ 0}(0,b)=\frac{|b|}{n}N_n(0,b)$

不返回出发点: $N_n^{not\ return\ 0}(0)=\sum\limits_b N_n^{not\ return\ 0}(0,b)=\frac{\mathbb{E}|S_n|}{n}$

首中时: $\tau_b=\min\limits_{k\ge 0}\{X_k=b\}$

$N_n^{\tau_b=n}(0,b)=N_n^{not\ attain\ b\ before\ n}(0,b)=N_n^{not\ return\ 0}(0,b)=\frac{b}{n}N_n(0,b)$

11. 游走的最远距离: $M_n=\max\limits_{1\sim n}S_k$

for 对称随机游走: $P(M_n\ge r|S_0=0)=P(S_n\ge r|S_0=0)+P(S_n\ge r-1|S_0=0)$

最后一次访问原点时间: $T_{2n}=\max\limits_{k=0\sim 2n}\{X_k=0\}$

for 对称随机游走: $P(T_{2n}=T_{2k})=P(S_{2k}=0)P(S_{2n-2k}=0)$

1.8.21

记 $X_n = \begin{cases} k, 第n次是连续的第k个head \\ -k, 第n次是连续的第k个tail \end{cases}$，则题目即求 $P(X_n在到-s之前先到r \mid X_0=0)$

有 $P(X_{n+1}=j \mid X_n=i) = \begin{cases} p, & j=i+1 且 i\geq 0 \\ p, & j=1 且 i<0 \\ q, & j=i-1 且 i\leq 0 \\ q, & j=-1 且 i>0 \end{cases}$

令 $h(x) = P(X_n在到-s之前先到r \mid X_0=x)$

则 $\begin{cases} h(x) = ph(x+1)+qh(-1), & x=1,2 \cdots r-1 & ① \\ h(x) = ph(1)+qh(x-1), & x=-1,-2 \cdots -(s-1) & ② \\ h(0) = ph(1)+qh(-1) & & ③ \\ h(r)=1 & & ④ \\ h(-s)=0 & & ⑤ \end{cases}$

由①④ $\xRightarrow{递推}$ $h(x) = p^{r-x} + (1-p^{r-x})h(-1)$，$x=1 \sim r-1$

由②⑤ $\xRightarrow{递推}$ $h(x) = (1-q^{x+s})h(1)$，$x=-1 \sim -(s-1)$

特别地 $\begin{cases} h(1) = p^{r-1}+(1-p^{r-1})h(-1) \\ h(-1) = (1-q^{s-1})h(1) \end{cases}$

$\Rightarrow \begin{cases} h(1) = \dfrac{p^{r-1}}{p^{r-1}+q^{s-1}-p^{r-1}q^{s-1}} \\ h(-1) = \dfrac{p^{r-1}(1-q^{s-1})}{p^{r-1}+q^{s-1}-p^{r-1}q^{s-1}} \end{cases}$

$\Rightarrow h(0) = ph(1)+qh(-1) = \dfrac{p^{r-1}(1-q^{s})}{p^{r-1}+q^{s-1}-p^{r-1}q^{s-1}}$

补充: 取 $\varepsilon_1 \cdots \varepsilon_n$ 是独立同分布的随机变量且 $P(\varepsilon_k=1)=P(\varepsilon_k=-1)=\frac{1}{2}$

$X_k = \varepsilon_k a_k$

则 $E[X_k]=0 \quad E[X_k^2]=a_k^2$

由 X_k 相互独立 $\Rightarrow E[X_jX_k] = E[X_j]E[X_k] = 0 \quad (j\neq k)$

$\therefore E\left(\sum_{k=1}^n \varepsilon_k a_k\right)^2 = E\left(\sum_{k=1}^n X_k\right)^2 = \sum_{k=1}^n EX_k^2 = \sum_{k=1}^n a_k^2$

$\Rightarrow \exists\, \varepsilon_1 \cdots \varepsilon_n$ 使 $\sum_{k=1}^n \varepsilon_k a_k \leq \sum_{k=1}^n a_k^2$

3.8.6 (2) $S=\sum_{r=1}^{N} X_r$, X_r iid

指标 N 是随机变量，不好做 → 关于 $N=n$ 分类讨论然后再加起来

$$E[Sg(S)] = E[E[Sg(S)|N]]$$

其中 $E[Sg(S)|N] = E[\sum_{r=1}^{n} X_r g(\sum_{r=1}^{n} X_r) | N=n] = nE[X_n g(\sum_{r=1}^{n} X_r) | N=n]$

$$\therefore E[Sg(S)] = \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} n E[X_n g(\sum_{r=1}^{n} X_r) | N=n]$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^{n+1}}{n!} E[X_{n+1} g(\sum_{r=1}^{n+1} X_r) | N=n]$$

不用把这个E展开

~~考虑到 $S=\sum_{r=1}^{n} X_r$，想把 X_{n+1} 去掉~~

~~本来期望是 $\sum_{x_1 - x_{n+1}} x_{n+1} g(\sum_{r=1}^{n+1} x_r) f(x_1) f(x_2) \cdots f(x_{n+1})$~~

~~这里用 $E[X_{n+1} g(\sum_{r=1}^{n+1} X_r)] = E[E[X_{n+1} g(X_{n+1}+S)|S]] = \sum f(x_{n+1}) x_{n+1} E[g(x_{n+1}+S) | X_{n+1}=x_{n+1}]$~~

~~$\therefore E[Sg(S)]$~~ $=$

$$= \lambda E[E[X_{n+1} g(\sum_{r=1}^{n+1} X_r) | N=n]]$$

$$= \lambda E[X_{n+1} g(X_{n+1}+S)]$$

$$= \lambda E[X_0 g(X_0+S)]$$

4.1.1 4.4.6 ~~4.5.4~~ 4.5.7 4.6.6

$\boxed{P1}$

4.26. 4.1.1. 4.1.2. 4.2.2. 4.2.4.

4.29. 4.4.6. 4.5.4. 4.5.7. 4.6.6.

5.6. 4.7.8. 4.7.14. 4.8.6

5.8. 4.9.4. 4.14.37. 补充: $Z\sim N(0,1)$, $E[Z^k\bar{Z}^l]=\begin{cases}k!, & k=l\\ 0, & k\neq l.\end{cases}$

5.10. 4.10.1. 4.14.16. 补充: 设 $X_1,\cdots,X_n$ 独立同 $N(\mu,\sigma^2)$.

$\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$, 试求 $\rho(X_i,\bar{X})$.

4.1.1. 见答案. 4.1.2. 见答案. 4.2.2. 见答案. ~~4.2.4 见答案~~.

4.2.4. $P(Y(y)>k)=P(X_i<y,\ i=1,\cdots,k)\overset{\text{独立}}{\underset{\text{同分}}{=}}F(y)^k.\quad k\geq 1.$

~~$EY(y)=$~~ see 3.11.13 (a) $\boxed{EY(y)=\sum_{k=0}^{+\infty}P(Y(y)>k)}$

↑这是使用的条件.

$P(Y(y)>0)=1.$

$\therefore EY(y)=1+F(y)\frac{1}{1-F(y)}=\frac{1}{1-F(y)}$.

$P(Y(y)>EY(y))=\{1-[1-F(y)]\}^{\left[\frac{1}{1-F(y)}\right]}$

~~$\sim\exp\{1-[1-F(y)]\}\left[\frac{1}{1-F(y)}\right]$~~

对于取值为非负整数的随机变量 X,

$E(X)=\sum_{n=0}^{+\infty}P(X>n)$.

Let $x=1-F(y)$. $\lim_{y\to+\infty}x=0$.

$\lim_{y\to+\infty}P(Y(y)>EY(y))=\lim_{x\to0}\{1-x\}^{\frac{1}{x}}=\lim_{x\to0}e^{\frac{1}{x}\ln(1-x)}$

$=e^{\lim_{x\to0}\frac{\frac{-1}{1-x}}{1}}=e^{-1}$.

4.4.6. 见答案. (Stein's method)

4.5.4. (i) $F_U(u)=1-P(U>u)=1-(1-u)^2.\quad 0<u<1.$

$\therefore EU=\frac{1}{3}$.

11.1. 4.4.6. 4.5.4. 4.5.7. 4.6.6.

$\boxed{4.4.6.}$ 见答案. 用分部积分.

1(i) $\boxed{4.5.4}$ 1°. $F_U(u) = P(X \le u \text{ or } Y \le u) = 1-(1-u)^2 \quad 0 \le u \le 1.$

$\therefore EU = \int_0^1 u f_U(u)\,du = 2\int_0^1 u(1-u)\,du = \frac{1}{3}$

2°. $Cov(U,V) = E(UV) - EU \cdot EV = EXY - \frac{1}{3}EV = EX \cdot EY - \frac{1}{3}EV$

$UV = XY$

4. $= (\frac{1}{2})^2 - \frac{1}{3} \times \frac{2}{3} = \frac{1}{36}$

$U+V = X+Y. \Rightarrow EV = E(X+Y) - EU = 1 - \frac{1}{3} = \frac{2}{3}$

$\boxed{4.5.7}$ 令. $EX_1 = \mu$. $Var X_1 = \sigma^2$. then. $EX_1^2 = \sigma^2 + \mu^2$.

$Cov(\bar{X}, X_r) = \frac{1}{n} Cov(X_r, X_r) = \frac{\sigma^2}{n}$. 独立一定不相关.

($Cov(\bar{X}, \bar{X}) = \frac{1}{n^2} Cov(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i) = \frac{n Var(X_1)}{n^2} = \frac{\sigma^2}{n}$.

$\therefore Cov(\bar{X}, X_r - \bar{X}) = Cov(\bar{X}, X_r) - Cov(\bar{X}, \bar{X}) = 0$.

$\boxed{4.6.6.}$ $\underbrace{G_n(x)}_{\text{function of } x} = P(N>n) = P(X_1 + \cdots + X_n \le x) = E[E[I_{(X_2+\cdots+X_n \le x - X_1)} | X_1]]$.

$\overset{\text{As } X_1 \text{ 与 } (X_2, \cdots, X_n) \text{ 独立.}}{=\!=\!=} E[E[I_{(X_2+\cdots+X_n \le x-y)}]|_{y=X_1}] = E[G_{n-1}(x-X_1)] = \int_0^x G_{n-1}(x-u)\,du$

$\overset{v=x-u}{=\!=\!=} \int_0^x G_{n-1}(v)\,dv$.

Note: if. $x<0$. $I_{(X_1+\cdots+X_n \le x)} = I_\phi = 0$. $\therefore G_n(x) = P(\phi) = 0$.

$\therefore$ as. $G_0(v) = P(N>0) = 1$. $\forall v \in (0,1]$. by 定义.

by induction, $G_n(x) = \frac{x^n}{n!}$.

$EN = \sum_{n=0}^{+\infty} P(N>n) = e^x$. $\quad EN^2 = \sum_{n=1}^{+\infty} n^2 P(N=n) = \sum_{n=1}^{+\infty} n^2 [P(N>n-1) - P(N>n)]$

虽然拆分不严格, 但 → $= \sum_{n=1}^{+\infty} n^2 P(N>n-1) - \sum_{n=0}^{+\infty} n^2 P(N>n) = \sum_{n=0}^{+\infty} (2n+1) P(N>n)$

结论是对的.

$\therefore Var N = EN^2 - (EN)^2 = 2xe^x + e^x - e^{2x}$.

P2

1(ii). $E(U+V)=E(X+Y)=1$. $\therefore EV=\frac{2}{3}$.

$UV=XY$. $\therefore E(UV)=EX\cdot EY=\frac{1}{4}$.

$\therefore Cov(U,V)=E(UV)-EU\cdot EV=\frac{1}{4}-\frac{1}{3}(1-\frac{1}{3})=\frac{1}{36}$.

4.5.7. 设 $EX_1=\mu$, $Var X_1=\sigma^2$ i.e. $EX_1^2=\sigma^2+\mu^2$.

$E\bar{X}=\mu$, $E(X_r-\bar{X})=0$.

$\therefore Cov(\bar{X},X_r-\bar{X})=E\bar{X}(X_r-\bar{X})=\frac{1}{n}E(\sum_s X_rX_s)-E(\bar{X}^2)$

$=\frac{1}{n}\{\sigma^2+n\mu^2\}-(Var(\bar{X})+[E(\bar{X})]^2)=\frac{1}{n}\{\sigma^2+n\mu^2\}-\{\frac{n\sigma^2}{n^2}+\mu^2\}=0$.

$(E\sum_r X_rX_s=\underbrace{EX_r^2}_{\sigma^2+\mu^2}+\underbrace{\sum_{r\neq s}EX_rX_s}_{(n-1)\mu^2})$

4.6.6. ①. $P(N>n)=P(X_1+\cdots+X_n\le x)=E[I_{(X_1+\cdots+X_n\le x)}]$.

$=E[\underbrace{E[I_{(X_1+\cdots+X_n\le x}|X_1]}_{\text{关于}X_1\text{的随机变量}}]=\int E[I_{\underbrace{(X_1+\cdots+X_n\le x)|X_1=u}_{\Downarrow\ X_1\le x}}]f_{X_1}(u)du$

均匀 $=\int_0^x E[I_{(X_2+\cdots+X_n\le x-u)}]du \overset{u=x-v}{=\!=\!=} \int_0^x E[I_{(X_2+\cdots+X_n\le v)}]dv$.

令 $G_n(x)=P(X_1+\cdots+X_n\le x)$.

由 i.i.d. $G_n(x)=\int_0^x G_{n-1}(v)dv$. $G_0(v)=1$. $\forall\ v\in(0,1]$.

as 指标为0的求和为0.

递归, $\Rightarrow G_n(v)=\frac{x^n}{n!}$.

再取极限

② $\therefore EN=\sum_{n=0}^{+\infty}P(N>n)=e^x$.

严谨的做法是利用前n项和 $(\sum_{n=1}^{+\infty}n^2P(N>n))$

$EN^2=\sum_{n=1}^{+\infty}n^2P(N=n)=\sum_{n=1}^{+\infty}n^2[P(N>n-1)-P(N>n)]=\sum_{n=1}^{+\infty}n^2P(N>n-1)-$

$=\sum_{n=0}^{+\infty}(n+1)^2P(N>n)-\sum_{n=0}^{+\infty}n^2P(N>n)=2\sum_{n=0}^{+\infty}nP(N>n)+\sum_{n=0}^{+\infty}P(N>n)$.

P3

$$= 2\sum_{n=1}^{+\infty} n\frac{x^n}{n!} + e^x = 2xe^x + e^x$$

$$\therefore \mathrm{Var}(N) = EN^2 - (EN)^2 = 2xe^x + e^x - e^{2x}.$$

4.7.8. ① 答案直接算.

注意: 教材 P_{108} ex(2) 后一段中第4行: one-one and onto.

② $f_{X,Y}(x,y) = \frac{1}{\pi} I(x^2+y^2<1) = \frac{1}{\pi} I(x^2+y^2<1, y>0) + \frac{1}{\pi} I(x^2+y^2<1, y\le 0)$.

在圆盘内部, f 光滑, $\begin{cases} r = \sqrt{x^2+y^2} \\ x = x \end{cases}$

$$\left|\frac{\partial(r,x)}{\partial(x,y)}\right| = \left\|\begin{matrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ 1 & 0 \end{matrix}\right\| = \frac{\sqrt{r^2-x^2}}{r}.$$

为使变换光滑, 需 $|x| < r < 1$

$$f_{R,X}(r,x) = f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(r,x)}\right| = \frac{1}{\pi}\frac{r}{\sqrt{r^2-x^2}} + \frac{1}{\pi}\frac{r}{\sqrt{r^2-x^2}} = \frac{1}{\pi}\frac{2r}{\sqrt{r^2-x^2}}.$$

Y消失了, 但由对称性, 计算了两次.

容易混淆, 建议用第一种方法.

4.7.14. 见答案.

4.8.6. 见答案 "独立同的 Gaussian 随机变量的联合正态分析".

4.9.4. $\vec{X} \sim N(\vec{\mu}, V)$. $f(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n |V|}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu}) V^{-1} (\vec{x}-\vec{\mu})^t\right\}$.

Now let $n=2$. $\vec{\mu}=0$. $V = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. $\det V = 1-\rho^2$.

$V^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$. $\vec{x} = (x, y)$.

$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\}$

let $\begin{cases} U = X+Y \\ V = X-Y \end{cases}$ i.e. $\begin{cases} X = \frac{1}{2}(U+V) \\ Y = \frac{1}{2}(U-V) \end{cases}$ $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$

$$f_{U,V}(u,v) = \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{u^2}{4(1+\rho)} - \frac{v^2}{4(1-\rho)}\right\},$$

$N(\mu, \sigma^2)$: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\therefore U \sim N(0, 2(1+\rho))$

$V \sim N(0, 2(1-\rho))$.

P4

4.14.37. 见答案.(Hermite 多项式).

补充: $f_Z(z)=\frac{1}{\pi}e^{-|z|^2}$

$$E[Z^k\bar{Z}^l]=\int_{\mathbb{C}} z^k\bar{z}^l\frac{1}{\pi}e^{-|z|^2}dz \xlongequal{z=re^{i\theta}} \frac{1}{\pi}\int_0^{+\infty}rdr\int_0^{2\pi}r^{k+l}e^{i(k-l)\theta}e^{-r^2}d\theta$$

$$=2\delta_{k,l}\int_0^{+\infty}r^{k+l+1}e^{-r^2}dr \underset{\substack{r=\sqrt{u}\\ dr=\frac{du}{2\sqrt{u}}}}{\overset{u=r^2}{=}} \delta_{k,l}\int_0^{+\infty}u^{\frac{k+l}{2}}e^{-u}du=\Gamma(k+1)\delta_{k,l}$$

4.10.1. 见答案. $X_1\overset{d}{=}Z_1^2+Z_2^2+\cdots+Z_m^2$. $Z_1,\cdots,Z_m$ i.i.d. $N(0,1)_{\mathbb{R}}$.

iff: $X_1\sim\chi^2(m)$

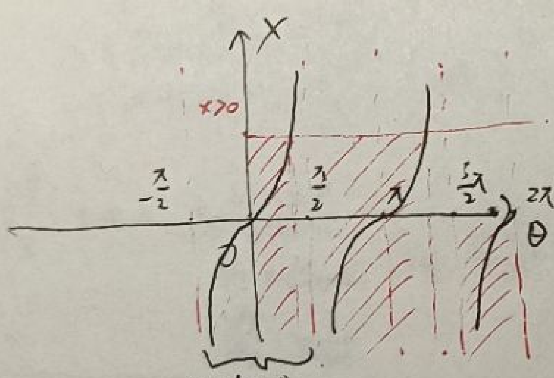
4.14.16. ① $\begin{cases}X=R\cos\Theta\\ Y=R\sin\Theta\end{cases}$ $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=r$.

$\therefore f_{R,\Theta}(r,\theta)=f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=\frac{r}{2\pi}e^{-\frac{r^2}{2}}$. $r>0$. $0\le\theta<2\pi$

② 1° 从而 Θ 为 $(0,2\pi)$ 上均匀分布.

$$P(\tan\Theta\le x)=\cancel{\int_0^{\arctan}\frac{1}{2\pi}\arctan x}\begin{cases}\frac{1}{2\pi}\left[\arctan x+\arctan x+\frac{\pi}{2}+\frac{\pi}{2}\right]. & x\ge 0.\\ \frac{1}{2\pi}\left[\arctan x+\frac{\pi}{2}+\arctan x+\frac{\pi}{2}\right] & x<0\end{cases}$$

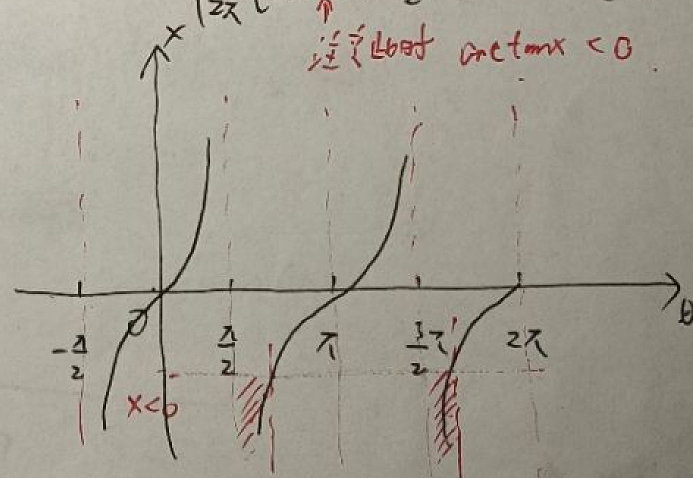
注意此时 $\arctan x<0$



$\underbrace{\quad}$ $x=\tan\theta$

$\theta=\arctan x$

$\in(-\frac{\pi}{2},\frac{\pi}{2})$.



$(\arctan x)'=\frac{1}{1+x^2}$. $(u=\arctan x.\ \ x=\tan u.\ \ x'_u=\cos^2 u.\ \ u'_x=\cos^{-2}u=\frac{1}{1+x^2})$.

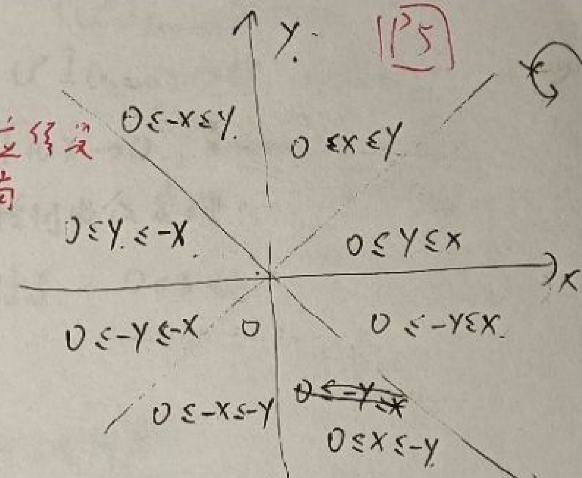
$\therefore f_{\tan\Theta}(x)=\frac{1}{\pi}\frac{1}{1+x^2}$.

2°. 由4.10.1. $R^2\sim\chi^2(2)$ or 见答案. 直接算.

P5

③ 由对称性. $E\left(\frac{X^2}{R^2}\right) = \frac{1}{2}E\left(\frac{X^2+Y^2}{R^2}\right) = \frac{1}{2}$.

虽然 $f_{X,Y}$ 旋转不变, 但这玩意儿应该没有



④ $\frac{\min\{|X|,|Y|\}}{\max\{|X|,|Y|\}} = \frac{Y}{X} 1_{\{0\le Y\le X\}} + \frac{X}{Y} 1_{\{0\le X\le Y\}}$

$+ \frac{-X}{Y} 1_{\{0\le -X\le Y\}} + \frac{Y}{-X} 1_{\{0\le Y\le -X\}} + \frac{Y}{X} 1_{\{0\le -Y\le -X\}}$

$\cancel{= \tan\theta\, 1_{[0,\frac{\pi}{4})} + \tan^{-1}\theta}$

$+ \frac{X}{Y} 1_{\{0\le -X\le -Y\}} + \frac{X}{-Y} 1_{\{0\le X\le -Y\}} + \frac{-Y}{X} 1_{\{0\le -Y\le X\}}$.

$= \tan\theta\, 1_{[0,\frac{\pi}{4})} + \underbrace{\tan^{-1}\theta\, 1_{[\frac{\pi}{4},\frac{\pi}{2})}}_{①} \underbrace{- \tan^{-1}\theta\, 1_{[\frac{\pi}{2},\frac{3}{4}\pi)}}_{②} \underbrace{- \tan\theta\, 1_{[\frac{3}{4}\pi,\pi)}}_{③}$

$\underbrace{+ \tan\theta\, 1_{[\pi,\frac{5}{4}\pi)}}_{④} + \underbrace{\tan^{-1}\theta\, 1_{[\frac{5}{4}\pi,\frac{3}{2}\pi)}}_{⑤} \underbrace{- \tan^{-1}\theta\, 1_{[\frac{3}{2}\pi,\frac{7}{4}\pi)}}_{⑥} \underbrace{- \tan\theta\, 1_{[\frac{7\pi}{4},2\pi)}}_{⑦}$

①: $\tan^{-1}\left(\frac{\pi}{2}-\theta\right) = \tan\theta$. ② $-\tan^{-1}\left(\theta+\frac{\pi}{2}\right) = \tan\theta$, ③ $-\tan(\pi-\theta) = \tan\theta$

④ $\tan(\theta+\pi) = \tan\theta$. ⑤ $\tan^{-1}\left(\frac{3}{2}\pi-\theta\right) = \tan\theta$. ⑥ $-\tan^{-1}\left(\theta+\frac{3}{2}\pi\right) = \tan\theta$.

⑦ $-\tan(2\pi-\theta) = \tan\theta$.

$\therefore E\left\{\frac{\min\{|X|,|Y|\}}{\max\{|X|,|Y|\}}\right\} = 8\int_0^{\frac{\pi}{4}} d\theta \frac{\tan\theta}{2\pi} \int_0^{+\infty} re^{-\frac{1}{2}r^2} dr = \frac{2}{\pi}\log 2$.

$\int_0^{\frac{\pi}{4}} \tan\theta \cdot d\theta \overset{u=\tan\theta}{=\!=\!=} \int_0^1 \frac{u}{1+u^2} du = \frac{1}{2}\log 2$.

$\theta = \arctan u$.

$\frac{d\theta}{du} = \frac{1}{1+u^2}$

补充: $\rho(X_i, \bar{X}) = \frac{Cov(X_i,\bar{X})}{\sqrt{Var X_i \cdot Var\bar{X}}}$ 见 Ex 4.5.7

$\Downarrow$

$Cov(X_i,\bar{X}) = Cov(X_i-\bar{X},\bar{X}) + Var\bar{X} = Var\bar{X} = \frac{1}{n}Var X_1$

$\therefore \rho(X_i,\bar{X}) = \sqrt{\frac{Var\bar{X}}{Var X_1}} = \sqrt{\frac{1}{n}}$.

P6

$|u^r I_{[x,+\infty)}(u)| \le u^r$.
$u^r I_{[x,+\infty)} \to 0$, $x \to +\infty$.

5.6.4. ① ⇒ 若 $E|X^r| < +\infty$

则 $x^r \cdot P(|X| \ge x) \le \int_x^{+\infty} u^r \, dF(u) \to 0$, $x \to +\infty$

↑ 控制收敛定理. $F = F_{|X|}$.

② 现设 $x^r P(|X| \ge x) \to 0$; $r > 0$. 对于 $0 \le s < r$.

法(1): $E|X^s| = \lim_{M \to +\infty} \int_0^M u^s \, dF(u)$.

关于 Lebesgue stieltjes 积分的分部积分.

$$\int_{(a,b]} F \, dG + \int_{(a,b]} G \, dF = F(b)G(b) - F(a)G(a).$$

if. F, G 有界变差. 且至少有一个为连续的.

$$\therefore \int_\delta^M u^s \, dF(u) = \underbrace{\Big[-u^s[1-F(u)]\Big]_\delta^M} + \int_\delta^M s u^{s-1} \cdot [1-F(u)] \, du.$$

$1 - F(u) \le C \cdot u^{-r}$, 对充分大的 u、 $\int_0^\delta u^s \, dF(u) < +\infty$.

$$\{-u^s[1-F(u)]\}\Big|_\delta^M = \underbrace{-M^s[1-F(M)]}_{\le 0} + \underbrace{\delta^s[1-F(\delta)]}_{\text{Const}}.$$

$\int_\delta^M s u^{s-1}[1-F(u)] \, du \le C \cdot \int_\delta^M u^{s-r-1} \, du < +\infty$ uniformly in M.

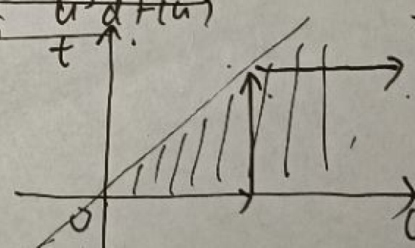
$(s-r-1 < -1)$

Fubini. $\int_0^{+\infty} du \int_0^u dt \cdots = \int_0^{+\infty} dt \int_t^{+\infty} du$

法(2): $\cancel{\int_0^{+\infty} u^s \, dF(u) = \underbrace{\int_0^\delta u^s \, dF(u)}_{<+\infty} + \int_\delta^{+\infty} u^s \, dF(u)}$

$s = 0$, $\int_0^{+\infty} u^0 \, dF(u) = 1$.

$\cancel{\int_\delta^{+\infty} u^s \, dF(u) = \int_\delta^{+\infty}}$



$s > 0$. $\int_0^{+\infty} u^s \, dF(u) = \int_0^{+\infty} s \int_0^u t^{s-1} dt \cdot dF(u)$.

$$= \int_0^{+\infty} s t^{s-1} \int_t^{+\infty} 1 \, dF(u) \, dt = \underbrace{\int_0^{+\infty}}_{\underbrace{\int_0^\delta}_{<+\infty} + \int_\delta^{+\infty}} s \cdot t^{s-1} (1 - F(t)) \, dt.$$

if $t > \delta$. $1 - F(t) \le C \cdot t^{-r}$.

. same as 法①.

Problem 1. $\int_{R} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} \Rightarrow \int_{R^n} e^{-\frac{1}{2}\vec{x}\cdot\vec{x}^t} d\vec{x} = \sqrt{(2\pi)^n}$

$$\Rightarrow \int_{R^n} e^{-\frac{1}{2}(\vec{x}A\vec{x}^t + 2\vec{x}B\vec{y}^t)} d\vec{x} = \sqrt{(2\pi)^n(\det A)^{-1}}\, e^{\frac{1}{2}\vec{y}B^tA^{-1}B\vec{y}^t} \quad (1).$$

$\vec{x} = (x_1, \cdots, x_n)$. row vector, $\quad -\frac{1}{2}(\vec{x}+\vec{y}B^tA^{-\frac{1}{2}})(\vec{x}+\vec{y}B^tA^{-\frac{1}{2}})^t + \frac{1}{2}\vec{y}B^tA^{-1}B\vec{y}^t$.

$\vec{y} = (y_1, \cdots, y_n)$.

Problem 2. for $1 \le i_1 < i_2 < \cdots < i_k \le n$. $\vec{i} := (i_1, i_2, \cdots, i_k)$.

$\vec{\mu} = (\mu_1, \cdots, \mu_n)$. $V = (v_{ij})_{i,j=1,\cdots,n}$. let $V = V^t$. & $V > 0$. i.e. $\det V > 0$

denote. $\vec{\mu}_{\vec{i}} = (\mu_{i_1}, \cdots, \mu_{i_k})$. $V_{\vec{i}} = (v_{i_\alpha, i_\beta})_{\alpha,\beta=1,\cdots,k}$.

Also. let $\det V_{\vec{i}} > 0$, $\forall$ k & $\vec{i}$. Then,

if $\vec{X} \sim N(\vec{\mu}, V)$. i.e. $f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n|V|}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})V^{-1}(\vec{x}-\vec{\mu})^t}$.

then. $\vec{X}_{\vec{i}} \sim N(\vec{\mu}_{\vec{i}}, V_{\vec{i}})$.

Pf: w.l.g. let $\vec{i} = (1, 2, \cdots, k)$. denote. $X := (X_I, X_{II})$.

$X_I := (x_1, \cdots, x_k)$. $X_{II} := (x_{k+1}, \cdots, x_n)$. then:

$$f_{X_I}(\vec{x}_I) = \frac{1}{\sqrt{(2\pi)^n|V|}} \int_{R^{n-k}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})V^{-1}(\vec{x}-\vec{\mu})^t} \underbrace{dx_{II}}_{dx_{k+1}\cdots dx_n}$$

①. (regularity). let $\Sigma := V^{-1} =: \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^t & \Sigma_{22} \end{pmatrix}$. $V := \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^t & V_{22} \end{pmatrix}$.

then: $\Sigma = \begin{pmatrix} (V_{11} - V_{12}V_{22}^{-1}V_{12}^t)^{-1} & -\Sigma_{11}V_{12}V_{22}^{-1} \\ -\Sigma_{22}V_{12}^tV_{11}^{-1} & (V_{22} - V_{12}^tV_{11}^{-1}V_{12})^{-1} \end{pmatrix}$ ← solve: $\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^t & \Sigma_{22} \end{pmatrix}\begin{pmatrix} V_{11} & V_{12} \\ V_{12}^t & V_{22} \end{pmatrix} = \begin{pmatrix} I & \\ & I \end{pmatrix}$

$\det V = \det V_{11} \det(V_{22} - V_{12}^tV_{11}^{-1}V_{12}) = \det V_{22} \det(V_{11} - V_{12}V_{22}^{-1}V_{12}^t)$.

As. $\det V \neq 0$, $\det V_{11} \neq 0$, $\det V_{22} \neq 0$, $\Rightarrow \det(V_{22} - V_{12}^tV_{11}^{-1}V_{12})\det(V_{11} - V_{12}V_{22}^{-1}V_{12}^t) \neq 0$.

Hence, $(V_{11} - V_{12}V_{22}^{-1}V_{12}^t)^{-1}$, $(V_{22} - V_{12}^tV_{11}^{-1}V_{12})^{-1}$. exist.

therefore, Σ_{11}^{-1}, Σ_{22}^{-1} exists. Hence.

$$V = \begin{pmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^t)^{-1} & * \\ * & * \end{pmatrix}$$

②. (Computation).

$$(\vec{x}-\vec{\mu})\Sigma(\vec{x}-\vec{\mu})^t = (X_I-\mu_I)\Sigma_{11}(X_I-\mu_I)^t + 2(X_I-\mu_I)\Sigma_{12}(X_{II}-\mu_{II})^t + (X_{II}-\mu_{II})\Sigma_{22}(X_{II}-\mu_{II})^t.$$

$$f_{X_I}(\vec{X}_I) = \frac{1}{\sqrt{(2\pi)^n|V|}} e^{-\frac{1}{2}(X_I-\mu_I)\Sigma_{11}(X_I-\mu_I)^t} \times$$

$$\underline{\int_{R^{n-k}} e^{-\frac{1}{2}\left[(X_{II}-\mu_{II})\Sigma_{22}(X_{II}-\mu_{II})^t + 2(X_I-\mu_I)\Sigma_{12}(X_{II}-\mu_{II})^t\right]} dX_{II}}$$

$$\left(= \int_{R^{n-k}} e^{-\frac{1}{2}[X_{II}\Sigma_{22}X_{II}^t + 2X_{II}\Sigma_{12}^t(X_I-\mu_I)^t} dX_{II}\right).$$

$$\overset{\text{eq.(1)}}{=} \frac{1}{\sqrt{(2\pi)^n|\vec{V}|}} e^{-\frac{1}{2}(X_I-\mu_I)\Sigma_{11}(X_I-\mu_I)^t} \times \underbrace{\sqrt{(2\pi)^{n-k}(\det\Sigma_{22})^{-1}}\, e^{\frac{1}{2}(X_I-\mu_I)\Sigma_{12}\Sigma_{11}^{-1}\Sigma_{12}^t(X_I-\mu_I)^t}}_{=}$$

$$= \frac{\sqrt{\det\Sigma_{22}}}{\sqrt{(2\pi)^k\det V}} \exp\left\{-\frac{1}{2}(X_I-\mu_I)(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^t)(X_I-\mu_I)^t\right\}$$

$$\det V^{-1} = \det\Sigma = \det\Sigma_{22}\cdot\det(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^t) = \det\Sigma_{22}\det V_{11}^{-1}.$$

as. $V_{11}^{-1} = \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^t$.

$$= \frac{1}{\sqrt{(2\pi)^k\det V_{11}}} \exp\left\{-\frac{1}{2}(X_I-\mu_I)V_{11}^{-1}(X_I-\mu_I)^t\right\} \qquad \square.$$

Problem 3. If $\vec{X}=(X_1,X_2,\dots,X_n)\sim N(\vec{0},V)$. $\vec{Y}=(Y_1,\dots,Y_m)=XD_{n\times m}$, $m\le n$.
then $\vec{Y}\sim N(\vec{0},D^tVD)$. if. $\operatorname{rank}D=m$.

Pf: ~~① m=n. simple~~ ~~② m<n.~~ $\phi_{\vec{Y}}(\vec{t}) = E[e^{i\vec{t}\vec{Y}^t}]$ by e.g. (5.8.6).

$$= E[e^{i\vec{t}D^t\vec{X}^t}] = E[e^{-\frac{1}{2}\vec{t}D^tVD\vec{t}^t}]$$

definition of rank.

As $\operatorname{rank}D=m$. $\exists$ Permutation matrix R. s.t. $RD = \begin{pmatrix}D_1\\D_2\end{pmatrix}$.
D_1. invertible. then: $\underbrace{\begin{pmatrix}D_1^{-1} & \\ & I\end{pmatrix}\begin{pmatrix}I & \\ -D_2D_1^{-1} & I\end{pmatrix}R}_{:=P^{-1}} D = \begin{pmatrix}I_m\\0\end{pmatrix}$. $D=P\begin{pmatrix}I_m\\0\end{pmatrix}$

$$\operatorname{rank}(D^tVD) = \operatorname{rank}[(V^{\frac{1}{2}}D)^t(V^{\frac{1}{2}}D)] = \operatorname{rank}(V^{\frac{1}{2}}D) = \operatorname{rank}D = m.$$

$$\operatorname{rank}(A^tA) = \operatorname{rank}(AA^t) = \operatorname{rank}A = \operatorname{rank}A^t$$

that is to say. D^tVD. is invertible. therefore $\underline{\vec{Y}\sim N(\vec{0},D^tVD)}$

Remark 1: Inversion theorem for multidimensional case also holds. $\leftarrow$ characteristic function determines its distribution

Remark 2: For $\boxed{\text{m-dimensional}}$ random vector $\vec{X}$, if $\phi_{\vec{X}}(\vec{t}) = e^{-\frac{1}{2}\vec{t}V\vec{t}^t}$ key $\boxed{m\times m}$. $\boxed{V^{-1}\text{ exists}}$ key, then $\vec{X}\sim N(\vec{0},\vec{V})$

Problem 4. If $\vec{X}=(X_1, X_2, \cdots, X_n) \sim N(\vec{o}, V)$. $\vec{Y}=(Y_1, \cdots, Y_m) = XD_{n\times m}$, ~~[illegible]~~

~~[illegible]~~ while $n<m$. $\operatorname{rank} D = n$. In this case, we say $\vec{Y}$ is not completely random, or in m-dimension, $\vec{Y}$ can't be viewed as a random vector.

Pf: $\exists$ Permutation matrix R. s.t. $R^{-1}=R^t$. also permutation.

$DR=(D_1, D_2)$. D_1^{-1}. exists. definition of Rank.

$\vec{Y} = \vec{X}D = (XD_1, XD_2)R^{-1}$. denote. $\vec{X}_1 := \vec{X}D_1$

From problem. 3. $\vec{X}_1 \sim N(\vec{o}, D_1^t V D_1)$.

$\vec{Y}R \cancel{R} = (\vec{X}_1, \vec{X}_1 D_1^{-1} D_2)$. entries of $\vec{X}_1 D_1^{-1} D_2$ are linear combinations of entries of $\vec{X}_1$.

that is to say $\vec{Y}$ only has $n (<m)$ independent $\boxed{\text{integration}}$ variables.

After some ~~[illegible]~~ permutation of entries of $\vec{Y}$, takes the form. $(\vec{X}_1, \vec{X}_1 D_1^{-1} D_2)$. $\square$.

Problem 5. If $\vec{X}=(X_1, \cdots, X_n) \sim N(\vec{o}, V)$. $\vec{Y}=(Y_1, \cdots, Y_m) = XD_{n\times m}$.

while $\operatorname{rank} D = r < m \le n$. In this case. $\exists$ invertible ~~Permutation~~ $P_{n\times n}$. $Q_{m\times m}$.

s.t. $D = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$.

$$\vec{Y} = \vec{X}P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q, \quad \text{let } \vec{X}_1 = \vec{X}P. \quad \text{then.}$$

$\vec{X}_1 \sim N(\vec{o}, P^t V P)$. $\vec{X}_1 := (\overset{r}{\vec{X}_I}, \overset{n-r}{\vec{X}_{II}})$. $Q = \begin{array}{c} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \end{array} \begin{array}{l} r \\ m-r \end{array}$

$\vec{Y} = (X_I, 0)Q = \vec{X}_I Q_1$. $\operatorname{rank} Q_1 = \operatorname{rank}[(I_r, 0)Q] = r$. $Q_1 = (Q_1)_{r\times m}$

denote. $P^t V P = \begin{pmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{pmatrix} \begin{array}{l} r \\ n-r \end{array}$. then from problem 2. $\vec{X}_I \sim N(\vec{o}, \tilde{V}_{11})$.

reduce to "Problem 4."

概率论习题课

5.6(1,4), 5.7.7.

5.8.(7,8,9,10). 5.9.(2,5)

a.s 对应的随机变量分布函数

5.10(1(b),3).

~~5.12.33.~~

特征函数:

对测度的Fourier变换.

Fourier变换有如下较好的性质,

- 唯一性, 不同测度的Fourier变换结果不同.
- 对较好函数空间, Fourier反变换可写出.

以及一些较好的性质(概率论中能将随机变量和(卷积)变为乘).

~~特子~~ ~~Def~~ 我们接下来对Schwartz空间作Fourier变换基本结果.

Def. Schwartz空间为如下函数空间(记为 S)

$f \in S(\mathbb{R})$ 计 $\forall_{k,t}$ 有 $\|(x^k+1)^t \partial^k f\| < \infty$

易见 $S(\mathbb{R})$ 是一向量空间, 称 $f_n \to f$ 若 $\|f_n - f\|_{k,t} \to 0$ 对 $\forall_{k,t}$

Def (Fourier transform)

$$\mathcal{F}(f)(t) := \int_{-\infty}^{+\infty} f(x) e^{itx}\,dx := \hat{f}$$

则 $\mathcal{F}: S \longmapsto S.$

$f \longmapsto \hat{f}$

并且满足有如下一些性质. (Ref: Stein 的 fourier analysis 的 ch 5)

(Stein prop 1.2. Ch5!

(i). $f(x+h) \longrightarrow \hat{f}(t)e^{2\pi iht}$

(ii). $f e^{-ihx} \longrightarrow \hat{f}(t+h)$

(iii) $f(\delta x) \longrightarrow \delta^{-1}\hat{f}(\delta^{-1}x t)$

(iv) $f'(x) \longrightarrow it\hat{f}(t)$

(v) $-itf \longrightarrow \hat{f}'(t)$

(vi) $f*g \longrightarrow \hat{f}\hat{g}$.

- (Stein 1 Thm 1.3).

$$f \in S(\mathbb{R}) \longrightarrow \hat{f} \in S(\mathbb{R})$$

- 对于 Gaussian 型.

$$\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{-itx} dx = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-it)^2} e^{-\frac{1}{2}t^2} dx$$

$$= e^{-\frac{1}{2}t^2}.$$

即 Gaussian 是 Fourier 变换特征函数

- Fourier inversion.

prop 1.8.

$$\int f\hat{g} = \int \hat{f} g$$

Thm 1.9.

$$f = \int \hat{f} e^{-itx} dx$$

proof. ~~$\int \hat{f} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x}{\sigma})^2} = \int f e^{-\frac{1}{2}\sigma^2 t^2}$~~，令 $\sigma \to 0$，得到 $f(0) = \int \hat{f}$.

- Thm 1.12. (Plancherel)

If $f \in S(\mathbb{R})$ 则 $\|\hat{f}\|_{L^2} = \sqrt{2\pi}\|f\|_{L^2}$

proof 取 $f^b(x) = \overline{f(-x)}$，则 $\widehat{f^b}(t) = \overline{\hat{f}(t)}$

~~$\int f e^{-\frac{1}{2}\sigma^2 t^2} dt = \int \hat{f}$~~

$$\int \hat{f} e^{-\frac{1}{2}\sigma^2 t^2} dt = 2\pi \int f \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$$

令 $\sigma \to 0 \Rightarrow 2\pi f(0) = \int \hat{f}$.

Thm 1.12 (Plancherel).

If $f \in S(\mathbb{R})$. 则 $\|\hat{f}\|_{L^2} = 2\pi\|f\|_{L^2}$

proof. 考虑同分布随机变量 X, Y （假设其连续，因 $f \in S(\mathbb{R})$）

$Z = X - Y$，$f_Z(0) = \int |f_X(x)|^2$.

则有 $\phi_Z(t) = |\phi_X(t)|^2$.

则 $\int \phi_Z(t) = 2\pi f_Z(0) = \int |f_X(x)|^2$

即 $\|\hat{f}\|_{L^2} = 2\pi\|f\|_{L^2}$，即 Fourier 变换的 L^2 等距.

更多相关内容可参见 (Stein 1. fourier analysis.
再进一步可阅读调和分析
相关内容.如
Duoandikoetxea Fourier analysis)

- 特征函数收敛与依分布收敛 ~~(Ref: Folland Ch~~ ~~Real analysis~~

Ref: folland real analysis. ch 9 & 8.

本部分我们将建立 $\phi_{X_n}(t) \longrightarrow \phi$ point wise.

$+$ ϕ 在 0 处连续（X_n 不会在 ∞ 处 escape.）

则 $X_n \xrightarrow{d} X$

X_n 作为随机变量实际上诱导了一个 $\mathbb{R}$ 上的测度 μ_n，同样记 X 对应的测度为 μ。我们考虑什么是测度的依分布收敛。

- X_n 依分布收敛 $\Longleftrightarrow \int f \, d\mathbb{P}(X_n \le x) \longrightarrow \int f \, d\mathbb{P}(X \le x)$

  对所有 $f \in C_b(\mathbb{R})$ 有界连续函数.

故 $X_n \xrightarrow{d} X$ 我们可看成成 $\int f \mu_n(dx) \longrightarrow \int f \mu(dx)$

- 若将 μ_n 看成 $f \in C_b(\mathbb{R})$ 上的线性算子，便是（泛函中）算子的弱*收敛.

我们先考虑~~一类~~更好的函数空间 $S(\mathbb{R})$

Def Distribution $D'(\mathbb{R})$ 为 $S(\mathbb{R})$ 的对偶空间.

即 $T \in D'$ 是 S 上（连续）线性泛函. i.e. 若 $\|f_n - f\|_{\alpha,\beta} \to 0$，则 $Tf_n \to Tf$.

- 当 $T(f) = \int gf$，$g \in S(\mathbb{R})$ 时显然是.

将 Fourier 变换延拓到 D' 中

$\widehat{T}(f) = T(\hat{f})$ （由 $\int \hat{g} f = \int g \hat{f}$ 可看出）

并且对于概率测度 μ.

$\hat{\mu}(f) = \int \hat{\mu}(t) f(t) dt = \int f(t) e^{itx} \mu(dx) dt = \int \hat{f} \mu(dx)$.

若 $\hat{\mu}_n \longrightarrow \hat{\mu}$ pointwise. 由控制收敛定理有

$\int \hat{\mu}_n(t) f \longrightarrow \int \hat{\mu}(t) f$

$\Rightarrow \quad \widehat{T}_n(f) = T_n(\hat{f}) \longrightarrow \widehat{T}(f) = T(\hat{f})$

则对于 $\forall f \in S(\mathbb{R})$ 有 $\int f \mu_n(dx) \longrightarrow \int f \mu(dx)$.

这称为 vaugely convergence.

若加上 μ_n 是 tight 的（由 ϕ 在 0 处连续性可推出）

即 $\forall \varepsilon > 0$，$\exists M$ $\mathbb{P}(|X_n| \ge M) < \varepsilon$ 一致成立，即可证明（加上 $S(\mathbb{R})$ 的稠密性）

$\int f \mu_n(dx) \longrightarrow \int f \mu(dx) \quad \forall f \in C_b(\mathbb{R})$.

- 矩方法.（Ref. Durrett 3.3. The moment Problem）

  - x^r $r \ge 1$ 并不是 $\mathbb{R}$ 上所有函数一组基

  - Counterexample. 若 $\int_0^\infty x^r f(1+a\sin(2\pi \log x))\frac{1}{\sqrt{2\pi}\,x}e^{-\frac{(\log x)^2}{2}} = 0$

    proof see durrett 3.3.5.

- (Carleman's condition) If $\limsup_{k\to\infty} \frac{\mu_{2k}^{\frac{1}{2k}}}{2k} = r < \infty$

  则存在至多一个 d.f. F 使 $\int x^k dF(x) = \mu_k$.

- 一般来说我们通过随机变量列的紧性，通过取出一个子列依分布收敛到一个极限，同时这个极限的矩唯一且满足 Carleman's 条件，来说明这列随机变量存在极限且唯一。

- proof of Carleman's condition.

  通过特征函数唯一说明，设 F, G 都有 μ_k.

  由于 $\left|e^{i\theta x}\left(e^{itX} - \sum_{m=0}^{n-1}\frac{(itX)^m}{m!}\right)\right| \le \frac{|tX|^n}{n!}$

  故 $|\phi_F(t+\theta) - \phi_G(t+\theta)| = \left|\int e^{i\theta X}(e^{itX}dF - e^{itX}dG)\right|$

  $= \left|\int e^{i\theta X}\left(e^{itX} - \sum_{m=0}^{n-1}\frac{(itX)^m}{m!}\right)(dF - dG)\right|$

  $\le \int \frac{|t|^n|X|^n}{n!} dF - dG \le \frac{|t|^n}{n!}\sqrt{\mu_{2n}} \le |t|^n \frac{(2kr)^k}{(\frac{k}{e})^k}$ 取 $t < \frac{1}{2er}$，

- Some Application.

  1. 课上中心极限定理的证明。可自行适当减弱独立条件。

  2. 一类随机过程中随机测度的收敛性，如随机矩阵 wigner 半圆律，edge distribution 等。

  3. 概率数论中一些有趣结果。

     - Erdös - Kac Theorem.
     - ~~Selberg center limit theorem~~

     Kloosterman sum 的 Sato - Tate density

Gaussing Free Field.

One way to generate Random surface.

是 Brownian motion 的高维推广.

- <u>布朗运动的一种构造.</u>

物理中经常要依照某种作用量 I
来选取一些路径或表示某种状态出现的概率
一状态, S 出现的概率依统计物理会正比于 $e^{-\beta I}$.

~~类似~~ 为研究连续或无穷的情况 我们考虑离散和有限时情形. 考虑 $1\sim n$ 上的函数 f 我们定义其上的能量泛函 (作用量), 为 $I(f_n)=\frac{1}{2}\sum_i |f(i+1)-f(i)|^2$. (Dirichlet energy)

则考虑选 f 的概率与 $e^{-\frac{1}{2}\sum|f(i+1)-f(i)|^2}$ 成正比.

可以说明在这种概率测度下 $f(i+1)-f(i)\sim N(0,1)$ 且两两独立.

我们令 g_n 为 $[0,1]$ 上函数. $g_n(\frac{i}{n})=f_n(i)$. 其余线性连接.

作 scale limiting, $\frac{g_n(t)}{\sqrt{n}}\longrightarrow B([0,1])$. (Donsker's Theorem)

Ref. Durrett Ch8.

- 离散 GFF.

f 是 $(n+2)\times(n+2)$ 的网格上的函数
其上作用量 $I(f)=\frac{1}{4}\sum_{i\sim j}|f(i)-f(j)|^2$ (i 与 j 相邻), 边上的f值为0
则 f 依 $e^{-I}=e^{-\frac{1}{4}\sum_{i\sim j}|f(i)-f(j)|^2}$ 选取. 便是离散高斯自由场.

类似于布朗运动, 其可近似看成是一离散化之后的面.
不过其 Scale change 之后并不如 Brownian motion 那样有界.
而是~~[illegible]~~几乎处处无界. (但附近的会相互抵消)
通过适当的 renormalise.. 我们可以对其进行处理.
如取 $h_\varepsilon(z)$ 为离 z ε 远的值算, 并且将 $h(z)$ 看作是曲面局部等距坐标下的曲率(或其它).
我们能够从 GFF 得到其对应的一个随机面.
(类似于布朗运动, 这类曲面不会很光滑).

感兴趣可阅读

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Gaussian free field, Liouville quantum gravity and Gaussian multiplicative chaos. 一书.

1 Distribution

$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

X is a random variable $\Leftrightarrow \forall A \in \mathcal{B}(\mathbb{R}), X^{-1}(A) \in \mathcal{F}$

The **distribution function** of X: $F_X(x) = P(X \le x)$

F_X is right-continuous, monotone nondecreasing and $F_X(-\infty) = 0, F_X(\infty) = 1$

The **distribution** of X: $\forall A \in \mathcal{B}(\mathbb{R})$, define $\mu_X(A) = P(X^{-1}(A))$

μ_X is a probability measure on $\mathbb{R}, \mathcal{B}(\mathbb{R})$

Theorem: (1) $\forall F : \mathbb{R} \to [0,1]$ right-continuous, monotone nondecreasing and $F_X(-\infty) = 0, F_X(\infty) = 1$, there exists a random variable X s.t. $F = F_X$

(2) There is a 1-1 correspondence between distributions functions on $\mathbb{R}$ and probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

(3) $\forall \mu$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exists a random variable X s.t. $\mu = \mu_X$

Proof: (1) Let $Y \sim U([0,1])$ and $X = F_X^{-1}(Y)$ for $F_X^{-1}(u) = \sup\{x | F_X(x) \le u\}$

(2) If μ is a probability measure, let $F_\mu(a) = \mu((-\infty, a])$, then F_μ is a distribution function

If F is a distribution function, let $\mu_F((a,b]) = F(b) - F(a)$, then μ_F can be uniquely extended to a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

(3) By (1) and (2)

2 Singular Random Variable

1. Discrete random variable: It takes values only in a countable set

X has distribution mass function $f_X(x) = P(X = x)$

Distribution function: $F_X = \sum\limits_x f_X(x) I_{[x,\infty)}$ is a step function

Distribution: $\mu_X = \sum\limits_x f_X(x)\delta_x$ where $\delta_x(A) = \begin{cases} 0, x \notin A \\ 1, x \in A \end{cases}$ is the Dirac measure

Expectation: $\mathbb{E}[g(X)] = \sum\limits_x p(x)g(x)$

Entropy: Shannon Entropy $H(X) = -\sum\limits_x f_X(x) \log f_X(x)$

2. Continuous random variable: Its distribution function F is absolutely continuous

X has density function $f_X(x) \overset{a.s.}{=} F'(x)$

Distribution function: $F_X(x) = \int_{-\infty}^x f_X(t)dt$ is absolutely continuous [1]

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[1] f is absolutely continuous$\Leftrightarrow \forall \epsilon, \exists \delta$, s.t. if $\sum\limits_{i=1}^n |b_i - a_i| < \delta$, then $\sum\limits_{i=1}^n |f(b_i) - f(a_i)| < \epsilon$

If f is absolutely continuous, then f is a.e. differentiable and $f(b) - f(a) = \int_a^b f'(x)dx$ for $a < b$

Distribution: $\mu_X(A) = \int_A f(x)dx$ is absolutely continuous [2] w.r.t. Lebesgue measure

Expectation: $\mathbb{E}[g(X)] = \int_{\mathbb{R}} f_X(x)g(x)dx$

Entropy: Differential Entropy $h(X) = -\int_{\mathbb{R}} f_X(x)\log f_X(x)dx$

3. Singular random variable: Its distribution function F is continuous and $F' \overset{a.s.}{=} 0$

F and F' don't satisfy Newton-Leibniz formula

Distribution function: F continuous and $F' \overset{a.s.}{=} 0$

Distribution: μ_X is singular [3] w.r.t. Lebesgue measure

Expectation: Lebesgue-Stieltjes Integral

Entropy: can't define because it doesn't have something like distribution mass or density

Example (Cantor distribution)

$C_n = \bigcup_{k=1}^{2^n} C_{n,k}$ where $C_{n,k}$ is an interval of length $\frac{1}{3^n}$

Cantor set: $C = \lim_{n\to\infty} C_n = \{(0.a_1a_2\dots)_3 | a_i = 0 \text{ or } 2\}$

Cantor function: $c : [0,1] \to [0,1], x \mapsto \begin{cases} (0.a_1a_2\dots)_2, \text{ if } x = (0.(2a_1)(2a_2)\dots)_3 \\ \sup\limits_{y\in C, y\le x} c(y), \text{ if } x \notin C \end{cases}$

Cantor function is continuous, and $c' \overset{a.s.}{=} 0$ but it is not absolutely continuous

Cantor distribution function: $F(x) = \begin{cases} 0, x < 0 \\ c(x), x \in [0,1] \\ 1, x > 1 \end{cases}$

Cantor distribution is a singular distribution

4. **Lebesgue's Decomposition Theorem** on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$: If μ is a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then there exists a decomposition: $\mu = \mu_{cont} + \mu_{sing} + \mu_{disc}$ where $\mu_{cont}(\ll \mu_L)$[4] is the absolutely continuous part, $\mu_{sing}(\perp \mu_L)$ is the singular continuous part and μ_{disc} is the discrete part

Corollary: If X is a random variable, then there exists a decomposition: $X = X_{cont} + X_{sing} + X_{disc}$ where X_{cont} is a continuous random variable, X_{sing} is a singular random variable and X_{disc} is a discrete random variable

Example Suppose $X_0 \sim Poisson(\lambda), X_1 \sim N(0,1), Y \sim Ber(\frac{1}{2})$ and they are independent

Let $Z = X_Y = \begin{cases} X_0, \text{ if } Y = 0 \\ X_1, \text{ if } Y = 1 \end{cases}$, then Z is a mixed random variable

[2] If μ, ν are measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then μ is absolutely continuous w.r.t. ν (denoted by $\mu \ll \nu$) if $\mu(\mathbb{R}) < \infty$ and $\forall A, \nu(A) = 0 \Rightarrow \mu(A) = 0$
By Radon-Nikodym theorem, if ν is σ-finite, then $\mu \ll \nu \Leftrightarrow \exists f$ s.t. $d\mu = fd\nu$

[3] If μ, ν are measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then μ and ν are singular to each other (denoted by $\mu \perp \nu$) if $\exists E \subset \mathcal{B}(\mathbb{R})$ s.t. $\mu(E) = 0$ and $\nu(E^c) = 0$

[4] μ_L denotes the Lebesgue measure

Let $Z_0 = \begin{cases} X_0, \text{ if } Y = 0 \\ 0, \text{ if } Y = 1 \end{cases}$ and $Z_1 = \begin{cases} 0, \text{ if } Y = 0 \\ X_1, \text{ if } Y = 1 \end{cases}$, then $Z = Z_0 + Z_1$ is a Lebesgue decomposition

5. Any distribution can be weakly approximated by discrete distributions

Proof: Suppose F is a right continuous, monotone nondecreasing function and $F(-\infty) = 0, F(\infty) = 1$

Let $F_n(x) = \sum\limits_{i=-2^{2n}}^{2^{2n}-1} F(\frac{i}{2^n}) I_{[\frac{i}{2^n}, \frac{i+1}{2^n})}(x) + I_{[2^n, \infty)}(x)$, then F_n are distribution functions and $F_n \xrightarrow{W} F$

2.1 Expectation

The Lebesgue-Stieljes integral of $f(x)$ in $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$ is denoted by $\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x) dF_X = \int_{\mathbb{R}} f(x) \mu_X(dx)$

For indicators (I_A for $A \in \mathcal{B}(\mathbb{R})$): Let $\mathbb{E}[I_A(X)] = \int_X I_A(x) \mu_X(dx) = \mu_X(A)$

For simple functions ($S = \sum\limits_{i=1}^{n} a_i I_{A_i}$): Let $\mathbb{E}[S(X)] = \int_X S(x) \mu_X(dx) = \sum\limits_{i=1}^{n} a_i \mu_X(A_i)$

For non-negative measurable functions ($f \in L^+ \Rightarrow \exists f_n \in S^+$ s.t. $f_n \uparrow f$): Let $\mathbb{E}[f(X)] = \int_X f(x) \mu_X(dx) = \lim\limits_{n \to \infty} \mathbb{E}[f_n(X)]$

For measurable function ($f \in L^+ \Rightarrow f = f^+ - f^-$ for $f^+, f^- \in L^+$)

If $\mathbb{E}[f^+(X)] < \infty$ or $\mathbb{E}[f^-(X)] < \infty$, then $\mathbb{E}[f(X)]$ exists and $\mathbb{E}[f(X)] = \mathbb{E}[f^+(X)] - \mathbb{E}[f^-(X)]$

3 Sample Space

When we claim a random variable, we sometimes ignore its sample space, like we can simply say $X \sim N(0,1)$ without mentioning its sample space

For one single random variable, we can ignore its sample space if we only care about its distribution

But for several random variables, their relationship is closely related to their joint sample space.

1. Sample space is not unique

Example If we roll a die, let $X = \begin{cases} 1, \text{ the outcome is odd} \\ 2, \text{ the outcome is even} \end{cases}$, then we can let

(1) $\Omega = \{1,2,3,4,5,6\}, \mathcal{F} = 2^\Omega, P(A) = \frac{|A|}{6}$ and $X : \Omega \to \mathbb{R}, w \mapsto \begin{cases} 1, w \text{ odd} \\ 2, w \text{ even} \end{cases}$

(2) $\Omega = \{\text{odd, even}\}, \mathcal{F} = 2^\Omega, P(A) = \frac{|A|}{2}$ and $X : \Omega \to \mathbb{R}, w \mapsto \begin{cases} 1, w = \text{odd} \\ 2, w = \text{even} \end{cases}$

or we can even suppose that we roll it twice but we only care about the first toss:

(3) $\Omega = \{1,2,3,4,5,6\}^2, \mathcal{F} = 2^\Omega, P(A) = \frac{|A|}{36}$ and $X : \Omega \to \mathbb{R}, w \mapsto \begin{cases} 1, w_1 \text{ odd} \\ 2, w_1 \text{ even} \end{cases}$

2. Sample space can depict the relationship between random variables

Example Suppose $X_1, X_2, \dots$ and $X : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ are random variables

X_i, X_j are independent$\Leftrightarrow \forall A, B \in \mathcal{B}(\mathbb{R})$, we have $P(X_i^{-1}(A)X_j^{-1}(B)) = P(X_i^{-1}(A))P(X_j^{-1}(B))$

$\{X_n | n \geq 1\}$ are independent$\Leftrightarrow \forall i_1, \dots, i_n$ and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$, we have $P(X_{i_1}^{-1}(A_1) \dots X_{i_n}^{-1}(A_n)) = P(X_{i_1}^{-1}(A_1)) \dots P(X_{i_n}^{-1}(A_n)$

$X_n \xrightarrow{P} X \Leftrightarrow \forall \epsilon, P(w : |X_n(w) - X(w)| > \epsilon) \to 0$

$X_n \xrightarrow{a.s.} X \Leftrightarrow P(w : X_n(w) \to X(w)) = 1$

Example (7.2.8) If X_n are independent and $X_n \xrightarrow{P} X$, then X is almost surely constant

Proof: (1) If X is not a.s. constant, then we have $a < b$ s.t. $P(X < a) > 0$ and $P(X > b) > 0$

Then $\exists \epsilon$ s.t. $P(X < a) > \epsilon$ and $P(X > b) > \epsilon$

By $X_n \xrightarrow{P} X$, we have N s.t. $P(|X_n - X| > \frac{b-a}{4}) < \frac{\epsilon}{2}$ for $n > N$

Then $P(X_n < a + \frac{b-a}{4}) \geq P(X < a, |X_n - X| < \frac{b-a}{4}) = 1 - P(X \geq a) - P(X < a, |X_n - X| \geq \frac{b-a}{4}) > 1 - (1-\epsilon) - \frac{\epsilon}{2} = \frac{\epsilon}{2}$

Similarly, $P(X_n > b - \frac{b-a}{4}) > \frac{\epsilon}{2}$

So for $m, n > N$, we have $P(|X_m - X_n| > \frac{b-a}{2}) \geq P(X_m > b - \frac{b-a}{4}, X_n < a + \frac{b-a}{4}) > \frac{\epsilon}{4}$

(2) X_n is convergence in probability, so X_n is Cauchy convergence in probability (7.3.1)

Then $\exists N'$ s.t. for $m, n > N$, we have $P(|X_m - X_n| > \frac{b-a}{2}) < \frac{\epsilon}{4}$, contradicts.

Note: Independent and (Cauchy) convergence are somewhat opposite: independent says that different variables don't have mutual information, while Cauchy convergence says that variables in a sequence are close to each other in some extent.

4 Some Consequences about Convergence

1. On the functional space $L(\Omega, \mathcal{F}, P)$, define $d_1(X,Y) = \mathbb{E}|X-Y|$, $d_2(X,Y) = \mathbb{E}(|X-Y| \wedge 1)$, $d_3(X,Y) = \mathbb{E}(\frac{|X-Y|}{1+|X-Y|})$

(1) d_1, d_2, d_3 are metrics on Ω

(2) $X_n \xrightarrow{1} X \Leftrightarrow d_1(X_n, X) \to 0$

(3) $X_n \xrightarrow{P} X \Leftrightarrow d_2(X_n, X) \to 0 \Leftrightarrow d_3(X_n, X) \to 0$

Proof:(3) If $X_n \xrightarrow{P} X$, then $\forall \epsilon > 0$, $\exists N$ s.t. $P(|X_n - X| > \epsilon) < \epsilon$ for $n > N$

then $d_2(X_n, X) = \int_{|X_n - X| > \epsilon} |X_n - X| \wedge 1 + \int_{|X_n - X| \leq \epsilon} |X_n - X| \leq P(|X_n - X| > \epsilon) + \epsilon P(|X_n - X| \leq \epsilon) \leq \epsilon + \epsilon$ for $n > N$

then we have $d_2(X_n, X) \to 0$

Unless $X_n \xrightarrow{P} X$, then $\exists \epsilon, \delta > 0$ and subsequence X_{n_k} s.t. $P(|X_{n_k} - X| > \epsilon) > \delta$

Then $d_2(X_{n_k}, X) \geq \int_{|X_{n_k} - X| > \epsilon} |X_{n_k} - X| \wedge 1 \geq \epsilon P(|X_{n_k} - X| > \epsilon) \geq \epsilon\delta$

Then $d_2(X_n, X) \nrightarrow 0$

Note: We have $d_2, d_3 \leq d_1$, i.e., d_2, d_3 are weaker than d_1. This implies that $X_n \xrightarrow{d_1} X \Rightarrow X_n \xrightarrow{d_2} X$ and $X_n \xrightarrow{d_3} X$

2. There doesn't exists a metric d on $L(\Omega, \mathcal{F}, P)$ s.t. $X_n \xrightarrow{a.s.} X \Leftrightarrow d(X_n, X) \to 0$. Furthermore, there doesn't exists a topology on τ $L(\Omega, \mathcal{F}, P)$ s.t. $X_n \xrightarrow{a.s.} X \Leftrightarrow X_n \to X$ in τ

Lemma: In a metric/topology space, if x_n, x s.t. for any subsequence of x_n, there exists a further subsequence converges to x, then we have $x_n \to x$

Proof of Lemma: If $x_n \nrightarrow x$, then there exists an open neighborhood U of x and a subsequence x_{n_k} s.t. $x_{n_k} \notin U$

Then for the subsequence x_{n_k}, it doesn't have a further subsequence converges to x

Proof: Let $X_n = \begin{cases} 1, \text{ with probability} \frac{1}{n} \\ 0, \text{ with probability} 1 - \frac{1}{n} \end{cases}$

Then $X_n \xrightarrow{P} X$, by Egoroff theorem, for any subsequence of X_n, there exists a further subsequence almost surely convergences to X. However, we don't have $X_n \xrightarrow{a.s.} X$

3. For a continuous function $f : \mathbb{R} \to \mathbb{R}$

(1) If $X_n \xrightarrow{a.s.} X$, then $f(X_n) \xrightarrow{a.s.} f(X)$

(2) If $X_n \xrightarrow{P} X$, then $f(X_n) \xrightarrow{P} f(X)$

(3) If $X_n \xrightarrow{D} X$, then $f(X_n) \xrightarrow{D} f(X)$

Proof: (1) There exists $N \in \mathcal{F}$ s.t. $P(N) = 0$ and $X_n \to X$ on $\Omega \backslash N$

On $\Omega \backslash N$, we have $X_n(w) \to X(w)$, so $f(X_n(w)) \to f(X(w))$ since f is continuous

So $f(X_n) \xrightarrow{a.s.} f(X)$

(2) If $f(X_n) \xrightarrow{P} f(X)$ doesn't holds, then there exists $\epsilon, \tau > 0$ and $f(X_{k_n})$ s.t. $P(|f(X_{k_n}) - f(X)| > \epsilon) > \tau$

By $X_{k_n} \xrightarrow{P} X$, it has a subsequence $X_{k'_n} \xrightarrow{a.s.} X$ (Egoroff theorem)

By (1), we have $f(X_{k'_n}) \xrightarrow{a.s.} f(X)$, thus $f(X_{k'_n}) \xrightarrow{P} f(X)$. Contradicts.

(3) By Skorokhod's representation theorem, we have $(\Omega, \mathcal{F}, P)$ and $Y_n, Y : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ s.t. $Y_n \overset{D}{=} X_n, Y \overset{D}{=} Y$ and $Y_n \xrightarrow{a.s.} Y$

By (1), we have $f(Y_n) \xrightarrow{a.s.} f(Y)$, so $f(Y_n) \xrightarrow{D} f(Y)$, so $f(X_n) \xrightarrow{D} f(X)$

4. (7.12.25)

(1) If $X_n \xrightarrow{a.s.} X$ and $N_k \xrightarrow{a.s.} \infty$, then $X_{N_k} \xrightarrow{a.s.} X$

(2) If $X_n \xrightarrow{a.s.} X$ and $N_k \xrightarrow{P} \infty$, then $X_{N_k} \xrightarrow{P} X$

(3) If $X_n \xrightarrow{P} X$, $N_k \xrightarrow{P} \infty$ and X_n are independent of N_k, then $X_{N_k} \xrightarrow{P} X$

(4) If $X_n \xrightarrow{D} X$, $N_k \xrightarrow{P} \infty$ and X_n are independent of N_k, then $X_{N_k} \xrightarrow{D} X$

Proof: (1) $\exists E$ s.t. $P(E) = 0$ and $X_n \to X, N_k \to \infty$ on $\Omega \backslash E$

For $w \in \Omega \backslash E$, $\forall \epsilon > 0$, $\exists N$ s.t. $|X_n(w) - X(w)| < \epsilon$ for $n > N$, $\exists K$ s.t. $N_k(w) > n$ for $k > K$

So $|X_{N_k}(w) - X(w)| < \epsilon$ for $k > K$

So $X_{N_k}(w) \to X(w)$

So $X_{N_k} \to X$ on $\Omega \backslash E$

(2) $P(|X_{N_k} - X| > \epsilon) \leq P(N_k \leq n) + P(|X_{N_k} - X| > \epsilon, N_k > n) \leq P(N_k \leq n) + P(\sup_{m \geq n} |X_m - X| > \epsilon)$

Let $k \to \infty$, we have $\lim_{k \to \infty} P(|X_{N_k} - X| > \epsilon) \leq P(\sup_{m \geq n} |X_m - X| > \epsilon)$

Let $n \to \infty$, we have $\lim_{k \to \infty} P(|X_{N_k} - X| > \epsilon) \leq 0$

(3) $\forall \epsilon > 0, \exists N$ s.t. $P(|X_n - X| > \epsilon) < \delta$ for $n > N$, $\exists K$ s.t. $P(N_k > N) < \delta$ for $k > K$

$P(|X_{N_k} - X| > \epsilon) \leq P(N_k \leq N) + \sum_{n=N+1}^{\infty} P(|X_{N_k} - X| > \epsilon, N_k = n) = P(N_k \leq N) + \sum_{n=N+1}^{\infty} P(|X_n - X| > \epsilon) P(N_k = n) \leq 2\delta$

$\lim_{k\to\infty} P(|X_{N_k} - X| > \epsilon) = 0$

(4) Let $\phi_n(t) = \mathbb{E}[e^{iX_n t}]$, $\phi(t) = \mathbb{E}[e^{iXt}]$ and $\psi_k(t) = \mathbb{E}[e^{iX_{N_k}}(t)]$

$|\psi_k(t) - \phi(t)| = |\sum_{n=1}^{\infty} P(N_k = n)\phi_n(t) - \phi(t)| \leq \sum_{n=1}^{\infty} P(N_k = n)|\phi_n(t) - \phi(t)|$

$\exists N$ s.t. $|\phi_n(t) - \phi(t)| < \epsilon$ for $n > N$, $\exists K$ s.t. $P(N_k \leq N) < \frac{\epsilon}{M}$ for $k > K$

For $k > K$, we have $|\psi_k(t) - \phi(t)| \leq \frac{\epsilon}{M} \max_{1 \leq n \leq N} |\phi_n(t) - \phi(t)| + P(N_k > N)\epsilon$

Let $M = 1/\max_{1 \leq n \leq N} |\phi_n(t) - \phi(t)|$, we have $|\psi_k(t) - \phi(t)| < 2\epsilon$, so $\psi_k(t) \to \phi(t)$

1. 证明：(1) 标准正态分布被其矩序列决定，

(2) 求半圆律 $p(x)=\frac{1}{2\pi}\sqrt{4-x^2}$，$x\in[-2,2]$ 各阶矩．

(3) 且其各阶矩决定分布．

(4) 序列 $\gamma_{2k+1}=0$，$\gamma_{2k}=1$，是否对应随机变量矩序列？

pf: (1) 设 $X\sim N(0,1)$，则

$$EX^{2k}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}x^{2k}e^{-\frac{x^2}{2}}dx=\sqrt{\frac{2}{\pi}}\int_0^{+\infty}x^{2k}e^{-\frac{x^2}{2}}dx,\quad y=x^2$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^{+\infty}y^{k-\frac{1}{2}}e^{-\frac{y}{2}}dy=\sqrt{\frac{2}{\pi}}\frac{1}{2}\cdot 2\cdot 2^{k-\frac{1}{2}}\int_0^{+\infty}y^{(k+\frac{1}{2})-1}e^{-y}dy,\quad x=\sqrt{y}$$

$$=\frac{1}{\sqrt{\pi}}2^k\,\Gamma(k+\tfrac{1}{2})=\frac{1}{\sqrt{\pi}}2^k\cdot(k-\tfrac{1}{2})(k-\tfrac{3}{2})\cdots\tfrac{1}{2}\,\Gamma(\tfrac{1}{2})\qquad dx=\frac{dy}{2\sqrt{y}}$$

$$=(2k-1)!!\qquad \Gamma(\tfrac{1}{2})=\sqrt{\pi}.$$

$$(2k-1)!!=\frac{(2k)!}{2^k k!}\overset{\text{stirling}}{\sim}2^{-k}\frac{1}{\sqrt{2\pi k}}\left(\frac{e}{k}\right)^k\cancel{\frac{1}{\sqrt{2\pi\cdot 2k}}}\sqrt{2\pi\cdot 2k}\left(\frac{2k}{e}\right)^{2k}$$

$$=2^{-k}\sqrt{2}\,e^{-k}\cdot 2^{2k}\cdot k^k$$

$$\therefore\ \frac{1}{2k}\left[(2k-1)!!\right]^{\frac{1}{2k}}\sim\frac{1}{2k}\sqrt{2}\,e^{-\frac{1}{2}}\sqrt{k}\to 0.\quad k\to+\infty.$$

自然，$\limsup_k \frac{1}{2k}(\gamma_{2k})^{\frac{1}{2k}}<+\infty$．　　done.

(2). 由对称性，$\gamma_{2k+1}=0$．

$$\gamma_{2k}=\frac{1}{2\pi}\int_{-2}^{2}x^{2k}\sqrt{4-x^2}\,dx=\frac{1}{\pi}\int_0^2 x^{2k}\sqrt{4-x^2}\,dx,\qquad x^2=4r.$$

$$=\frac{2^{2k+1}}{\pi}\int_0^1 r^{(k+\frac{1}{2})-1}\sqrt{1-r}\,dr=\frac{2^{2k+1}}{\pi}B(k+\tfrac{1}{2},\tfrac{3}{2}).\qquad dx=\frac{dr}{\sqrt{r}}$$

$$=\frac{2^{2k+1}}{\pi}\frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(k+2)}=\frac{1}{k+1}\cdot\frac{(2k)!}{(k!)^2}=\frac{1}{k+1}\binom{2k}{k}.$$

(3) $\gamma_{2k}=\frac{1}{k+1}\frac{(2k)!}{(k!)^2}\sim\frac{1}{k+1}\frac{1}{2\pi k}\left(\frac{e}{k}\right)^{2k}\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}$

$$=\frac{1}{k+1}\frac{1}{\sqrt{\pi k}}2^{2k}.$$

易知：$\limsup_k \frac{1}{2k}(\gamma_{2k})^{\frac{1}{2k}}=0<+\infty$.

(3). $P(X=\pm1)=\frac{1}{2}$.　　satisfies.

2. 设 $\{X_k\}$ 为 i.i.d. 随机变量. $EX_1=0$. $\mathrm{Var}\, X_1=1$.

$E|X_1|^3<+\infty$. 试用 Lindeberg 替换原理 证明

$\forall\ t\in\mathbb{R}$, $|P(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}X_k\le t)-\Phi(t)|=O(n^{-\frac{1}{8}})$

Pf: ①. $Y_1, Y_2, \cdots, Y_n, \cdots$ i.i.d. $G\sim N(0,1)$ 与 $\{X_k\}$ 独立

let $Z_n=\frac{1}{\sqrt{n}}(X_1+\cdots+X_n)$, $W_n=\frac{1}{\sqrt{n}}(Y_1+Y_2+\cdots+Y_n)\overset{d}{=}G\sim N(0,1)$.

要证. $E\varphi(Z_n)=E\varphi(G)+O(\frac{1}{\sqrt{n}}E|X|^3\sup_{x\in\mathbb{R}}|\varphi'''(x)|)$ —— (*).

Now, $E[\varphi(Z_n)-\varphi(W_n)]=-\sum_{i=0}^{n-1}E[\varphi(Z_{n,i})-\varphi(Z_{n,i+1})]$

$Z_{n,i}:=\frac{1}{\sqrt{n}}(X_1+\cdots+X_i+Y_{i+1}+\cdots+Y_n)$

令. $Z_{n,i}=S_{n,i}+\frac{Y_{i+1}}{\sqrt{n}}$, $Z_{n,i+1}=S_{n,i}+\frac{X_{i+1}}{\sqrt{n}}$. $S_{n,i}=\frac{X_1+\cdots+X_i+Y_{i+2}+\cdots+Y_n}{\sqrt{n}}$

由 Taylor 展开,
$$\begin{cases}\varphi(Z_{n,i})=\varphi(S_{n,i})+\varphi'(S_{n,i})Y_{i+1}\frac{1}{\sqrt{n}}+\frac{1}{2}\varphi''(S_{n,i})Y_{i+1}^2\frac{1}{n}\\ \quad+O(|Y_{i+1}|^3/n^{\frac{3}{2}}\cdot\sup_{x\in\mathbb{R}}|\varphi'''(x)|)\\ \varphi(Z_{n,i+1})=\varphi(S_{n,i})+\varphi'(S_{n,i})X_{i+1}\frac{1}{\sqrt{n}}+\frac{1}{2}\varphi''(S_{n,i})\frac{X_{i+1}^2}{n}\\ \quad+O(E|X_{i+1}|^3\sup_{x\in\mathbb{R}}|\varphi'''(x)|/n^{\frac{3}{2}})\end{cases}$$

$\because$ $EX_{i+1}=EY_{i+1}=0$, $EX_{i+1}^2=EY_{i+1}^2=1$,

$\therefore$ $E[\varphi(Z_{n,i})-\varphi(Z_{n,i+1})]=O(E|Y_{i+1}|^3\sup_{x\in\mathbb{R}}|\varphi'''(x)|/n^{\frac{3}{2}})$

$+O(E|X_{i+1}|^3\sup_{x\in\mathbb{R}}|\varphi'''(x)|/n^{\frac{3}{2}})=O(E|X|^3\sup_{x\in\mathbb{R}}|\varphi'''(x)|/n^{\frac{3}{2}})$

($E|G|^3=2^{\frac{3}{2}}\pi^{-\frac{1}{2}}=O(1)$, $1=E|X|^2\le\{E|X|^3\}^{\frac{3}{3}}$.).

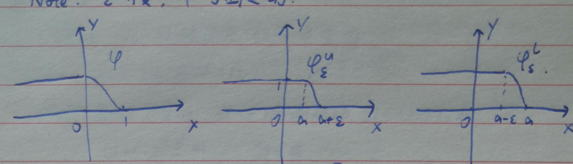
对 i 求和即得 (*).

②. Claim: $\exists\ \varphi\in C^3(\mathbb{R})$. s.t. $\begin{cases}\varphi=1. & x\le 0\\ \varphi=0. & x>1\\ 0\le\varphi\le 1, & 0\le x\le 1.\end{cases}$ of course: $\sup_{x\in\mathbb{R}}\{|\varphi(x)|+|\varphi'(x)|+|\varphi''(x)|+|\varphi'''(x)|\}<+\infty$.

令. $\varphi_\varepsilon^u(x)\triangleq\varphi(\frac{x-a}{\varepsilon})$. $\varphi_\varepsilon^l(x)\triangleq\varphi(\frac{x-a+\varepsilon}{\varepsilon})$.

$\varphi_\varepsilon^u, \varphi_\varepsilon^l$ 满足： $\sup_{x\in\mathbb{R}}\{|[\varphi_\varepsilon^u(x)]'''| + |[\varphi_\varepsilon^l(x)]'''|\} = O(\varepsilon^{-3})$.

Note：ε 可变，φ 为固定的.



1°. $P(Z_n \le a) \le E\varphi_\varepsilon^u(Z_n) \overset{①}{\le} E\varphi_\varepsilon^u(G) + O(\frac{1}{\sqrt{n}}E|X|^3\varepsilon^{-3})$.

同时 $E\varphi_\varepsilon^u(G) = P(G \le a) + O(\varepsilon)$.

$\left(E\underbrace{\varphi_\varepsilon^u(G)}_{\le 1} I(a<G<a+\varepsilon) \le P(a<G<a+\varepsilon) = \frac{1}{\sqrt{2\pi}}\int_a^{a+\varepsilon} \underbrace{e^{-\frac{x^2}{2}}}_{\le 1} dx \le \frac{\varepsilon}{\sqrt{2\pi}} \right)$.

$\therefore P(Z_n \le a) \le P(G \le a) + O(\varepsilon + \frac{1}{\sqrt{n}}E|X|^3\varepsilon^{-3})$.

这里，我们认定 $O(\cdot)$ 的常数 C 为大于 0 的.

2°. $P(Z_n \le a) \ge E\varphi_\varepsilon^l(Z_n) \ge E\varphi_\varepsilon^l - O(\frac{1}{\sqrt{n}}E|X|^3\varepsilon^{-3})$

$\ge P(G \le a) - O(\varepsilon) - O(\frac{1}{\sqrt{n}}E|X|^3\varepsilon^{-3})$.

Therefore, $P(Z_n \le a) - P(G \le a) = O(\varepsilon + \frac{1}{\sqrt{n}}E|X|^3\varepsilon^{-3})$. $\forall \varepsilon > 0$

根据以上分析，$O(\cdot)$ 的 bound 与 ε 无关.

令 $\varepsilon = n^{-\frac{1}{8}}(E|X|^3)^{\frac{1}{4}} \cdot 3^{\frac{1}{4}}$

$\varepsilon + \frac{1}{\sqrt{n}}E|X|^3\varepsilon^{-3} = (3^{\frac{1}{4}} + 3^{-\frac{3}{4}})(E|X|^3)^{\frac{1}{4}} n^{-\frac{1}{8}}$. done.

(In fact, let $f(\varepsilon) = \varepsilon + a\varepsilon^{-3}$, $0<\varepsilon<1$, $0<a<\frac{1}{100}$,

Then $f_{min}(\varepsilon) = f((3a)^{\frac{1}{4}})$.)

3. GUE. $X=(x_{ij})_{i,j=1}^{n}\in\mathbb{C}^{n\times n}$. $\{\mathrm{Re}\,x_{ij},\ \mathrm{Im}\,x_{ij}\}_{i,j=1}^{n}$ 为 i.i.d $N(0,1)$.

令. $H=\frac{1}{2}(X+X^{+})$. $X^{+}=(\overline{X^{t}})$.

试证 (i) H 矩阵元联合密度为:

$$f(H)=\frac{1}{(\sqrt{2\pi})^{n^2}}(\sqrt{2})^{n(n-1)}e^{-\frac{1}{2}\mathrm{tr}H^2}.$$

(ii) 酉群不变性, 即任给酉矩阵 $U\in U(n)$, 有:

$UHU^{+}\overset{d}{=}H$. 同分布.

(i). $H=(h_{ij})_{i,j=1}^{n}=H^{+}$, 其独立元为: $\{h_{ii}\}_{i=1}^{n}\subseteq\mathbb{R}$,

$\{h_{ij}\}_{1\le i<j\le n}\subseteq\mathbb{C}$, $h_{ii}=\frac{1}{2}(x_{ii}+\bar{x}_{ii})=\mathrm{Re}\,x_{ii}.\sim N(0,1)$

$$h_{ij}=\frac{1}{2}(x_{ij}+\bar{x}_{ji})=\frac{1}{2}(\mathrm{Re}\,x_{ij}+\mathrm{Re}\,x_{ji})+\frac{\sqrt{-1}}{2}(\mathrm{Im}\,x_{ij}-\mathrm{Im}\,x_{ji})$$

$$\left.\begin{aligned}\mathrm{Re}\,h_{ij}&=\tfrac{1}{2}(\mathrm{Re}\,x_{ij}+\mathrm{Re}\,x_{ji})\sim N(0,\tfrac{1}{2})\\ \mathrm{Im}\,h_{ij}&=\tfrac{1}{2}(\mathrm{Im}\,x_{ij}-\mathrm{Im}\,x_{ji})\sim N(0,\tfrac{1}{2})\end{aligned}\right\}\text{独立性.}$$

$$\therefore f(H)=\prod_{i=1}^{n}f_{h_{ii}}(h_{ii})\prod_{1\le i<j\le n}f_{\mathrm{Re}\,h_{ij}}(\mathrm{Re}\,h_{ij})\,f_{\mathrm{Im}\,h_{ij}}(\mathrm{Im}\,h_{ij}).$$

$$=e^{-\sum_{i=1}^{n}h_{ii}^2/2}\left(\frac{1}{\sqrt{2\pi}}\right)^{n}\times\left(\frac{1}{\sqrt{\pi}}\right)^{n(n-1)}e^{-\sum_{i<j}|h_{ij}|^2}$$

$$=\frac{1}{(\sqrt{2\pi})^{n^2}}(\sqrt{2})^{n(n-1)}e^{-\sum_{i=1}^{n}\frac{1}{2}h_{ii}^2-\frac{1}{2}\sum_{i\neq j}|h_{ij}|^2}=C_n e^{-\mathrm{tr}H^2}.\ \text{done.}$$

(ii) 证明本质上是利用. P_{117}. Thm 4.8.6 的复值情形.

如果要严格证, 就要从最基本的特征函数开始.

(i.e. if X is as in 3. Then so is UX, fixed $U\in U(n)$)

这里引入一个系统的处理问题的方法: Matrix-valued variables, transformation. via. Jacobians. nothing new.

by elementary mathematical analysis, we can get. the following results:

Definition:

① $X \in R^{m\times n}$. $dX := \begin{bmatrix} dx_{11} & \cdots & dx_{1n} \\ \vdots & & \vdots \\ dx_{m1} & \cdots & dx_{mn} \end{bmatrix}$. if $X = (x_{ij})_{i=1, j=1}^{m\ \ n}$

$[dX] := \prod_{i=1}^{m} \prod_{j=1}^{n} dx_{ij}$, (Compare with multi-variable integration).

② let $m=n$. $X = X^t$, $[dX] := \prod_{i \le j} dx_{ij}$. (independent variable).

③ $\tilde{X} \in \mathbb{C}^{n\times n}$. $\tilde{X} = X_1 + \sqrt{-1} X_2$, $X_1, X_2 \in R^{n\times n}$.

$[d\tilde{X}] := [dX_1][dX_2]$ For general complex matrix $\tilde{X}$,

$[dX_1] = \prod_{i,j} d(X_1)_{ij}$ $[dX_2] = \prod_{i,j=1}^{n} d(X_2)_{ij}$.

④ but for Hermitian. $\tilde{X} = \tilde{X}^+$

$[dX_1] = \prod_{i \ge j} d(X_1)_{ij}$. $[dX_2] = \prod_{i>j} d(X_2)_{ij}$. because $X_1^t = X_1$, $X_2^t = -X_2$.

Now properties:

zenzen no restrictions → ① $X, Y \in R^{m\times n}$, matrix of variables. for fixed $A \in R^{m\times m}$, $B \in R^{n\times n}$.

If $Y = AXB$. then:

$$[dY] = (\det A)^n (\det B)^m [dX].$$

restrictions come → ②. X, Y now be $n\times n$ real symmetric. for fixed $A \in R^{n\times n}$,

If $Y = AXA^T$, then:

$$[dY] = (\det A)^{n+1} [dX].$$

③ if in ② instead, X, Y, real skew-symmetric, then:

if $Y = AXA^T$, $[dY] = (\det A)^{n-1} [dX]$.

↑real
↓complex

④ $X, Y \in \mathbb{C}^{m\times n}$, matrix of completely independent variables, for fixed $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$. if $Y = AXB$.

then $[dY] = |\det(AA^+)|^n |\det(BB^+)|^m [dX]$.

⑤. let $X \in \mathbb{C}^{n\times n}$, hermitian. For fixed $A \in \mathbb{C}^{n\times n}$.

If $Y = AXA^+$. ($Y = Y^+$, surely). then:

$$[dY] = |\det(AA^+)|^n [dX]$$

For more results. (Jacobians of various matrix decomposition, we do not state here).

Come back:

H GUE. i.e. $P(H) = C_n e^{-\frac{1}{2}\mathrm{tr} H^2}$.

let $H_1 = UHU^+$. Then $H = U^+ H_1 U$.

P108. Thm 4.7.3. $[dH] = |\det(UU^+)|^n [dH_1] = [dH_1]$. i.e. Jacobian equals 1.

换元公式, Then: $P(H)[dH] = P(U^+ H_1 U)[dH_1]$.

$= C_n e^{-\frac{1}{2}\mathrm{tr}(U^+ H_1 U)^2}[dH_1] = C_n e^{-\frac{1}{2}\mathrm{tr}(H_1)^2}[dH_1]$.

i.e. $P(H) = P(H_1)$. 即. H 与 $H_1 = U^+ H U$. 同分布.

4. 若 X 与 Y 独立, (X, Y) 旋转不变, 即 $\forall \in [0, 2\pi)$.
$e^{i\theta}(X+iY)$ 与 $X+iY$ 同分布, 试证:
X 与 Y 必为零均值且有相同方差正态分布.

pf: (我们假定 X, Y 的一阶矩均存在).

由 P183. Thm (4) (b). 可知. $\phi_X'(t)$, $\phi_Y'(t)$ 均∃.

① 首先作一个观察, 后边要用, 令 $\theta = \pi$. $\Rightarrow X+iY \overset{d}{=} -X-iY$.

$\therefore X \overset{d}{=} -X$, $\phi_X(t) = Ee^{itX} = E\cos tX + \sqrt{-1}\, E\sin tX = E\cos tX) \in R$.

同理 $\phi_Y(t)$, $\phi_X'(t)$, $\phi_Y'(t)$. $t \in R$.

② $e^{i\theta}(X+iY) = (X\cos\theta - Y\sin\theta) + \sqrt{-1}\,(X\sin\theta + Y\cos\theta) \triangleq X_1 + iY_1$

$\because (X, Y) \overset{d}{=} (X_1, Y_1)$.

$\therefore \phi_{(X,Y)}(t_1, t_2) = \phi_{X_1, Y_1}(t_1, t_2)$.

See Durret.

P_{32}. (thm 1.6.9) $Ef(X) = \int_\Omega f(X(w)) P(dw) = \int_S f(x) P_X(dx)$ if $X(w): \Omega \to S$.

i.e. $\phi_{X,Y}(t_1, t_2) = \phi_{X_1, Y_1}(t_1, t_2) = \int_\Omega e^{i(t_1,t_2)(X_1,Y_1)(w)} \, dP(w)$ $\cancel{dP_{(X_1,Y_1)}(w)}$ $= \cancel{dP_{(X,Y)}(w)}$

$= \int_\Omega e^{i(t_1,t_2)\cdot(X\cos\theta - Y\sin\theta,\ X\sin\theta + Y\cos\theta)(w)} \, dP(w)$ $\cancel{dP_{(X_1,Y_1)}(w)}$

$= \int_\Omega e^{i(t_1\cos\theta + t_2\sin\theta,\ -t_1\sin\theta + t_2\cos\theta)\cdot(X,Y)(w)} \, dP(w)$ $\cancel{dP_{(X,Y)}(w)}$

$= \phi_{(X,Y)}(t_1\cos\theta + t_2\sin\theta, -t_1\sin\theta + t_2\cos\theta)$.

利用X, Y 独立:

$$\phi_X(t_1)\phi_Y(t_2) = \phi_X(\underbrace{t_1\cos\theta + t_2\sin\theta}_{t_1'})\phi_Y(\underbrace{-t_1\sin\theta + t_2\cos\theta}_{t_2'}).$$

两边作运算. $\frac{\partial}{\partial\theta}\log$, get:

$$0 = \frac{1}{\phi_X(t_1')}\phi_X'(t_1')[-\sin\theta\, t_1 + \cos\theta\, t_2] + \frac{1}{\phi_Y(t_2')}\phi_Y'(t_2')[\overset{-t_1\cos\theta - t_2\sin\theta}{\cancel{-t_1\sin\theta + t_2\cos\theta}}].$$

令 $\theta = 0$. then:

$$\frac{1}{t_1\phi_X(t_1)}\phi_X'(t_1) = \frac{1}{t_2\phi_X(t_2)}\phi_X'(t_2). \quad \forall\, t_1, t_2 \in R.$$

$\therefore \exists\, a \in R$. (by ①). s.t.

$$\frac{1}{t_1\phi_X(t_1)}\phi_X'(t_1) = \frac{1}{t_2\phi_X(t_2)}\phi_X'(t_2) = a$$

$\Rightarrow$. $\phi_X(t_1) = e^{\frac{at_1^2}{2} + b_1}$, $\phi_X(t_2) = e^{\frac{at_2^2}{2} + b_2}$

$\because \phi_X(0) = \phi_Y(0) = 1$. $\therefore b_1 = b_2 = 0$.

又 $\because |\phi_X(t_1)| \vee |\phi_Y(t_2)| \le 1$. $\therefore a \le 0$.

when $a = 0$. X, Y 均为原点平凡测度, 经检查. OK.

when $a < 0$. X, Y. 同分布 0均值 Gauss分布. 经检查. OK.

Now, done.

5. $X=(X_{ij})_{1\le i\le p,1\le j\le n}$，$\{X_{ij}\}$ 独立同 $N(0,1)$.

假设 $d=n-p\ge 0$，α 固定，证明：

$$\frac{1}{p}E\left[Tr\left(\frac{XX^t}{p}\right)^m\right]\longrightarrow C_m=\frac{1}{m+1}C_{2m}^m$$ when $m=2$.

pf：当 $m=2$ 时，$C_2=2$..

对一般 m，$X_{i_1j_1}$ 与 $X_{i_2j_2}$ 相同当且仅当 $i_1=i_2$，$j_1=j_2$.

$$\therefore \frac{1}{p}E\,tr\left(\frac{XX^t}{p}\right)^m=\frac{1}{p^{m+1}}\sum_{i_1,\dots,i_m}^{p}E\,a_{i_1i_2}\cdots a_{i_mi_1}$$

$$=\frac{1}{p^{m+1}}\sum_{\substack{1\le i_1,\dots,i_m\le p\\1\le j_1,\dots,j_m\le n}}(X_{i_1j_1}X_{i_2j_1})\cdots(X_{i_mj_m}X_{i_1j_m})$$

这里，$A=XX^t=(a_{ij})_{1\le i,j\le p}$，$a_{ij}=\sum_{k=1}^{n}x_{ik}x_{jk}$.

when $m=2$，$\frac{1}{p}E\,tr\left(\frac{XX^t}{p}\right)^2=\frac{1}{p^3}\sum_{\substack{1\le i_1,i_2\le p\\1\le j_1,j_2\le n}}E(X_{i_1j_1}X_{i_2j_1}X_{i_2j_2}X_{i_1j_2})$

(1). 若，i_1,i_2,j_1,j_2 互不相同，则：$X_{i_1j_1}$，$X_{i_2j_1}$，$X_{i_2j_2}$，$X_{i_1j_2}$ 独立.

$\therefore E(\cdots)=0$.

(2) 若，i_1,i_2,j_1,j_2 的自由度 ≤ 2，则 极限 $=0$，$p\to+\infty$.

(因为求和的项为 pn，$\frac{pn}{p^3}\to 0$)

(3) 只需考虑 i_1,i_2,j_1,j_2 有3个不同指标的情形.

(1°). 若，$i_1=i_2$，则：$E(\cdots)=EX_{i_1j_1}^2X_{i_1j_2}^2$. i_1,j_1,j_2 互不相同.

此时极限为：$\lim_{p\to+\infty}\frac{1}{p^3}p(n-1)(n-2)=1$.

(2°). 若 $i_1\ne i_2$，则 $X_{i_1j_1}$ 与 $X_{i_2j_1}$，$X_{i_2j_2}$ 独立，

$\therefore$ 必然有 $X_{i_1j_1}=X_{i_1j_2}$，$\therefore j_1=j_2$.

此时，$E(\cdots)=EX_{i_1j_1}^2X_{i_2j_1}^2$，$i_1,i_2,j_1$ 互不相同，

$\therefore$ 此时极限为：$\lim_{p\to+\infty}\frac{1}{p^3}p(p-1)(n-2)=1$.

总体上：$1+1=2$.

done.

6. 对所定义的实 Wigner 矩阵，令：

$$\|A_n\|_2 = \sup_{v \in \mathbb{R}^n, \|v\|=1} \|A_n v\|.$$

试证：$\lim_{n\to+\infty} P(\|A_n\| \geq n^{\frac{1}{2}+\delta}) = 0$. $\forall\ \delta > 0$.

pf: ① ~~$\|A_n\| = |\lambda|_{max}$~~. $\|A_n v\|^2 = v^t A_n^t A_n v$

$$= v^t O \begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{pmatrix} O^T v. \quad A_n = O \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} O^T$$

正交分解.

注意到：$\|O^t v\|^2 = v^t O O^t v = v^t v = \|v\|^2 = 1$.

$\therefore \|A_n\|^2 = \sup_{v \in \mathbb{R}^n, \|v\|=1} (v_1^2 \lambda_1^2 + \cdots + v_n^2 \lambda_n^2) = |\lambda|_{max}^2$.

$|\lambda|_{max}^2 := \max\{|\lambda_1|^2, \cdots, |\lambda_n|^2\}$

② $\frac{1}{n} E\left(\frac{|\lambda|_{max}}{\sqrt{n}}\right)^{2k} \leq \frac{1}{n^{k+1}} E[\lambda_1^{2k} + \cdots + \lambda_n^{2k}] = \frac{1}{n} E\left[tr\left(\frac{A_n}{\sqrt{n}}\right)^{2k}\right]$

$\to f(k)$. finite number. $\forall\ k \in N^*$.

③ 对 $\forall\ \delta > 0$. 取. $k \in N^*$. s.t. $2k\delta > 1$.

$P(\|A_n\| \geq n^{\frac{1}{2}+\delta}) = P\left(\frac{|\lambda|_{max}}{\sqrt{n}} \geq n^{\delta}\right) \leq \frac{1}{n^{2k\delta}} E\left(\frac{|\lambda|_{max}}{\sqrt{n}}\right)^{2k}$

$= \frac{1}{n^{2k\delta-1}} \cdot \frac{1}{n} E\left(\frac{|\lambda|_{max}}{\sqrt{n}}\right)^{2k}$

由 ② 知：$\limsup_{n} P(\|A_n\| \geq n^{\frac{1}{2}+\delta}) = 0$.

7. Hermite Wigner 矩阵.

设 $A_n = (a_{ij})_{i,j=1}^{n}$ $A_n = A_n^{+}$ (Hermitian) 满足：

①. 实值. $\{a_{ii}\}$ 独立同与 Y 同分布.

② . $\{Re\, a_{ij}, Im\, a_{ij}\}_{i<j}$ 独立与 Z 同分布.

③ . $\{a_{ii}: 1 \leq i \leq n\}$ 与 $\{Re\, a_{ij}, Im\, a_{ij}, 1 \leq i < j \leq n\}$ 独立.

④. $EY=0$, $EZ=0$, $EY^2<+\infty$, $EZ^2=\frac{1}{2}$.

⑤. $\forall k\geq 3$, $E|Y|^k$, $E|Z|^k<+\infty$,

试证明：$\frac{1}{n}E[tr(\frac{A_n}{\sqrt{n}})^k]\to \gamma_k=\int_{-2}^{2}x^k\frac{1}{2\pi}\sqrt{4-x^2}\,dx$

pf: 由第一题，$\gamma_{2k-1}=0$，$k\in N^*$，$\gamma_{2k}=\frac{1}{k+1}\binom{2k}{k}$

(1). 令 $B=A_n^k=(b_{ij})_{i,j=1}^{n}$，$b_{ij}=\sum_{i_2,\cdots,i_k=1}^{n}a_{ii_2}a_{i_2i_3}\cdots a_{i_kj}$.

$\therefore tr A_n^k=\sum_{i_1=1}^{n}b_{i_1i_1}=\sum_{i_1,i_2,\cdots,i_k=1}^{n}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$.

$\therefore I_n^k=\frac{1}{n}E[tr(\frac{A_n}{\sqrt{n}})^k]=\frac{1}{n^{1+\frac{k}{2}}}\sum_{i_1,i_2,\cdots,i_k=1}^{n}E\,a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}$

(2). 对 $k=1,2,3,4$ 进行讨论.

1°. $k=1$. $\gamma_1=0$. $I_n^1=n^{-\frac{3}{2}}\sum_{i_1=1}^{n}E\,a_{i_1i_1}=0$.

2°. $k=2$. $\gamma_2=1$, $I_n^2=n^{-2}\sum_{i_1,i_2=1}^{n}E\,a_{i_1i_2}a_{i_2i_1}=n^{-2}\sum_{i_1,i_2=1}^{n}E|a_{i_1i_2}|^2$

$=n^{-2}\sum_{i_1,i_2=1}^{n}E[(Re\,a_{i_1i_2})^2+(Im\,a_{i_1i_2})^2]=n^{-2}(\sum_{i=1}^{n}E(a_{ii})^2+\sum_{i\neq j}E[(Re\,a_{ij})^2+(Im\,a_{ij})^2]$

$=n^{-2}[nEY^2+\sum_{i\neq j}2EZ^2]=n^{-1}EY^2+\frac{n-1}{n}\to 1$. $n\to+\infty$.

3°. $k=3$. $\gamma_3=0$. $I_n^3=n^{-\frac{5}{2}}\sum_{i_1,i_2,i_3=1}^{n}E\,a_{i_1i_2}a_{i_2i_3}a_{i_3i_1}$

$\because 2<\frac{5}{2}<3$. $\therefore$ 要使 贡献项不为 0，$n\to+\infty$ 时，~~$\lim_{n\to+\infty}I_n^3\neq 0$；必须~~.

i_1,i_2,i_3 互不相同，则：$a_{i_1i_2}$, $a_{i_2i_3}$, $a_{i_3i_1}$ 独立.

$\therefore E\,a_{i_1i_2}a_{i_2i_3}a_{i_3i_1}=0$. $\therefore \lim_{n\to+\infty}I_n^3=0$.

4°. $k=4$. $\gamma_4=\frac{1}{3}\binom{4}{2}=2$. $I_n^4=n^{-3}\sum_{i_1,i_2,i_3,i_4=1}^{n}E\,a_{i_1i_2}a_{i_2i_3}a_{i_3i_4}a_{i_4i_1}]$

以下称 自由指标个数 为 i_1,i_2,i_3,i_4 不同的个数.

若自由指标数 ≤ 2. 则 $n\to+\infty$ 时，贡献为 0.

若自由指标数为 4. 则 $E(\cdots)=0$.

$\therefore$ 非平凡贡献当且仅当 自由指标数为 3.

①. 若有相邻指标相同. 不妨设 $i_1=i_2$.

$E(a_{i_1i_1}a_{i_1i_3}a_{i_3i_4}a_{i_4i_1})=E\,a_{i_1i_1}\,E(\cdots)=0$.

② ∵ 只剩下两种情形

(1°). $i_1=i_3$，i_1, i_2, i_4 互不相同：

$E(a_{i_1i_2}a_{i_2i_3}a_{i_3i_4}a_{i_4i_1}) = E|a_{i_1i_2}|^2 E|a_{i_3i_4}|^2 = (\frac{1}{2}+\frac{1}{2})\cdot 1 = 1$.

(2°). $i_1 \neq i_3$. 即 i_1 与 i_2, i_3, i_4 均不等.

考虑 i_2，若 $i_2 \neq i_4$，则 i_1, i_2, i_3, i_4 互不相同，则 $E(\cdots)=0$.

若 $i_2=i_4$，由轮换对称性，$E(\cdots)=1$.

$\therefore I_n^4 = n^{-3}\sum_{\substack{i_1\neq i_2, i_4\\ i_1=i_3}} 1 + n^{-3}\sum_{\substack{i_1\neq i_3\\ i_2=i_4}} 1 = n^{-3}\times 2n(n-1)(n-2) \to 2$. $n\to+\infty$.

(3) 对一般的 k. $I_n^k = \frac{1}{n^{1+\frac{k}{2}}}\sum_{i_1,\cdots,i_k} E(a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1})$.

1°. 若 $k=2m+1$. $m\in N^*$.

$\because EY=EZ=0$. $\therefore$ 要使 $E(\cdots)\neq 0$，必须有：

$\{i_1i_2, i_2i_3, \cdots, i_ki_1\}$ k 边集的不同边个数 $\leq m$，

由于对应的图为连通的，$\therefore$ 不同自由指标个数 $\leq m+1$.

而 $n^{-(1+\frac{k}{2})} = n^{-1-m-\frac{1}{2}}$，由于 Y, Z 高阶矩均 ∃.

$\therefore \lim_{n\to+\infty} I_n^k = 0$.

2°. 若 $k=2m$. $m\in N^*$. 同 1° 中讨论.

当且仅当自由指标数为 $m+1$，此时 $E(\cdots)$ 中不同边的个数恰为 $m=\frac{k}{2}$.

即每条边(非定向)恰好出现 2 次. 注意到 circle 是不允许出现的.

(为使 m 边对应 $m+1$ 个顶点，满足条件的连通图必然为树).

$\therefore E(\cdots) = E|a_{\alpha\beta}|^2\cdots E|a_{\omega\gamma}|^2 = (\frac{1}{2}+\frac{1}{2})\cdots(\frac{1}{2}+\frac{1}{2}) = 1$.

$\therefore$ 只需计算 $\#\{(i_1,\cdots,i_k) \mid \{i_1,i_2\}, \{i_2,i_3\}, \cdots \{i_k,i_1\}$ 恰有 m 条不同边$\}$

我们将满足这样条件的 $(i_1,\cdots,i_k)\in[n]^k$ 分类：

对 ∀ 边 $\{\{i_1,i_2\}, \{i_2,i_3\}, \cdots, \{i_{2m-1},i_{2m}\}, \{i_{2m},i_1\}\}$. 定义：

$$a_j = \begin{cases} 1. & 若 \{i_j, i_{j+1}\} 第一次出现. \\ -1. & \text{else}. \end{cases}$$

令 $S_0 = 0$. $S_i = S_{i-1} + a_i$.

then, 满足要求的 $(i_1, i_2, \cdots, i_{2m})$ 等价于. $S_0 = S_{2m} = 0$.

$S_1, \cdots, S_{2m-1} \geq 0$.

$\{S_i; 1 \leq i \leq 2m\}$

对每一个固定的轨道, $(i_1, i_2, \cdots, i_{2m})$ 中的 $m+1$ 个自由顶点被唯一确定.

$$\therefore I_n^{2m} = n^{-1-m}\, n(n-1)\cdots(n-m) \times \underbrace{\#\left\{\{S_i : 0 \leq i \leq 2m\} \,\middle|\, S_0 = S_{2m} = 0.\ S_1, \cdots, S_{2m-1} \geq 0\right\}}_{\triangleq P_m}$$

$\longrightarrow P_m$. $n \to +\infty$ 时.

由教材 Corollary 3.10.6 知, 简单随机游走, 从 $(0,0)$ 至 (n, b) $b > 0$, 中途不返回 x 轴的路径数为 $\frac{b}{n} N_n(0, b)$.

将 x 轴在 x-y 平面内向下平移一格, 则有:

$$P_m = \frac{1}{2m+1}\binom{2m+1}{m} = \frac{(2m)!}{m!\,(m+1)!} = \gamma_{2m}.$$

done.