$\partial_t^2 u-\Delta u=0 \qquad u(x,0)=\varphi(x),\quad \partial_t u(x,0)=\psi(x) \qquad (W)$

用极坐标 $u(x,t)=u(r,\omega,t)$ $\qquad \partial_t^2 u-\Delta u=\partial_t^2 u-\left(\partial_r^2 u+\frac{2}{r}\partial_r u+\frac{1}{r^2}\Delta_{S^2}u\right)=0$

两边同时在 $S^2$ 上积分，由于 $\int_{S^2}\Delta_{S^2}u\cdot d\sigma=0$，故

$$\partial_t^2\int_{S^2}u\,d\sigma-\left(\partial_r^2\int_{S^2}u\,d\sigma+\frac{2}{r}\partial_r\int_{S^2}u\,d\sigma\right)=0$$

令 $\bar{u}(r,t)$，则 $\begin{cases}\partial_t^2\bar{u}-\left(\partial_r^2\bar{u}+\frac{2}{r}\partial_r\bar{u}\right)=0\\ \bar{u}(r,0)=\bar{\varphi}(r),\quad \partial_t\bar{u}(r,0)=\bar{\psi}(r).\end{cases}$

将 $\bar{u}(r,t)$ 关于 $r$ 作偶延拓，令 $v=r\bar{u}$，则 $\begin{cases}\partial_t^2 v-\partial_r^2 v=0\\ v(r,0)=r\bar{\varphi}(r),\quad \partial_t v(r,0)=r\bar{\psi}(r).\end{cases}$

于是，$v(r,t)=\frac{1}{2}\left[(r+t)\bar{\varphi}(r+t)+(r-t)\bar{\varphi}(r-t)\right]+\frac{1}{2}\int_{r-t}^{r+t}s\bar{\psi}(s)\,ds$

由此可以计算出 $\bar{u}(r,t)$. 如何求得 $u(x,t)$？

Step1. $u(0,t)=\bar{u}(0,t)=\partial_r(r\bar{u})\big|_{r=0}$

$$=\frac{1}{2}\left(\bar{\varphi}(t)+t\bar{\varphi}'(t)+\bar{\varphi}(-t)-t\bar{\varphi}'(-t)\right)+\frac{1}{2}\left[t\bar{\psi}(t)-(-t)\bar{\psi}(-t)\right]$$

$$=\bar{\varphi}(t)+t\bar{\varphi}'(t)+t\bar{\psi}(t)$$

$$=\frac{d}{dt}\left(t\bar{\varphi}(t)\right)+t\bar{\psi}(t)$$

$$=\frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(t\omega)\,d\sigma\right)+\frac{t}{4\pi}\int_{S^2}\psi(t\omega)\,d\sigma$$

Step2. 波方程是平移不变的. 即若 $u(x,t)$ 是 $(W)$ 的解. 则 $\forall x_0\in\mathbb{R}^3$，$u(x+x_0,t)$ 也是 $(W)$ 的解. 初值为 $\varphi(x+x_0)$，$\psi(x+x_0)$.

对 $u(x+x_0,t)$ 应用Step1，可得

$$u(x_0,t)=\frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(x_0+t\omega)\,d\sigma\right)+\frac{t}{4\pi}\int_{S^2}\psi(x_0+t\omega)\,d\sigma$$

由于 $x_0\in\mathbb{R}^3$ 是任意的，故

$$u(x,t)=\frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(x+t\omega)\,d\sigma\right)+\frac{t}{4\pi}\int_{S^2}\psi(x_0+t\omega)\,d\sigma$$

$$=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|x-y|=t}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|x-y|=t}\psi(y)\,dS(y)$$

Kirchhoff 公式

其中 $dS(y)$ 是 $B(x,t)$ 上的面积元，$dS(y) = t^2 d\sigma$.

- $n=2$ 降维法.

考虑二维波动方程 $\partial_t^2 u - \cancel{a^2}(\partial_{x_1}^2 u + \partial_{x_2}^2 u) = 0$

$$u(x,0) = \varphi(x_1,x_2),\quad \partial_t u(x,0) = \psi(x_1,x_2).$$

将 $u(t,x_1,x_2)$ 看成三维空间中的函数 $\tilde{u}(t,x_1,x_2,x_3) = u(t,x_1,x_2)$.

$\tilde{\varphi}(t,x_1,x_2,x_3) = \varphi(x_1,x_2)$, $\tilde{\psi}(x_1,x_2,x_3) = \psi(x_1,x_2)$. 则

$$\begin{cases} \partial_t^2 \tilde{u} - \cancel{a}(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2)\tilde{u} = 0 \\ \tilde{u}(x,0) = \tilde{\varphi},\quad \partial_t \tilde{u}(0) = \tilde{\psi}. \end{cases}$$

由 Kirchhoff 公式，$\tilde{u}(x_1,x_2,x_3,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|x-y|=t} \tilde{\varphi}(y)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|x-y|=t} \tilde{\psi}(y)\,dS(y)$.

由于 $\tilde{u}$ 与 $x_3$ 无关. 不妨设 $x_3 = 0$. 由平移不变性，先求 $\tilde{u}(0,0,0,t)$.

$$\tilde{u}(0,0,\cancel{0},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y|=t} \varphi(y_1,y_2)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|y|=t} \psi(y_1,y_2)\,dS(y).$$

只需计算 $\int_{y_3=\pm\sqrt{t^2-y_1^2-y_2^2}} \varphi(y_1,y_2)\,dS(y) = 2\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}} \varphi(y_1,y_2)\,dS(y)$

$$= 2\int_{y_1^2+y_2^2\le t^2} \varphi(y_1,y_2)\sqrt{1+\left(\frac{\partial y_3}{\partial y_1}\right)^2+\left(\frac{\partial y_3}{\partial y_2}\right)^2}\,dy_1dy_2$$

$$= 2t\int_{y_1^2+y_2^2\le t^2} \varphi(y_1,y_2)\frac{1}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2$$

$$= 2t\int_{B(0,t)} \frac{\varphi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2.$$

即 $u(0,0,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{B(0,t)} \frac{\varphi(y)}{\sqrt{t^2-|y|^2}}\,dy_1dy_2\right) + \frac{1}{2\pi}\int_{B(0,t)} \frac{\psi(y)}{\sqrt{t^2-|y|^2}}\,dy_1dy_2$

对 $u(x+x_0,t)$ 应用上式，可得 $u(x_0,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{B(0,t)} \frac{\varphi(x_0+y)}{\sqrt{t^2-|y|^2}}\,dy_1dy_2\right) + \cdots\cdots$

故 $u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{|x-y|\le t} \frac{\varphi(y)}{\sqrt{t^2-|x-y|^2}}\,dy_1dy_2\right) + \frac{1}{2\pi}\int_{|x-y|\le t} \frac{\psi(y)}{\sqrt{t^2-|x-y|^2}}\,dy_1dy_2$.

$$\partial_t^2 u - \Delta u = 0 \qquad u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)$$

$d=3$ $$u(x,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|x-y|=t}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|x-y|=t}\psi(y)\,dS(y)$$

$d=2$ $$u(x,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{|x-y|\le t}\frac{\varphi(y)}{\sqrt{t^2-|x-y|^2}}\,dy_1dy_2\right)+\frac{1}{2\pi}\int_{|x-y|\le t}\frac{\psi(y)\,dy_1dy_2}{\sqrt{t^2-|x-y|^2}}$$

$(y,t)$

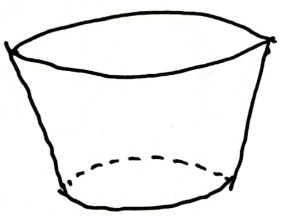
$|y-x|\le t$

依赖区域

$(x_0,R)$

$\{(x,t): |x-x_0| \le (R-t)\}$ 决定区域.

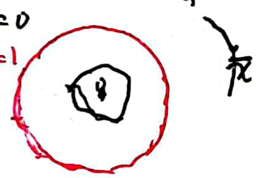
$|x-x_0|\le R$



$|x-x_0| \le |t|+R$

影响区域.

$d=3$



$t=0$

$t=1$

$t=t_1$.

Huygens principle.

$d=2$ 不成立.

非齐次方程
$$\begin{cases}\partial_t^2 u-\Delta u=f\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\end{cases}$$

叠加原理
$$\begin{cases}\partial_t^2 u-\Delta u=0\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\end{cases} \quad (\mathrm{I})$$
$$\begin{cases}\partial_t^2 u-\Delta u=f(x,t)\\ u(x,0)=0,\ \partial_t u(x,0)=0.\end{cases} \quad (\mathrm{II})$$

Claim: 令 $W(x,t;\tau)$ 是
$$\begin{cases}\partial_t^2 u-\Delta u=0\\ u(x,\tau)=0,\ \partial_t u(x,\tau)=f(x,\tau)\end{cases}$$

则 $u(x,t)=\int_0^t W(x,t;\tau)\,d\tau$ 是 (II) 的解.

扫描全能王 创建

$$\partial_t u = W(x,t;t) + \int_0^t \partial_t W(x,t;\tau)\,d\tau = \int_0^t \partial_t W(x,t;\tau)\,d\tau$$

$$\partial_t^2 u = \partial_t W(x;t;t) + \int_0^t \partial_t^2 W(x,t;\tau)\,d\tau$$

$$\Delta u = \int_0^t \Delta W(x,t;\tau)\,d\tau$$

$$\Rightarrow \begin{cases} \partial_t^2 u - \Delta u = \int_0^t (\partial_t^2 W - \Delta W)(x,t;\tau)\,d\tau + f(x,t) = f(x,t) \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0 \end{cases}$$

~~由Kirchhoff公式, $W(x,t) =$~~

令 $s = t - \tau$, 则 $u(x,t) = u(x, s+\tau) \triangleq v(x,s)$. $\partial_s^2 v - \Delta v = 0$

$\cancel{v(s,\ldots)}$ $v(x,0) = 0, \quad \partial_s v(x,0) = f(x,\tau)$

$$\Rightarrow v(x,s) = \frac{1}{4\pi s}\int_{|x-y|=s} f(y,\tau)\,dS(y)$$

$$\Rightarrow W(x,t) = \frac{1}{4\pi(t-\tau)}\int_{|x-y|=t-\tau} f(y,\tau)\,dS(y)$$

$$\Rightarrow u(x,t) = \int_0^t \frac{1}{4\pi(t-\tau)}\int_{|x-y|=t-\tau} f(y,\tau)\,dS(y)\,d\tau$$

$$\overset{t-\tau=s}{=} \frac{1}{4\pi}\int_0^t \frac{1}{s}\int_{|x-y|=s} f(y,t-s)\,dS(y)\,ds$$

$$= \frac{1}{4\pi}\int_{|x-y|\le t} \frac{f(y,t-|x-y|)}{|x-y|}\,dy$$

## §3.5 能量法. 解的唯一性与稳定性.

若 $u(x,t)$ 是 $\partial_t^2 u-\Delta u=0$ 的解, 则 $u(x,t+t_0)$, $u(x+x_0,t)$ 也是解.

$$\partial_t u\,(\partial_t^2 u-\Delta u)=\frac{1}{2}\partial_t(\partial_t u)^2-\operatorname{div}(\partial_t u\nabla u)+\partial_t\nabla u\cdot\nabla u$$

$$=\frac{1}{2}\partial_t(\partial_t u)^2+\frac{1}{2}\partial_t|\nabla u|^2-\operatorname{div}(\partial_t u\nabla u)$$

令 $e(u)=\frac{1}{2}\left[(\partial_t u)^2+|\nabla u|^2\right]$, 则 $\partial_t e(u)=\operatorname{div}(\partial_t u\nabla u)$

(I) 考虑波方程
$$\begin{cases}\partial_t^2 u-\Delta_{R^3}u=0 & \Omega\times\{t\geq 0\}\\ \left(\frac{\partial u}{\partial\nu}+\sigma u\right)\Big|_{\partial\Omega}=0 & t\geq 0\end{cases}\quad (3.5.2).$$

则 $\partial_t\int_\Omega e(u)dx=\int_\Omega \operatorname{div}(\partial_t u\nabla u)dx=\int_{\partial\Omega}\partial_t u\frac{\partial u}{\partial\nu}dS$

$$=-\sigma\int_{\partial\Omega}\partial_t u\,u\,dS=-\frac{\sigma}{2}\partial_t\int_{\partial\Omega}u^2dS$$

即 $\partial_t\left[\int_\Omega e(u)dx+\frac{\sigma}{2}\int_{\partial\Omega}u^2dS\right]=0$.

令 $E(u)=\int_\Omega e(u)dx+\frac{\sigma}{2}\int_{\partial\Omega}u^2dS$. 则 $E(u(t))=E(u(0))$.

定理3.1. 问题
$$\begin{cases}\partial_t^2 u-\Delta u=f & \text{in }\Omega\quad t\geq 0\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\\ \left(\frac{\partial u}{\partial\nu}+\sigma u\right)\Big|_{\partial\Omega}=\mu(x,t) & t\geq 0\end{cases}$$
最多有一个解.

证: 设 $u_1,u_2$ 是两个解. 令 $v=u_1-u_2$. 则 $v$ 满足
$$\begin{cases}\partial_t^2 v-\Delta v=0\\ v(x,0)=0,\ \partial_t v(x,0)=0\\ \left(\frac{\partial v}{\partial\nu}+\sigma v\right)\Big|_{\partial\Omega}=0\end{cases}$$

由能量估计,
$$E(v(t))=\frac{1}{2}\int_\Omega(\partial_t v)^2+|\nabla v|^2dx+\frac{\sigma}{2}\int_{\partial\Omega}|v|^2dS=E(v(0))=0.$$

故 $\partial_t v\equiv 0$, $\nabla v\equiv 0$. $\Rightarrow$ $v=const$ in $\bar\Omega\times\{t\geq 0\}$

又 $\int_S v^2dS=0$ $\Rightarrow$ $v\equiv 0$ on $\partial\Omega\times\{t\geq 0\}$  故 $v\equiv 0$ in $\bar\Omega\times\{t\geq 0\}$.

(II) 考虑波方程
$$\begin{cases}\partial_t^2 u-\Delta u=0 & \text{in }R^3\times\{t\geq 0\}\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x) & x\in R^3\end{cases}$$

若 $u\in C^2$ 且 $|u(x)|\to 0$, as $|x|\to+\infty$. 则

$$\partial_t\underbrace{\int_{R^3}\left(\frac{1}{2}|\partial_t u|^2+\frac{1}{2}|\nabla u|^2\right)dx}_{E(u(t))}=\int_{R^3}\operatorname{div}(\partial_t u\nabla u)dx=0\Rightarrow E(u(t))=E(u(0)).$$

考虑
$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x\in\Omega \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x) & x\in\Omega \\ \left(\frac{\partial u}{\partial \nu}+\sigma u\right)\Big|_{\partial\Omega} = 0 & t\geq 0 \end{cases} \qquad (3.5.10)$$

定理 (3.5.10) 的解 $u(x,t)$ 在下述意义下关于初值和右端项是稳定的:

$\forall \varepsilon>0$, $\exists \eta=\eta(\varepsilon,T)>0$, s.t. 若 $\|\varphi_1-\varphi_2\|_{L^2(\Omega)}\leq\eta$, $\|\nabla\varphi_1-\nabla\varphi_2\|_{L^2}\leq\eta$, $\|\psi_1-\psi_2\|_{L^2}\leq\eta$, $\|\varphi_1-\varphi_2\|_{L^2(\partial\Omega)}\leq\eta$, $\|f_1-f_2\|_{L^2([0,T]\times\Omega)}\leq\eta$, 则以 $(\varphi_1,\psi_1)$ 为初值, 右端项为 $f_1$ 的解 $u_1$ 与以 $(\varphi_2,\psi_2)$ 为初值, $f_2$ 为右端项的解 $u_2$ 的差在 $0\leq t\leq T$ 上满足

$$\|u_1-u_2\|_{L^2(\Omega)}+\|\nabla u_1-\nabla u_2\|_{L^2(\Omega)}\leq\varepsilon,\quad \|\partial_t u_1-\partial_t u_2\|_{L^2}\leq\varepsilon.$$

证明: 令 $u=u_1-u_2$, $f=f_1-f_2$, $\varphi=\varphi_1-\varphi_2$, $\psi=\psi_1-\psi_2$, 则 $\begin{cases}\partial_t^2 u-\Delta u=f\\ u(x,0)=\varphi,\ \partial_t u(x,0)=\psi\\ \left(\frac{\partial u}{\partial\nu}+\sigma u\right)\Big|_{\partial\Omega}=0\end{cases}$

且 $\|\varphi\|_{L^2(\Omega)}, \|\nabla\varphi\|_{L^2(\Omega)}\leq\eta$, $\|\varphi\|_{L^2(\partial\Omega)}\leq\eta$, $\|\psi\|_{L^2}\leq\eta$.

令 $E(t)=\int_\Omega \frac12\left((\partial_t u)^2+|\nabla u|^2\right)dx+\frac{\sigma}{2}\int_{\partial\Omega}|u|^2 dS$, 则

$$\frac{d}{dt}E(t)=\int_\Omega u_t f\,dx\leq\int_\Omega\frac{|f|^2+|u_t|^2}{2}dx=\frac12\int_\Omega|f|^2dx+\frac12\int_\Omega u_t^2dx\leq\frac12\int|f|^2dx+E(t)$$

$$\Rightarrow \frac{d}{dt}\left(e^{-t}E(t)\right)\leq\frac12 e^{-t}\int|f|^2dx$$

$$\Rightarrow e^{-t}E(t)-E(0)\leq\frac12\int_0^t e^{-s}\int|f|^2dx\,ds \Rightarrow E(t)\leq C_T\left(E(0)+\int_0^T\int|f(x,s)|^2dx\,ds\right).$$

令 $E_0(t)=\int_\Omega|u(x,t)|^2dx$, 则 $\frac{d}{dt}E_0(t)=2\int_\Omega u u_t\,dx\leq\int_\Omega|u|^2dx+\int_\Omega|u_t|^2dx$

$$\leq\int_\Omega|u|^2dx+2E(u(t)).$$

$$\leq\int_\Omega|u|^2dx+2C_T\left(E(0)+\int_0^T\int_\Omega|f|^2dx\,dt\right)$$

$$\Rightarrow E(t)+E_0(t)\leq E_0(0)+C_T\left(E(0)+\int_0^T\int_\Omega|f|^2dx\,dt\right)$$

$$\leq C\left(\int_\Omega|\varphi|^2dx+\int_\Omega(|\nabla\varphi|^2+|\psi|^2)dx+\int_{\partial\Omega}|\varphi|^2dS+\int_0^T\int_\Omega f^2dx\,dt\right)$$

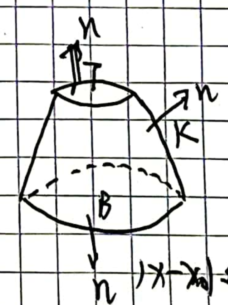
$$\leq C\eta,\quad \forall t\in[0,T].$$

取 $\eta$ s.t. $C\eta<\varepsilon$. 即可.

◆ 能量估计与有限传播速度

$$\partial_t e(u)+\partial_i(u_t\partial_i u)=0.$$

$$\int_\triangle\left[\partial_t e(u)+\partial_i(u_t\partial_i u)\right]dx\,dt=0$$




$|x-x_0|\leq R-t$.

在锥面 $|x-x_0|=|R-t|$ 上，令 $\phi(x,t)=|x-x_0|^2-|R-t|^2$. 则 $\nabla\phi=(2(x-x_0),-2(R-t))$

$$n=-\frac{\nabla\phi}{|\nabla\phi|}=\frac{(x-x_0,\,-(R-t))}{\sqrt{|x-x_0|^2+|R-t|^2}}=\frac{-1}{\sqrt{2}}\left(\frac{x-x_0}{|x-x_0|},\frac{-(R-t)}{R-t}\right)$$

于是，$\int_{\triangle}\left(\partial_t e(u)+\mathrm{div}(u_t\nabla u)\right)dx\,dt$

$$=\int_B (u_t\nabla u,\,e(u))\cdot(0,-1)\,dx+\int_T (u_t\nabla u,\,e(u))\cdot(0,1)\,dx$$

$$+\int_K \frac{1}{\sqrt{2}}\left(u_t\nabla u\cdot\frac{x-x_0}{|x-x_0|}+e(u)\frac{(R-t)}{|R-t|}\right)dS$$

$$=-\int_B e(u)(0)\,dx+\int_T e(u)(T)\,dx+\mathrm{Flux}[0,T],$$

$$\mathrm{Flux}[0,T]=-\frac{1}{\sqrt{2}}\int_K\left(u_t\nabla u\cdot\frac{x-x_0}{|x-x_0|}+e(u)\frac{-(R-t)}{|R-t|}\right)dS$$

$$=\frac{1}{\sqrt{2}}\int_K\left(e(u)-u_t\nabla u\,\frac{x-x_0}{|x-x_0|}\right)dS$$

$$=\frac{1}{2\sqrt{2}}\int_K\left(u_t-\frac{x-x_0}{|x-x_0|}\nabla u\right)^2dS+\frac{1}{2\sqrt{2}}\int_K\left(|\nabla u|^2-\left|\frac{x-x_0}{|x-x_0|}\nabla u\right|^2\right)dS\geq 0.$$

于是，$\int_B e(u)(0)\,dx=\int_T e(u)(T)\,dx+\mathrm{Flux}[0,T]$

Rmk: ① 时间反演不变.

② Minkowski 时空

③ Hamiltonian system.

# 第四章 热传导方程

考虑方程 $\partial_t u - \Delta u = 0 \qquad \Omega\times\{t\geq 0\}$

初值条件：$u(x,0)=\varphi(x) \qquad \Omega\times\{t=0\}$

边值条件：Dirichlet $\quad u|_{\partial\Omega}=g(x,t) \qquad t\geq 0.$

Neumann $\quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega}=g(x,t) \qquad t\geq 0$

Robin $\quad \left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial\Omega}=g(x,t) \qquad t\geq 0.$

解法：1. 分离变量法（$\Omega=[0,1]$，矩形，圆形域）.

2. Fourier 变换方法.

$$\begin{cases}\partial_t u - \Delta u = f(x,t) & \text{in } \mathbb{R}^n\times[0,T) \\ u(x,0)=g(x) & \text{in } \mathbb{R}^n\end{cases}$$

若 $f\in L^1(\mathbb{R}^n)$，可定义 $\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}dx$. 性质：若 $f\in S(\mathbb{R}^n)$，则有

(1) 令 $T_{x_0}f(x)=f(x-x_0)$，则 $\widehat{T_{x_0}f}(\xi)=e^{-ix_0\cdot\xi}\hat{f}(\xi)$

$$\widehat{T_{x_0}f}(\xi)=\int f(x-x_0)e^{-ix\cdot\xi}dx \overset{y=x-x_0}{=\!=} \int f(y)e^{-iy\cdot\xi}dy\, e^{-ix_0\cdot\xi}=\hat{f}(\xi)e^{-ix_0\cdot\xi}$$

(2) 令 $S_\lambda f(x)=f(\lambda x)$，则 $\widehat{S_\lambda f}(\xi)=\lambda^{-n}\hat{f}(\lambda^{-1}\xi)$

$$\widehat{S_\lambda f}(\xi)=\int f(\lambda x)e^{-ix\cdot\xi}dx \overset{y=\lambda x}{=\!=} \int f(y)e^{-iy\cdot\lambda^{-1}\xi}dy\,\lambda^{-n}=\lambda^{-n}\hat{f}(\lambda^{-1}\xi)$$

(3) 对于多重指标 $\alpha=(\alpha_1,\cdots,\alpha_n)$，记 $|\alpha|=|\alpha_1|+\cdots+|\alpha_n|$，$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$，

$\partial^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$. 则 $\widehat{\partial^\alpha f}(\xi)=(i\xi)^\alpha\hat{f}(\xi)$

$$\widehat{\partial^\alpha f}(\xi)=\int_{\mathbb{R}^n}\partial^\alpha f(x)e^{-ix\cdot\xi}dx=(-1)^{|\alpha|}\int f(x)\,\partial_x^\alpha(e^{-ix\cdot\xi})dx$$

$$=(-1)^{|\alpha|}(-i\xi)^\alpha\int_{\mathbb{R}^n}f(x)e^{-ix\cdot\xi}dx=(i\xi)^\alpha\hat{f}(\xi)$$

(4) $\widehat{(-ix)^\alpha f}(\xi) = \partial_\xi^\alpha \hat{f}(\xi)$

$$\widehat{(-ix)^\alpha f}(\xi) = \int (-ix)^\alpha f(x) e^{-ix\cdot\xi} dx = \int f(x)\, \partial_\xi^\alpha (e^{-ix\cdot\xi})\, dx = \partial_\xi^\alpha \int f(x) e^{-ix\cdot\xi} dx = \partial_\xi^\alpha \hat{f}(\xi)$$

(5) 令 $(f*g)(x) = \int f(x-y) g(y)\, dy = \int f(y) g(x-y)\, dy$，则 $\forall f, g \in \mathcal{S}(\mathbb{R}^n)$，

$$\widehat{f*g}(\xi) = \hat{f}(\xi)\, \hat{g}(\xi)$$

例：令 $f(x) = e^{-x^2}$，$x \in \mathbb{R}$，则 $\hat{f}(\xi) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{4}\xi^2}$

令 $F(\xi) = \hat{f}(\xi) = \int e^{-x^2} e^{-ix\cdot\xi} dx$，则 $F'(\xi) = \int e^{-x^2}(-ix)\, e^{-ix\cdot\xi} dx$

$$= \frac{i}{2}\int f'(x) e^{-ix\cdot\xi} dx = \frac{i}{2}\widehat{f'}(\xi) = -\frac{\xi}{2}\hat{f}(\xi) = -\frac{\xi}{2}F(\xi).$$

且 $F(0) = \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$ 故 $\begin{cases} F'(\xi) + \frac{\xi}{2} F(\xi) = 0 \\ F(0) = \sqrt{\pi} \end{cases}$

由ODE，$F(\xi) = \sqrt{\pi}\, e^{-\frac{\xi^2}{4}}$

定义：$\check{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} d\xi$

若 $f \in \mathcal{S}(\mathbb{R}^n)$，则 $\check{\hat{f}} = f$.

Claim: $\left(e^{-|\xi|^2 t}\right)^\vee(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$

$n=1$ 时，令 $G(x,t) = \frac{1}{(2\pi)}\int e^{-|\xi|^2 t} e^{ix\cdot\xi} d\xi$，则 $\frac{d}{dx}G = \frac{1}{(2\pi)}\int e^{-|\xi|^2 t}\, i\xi\, e^{ix\cdot\xi} d\xi$

$$= -\frac{i}{2t}\frac{1}{(2\pi)}\int \left(e^{-|\xi|^2 t}\right)' e^{ix\cdot\xi} d\xi = -\frac{x}{2t} G(x,t).$$

且 $G(0,t) = \frac{1}{(2\pi)}\int e^{-|\xi|^2 t} d\xi = \frac{1}{2\pi t^{\frac{1}{2}}}\int e^{-\eta^2} d\eta = \frac{1}{(4\pi t)^{\frac{1}{2}}}$

于是，$G(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}$

$n \geq 2$. $\frac{1}{(2\pi)^n}\int e^{-|\xi|^2 t} e^{ix\cdot\xi} d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}} e^{-\xi_1^2 t + ix_1\xi_1} d\xi_1 \cdots \int_{\mathbb{R}} e^{-\xi_n^2 t + ix_n\xi_n} d\xi_n$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{x_1^2}{4t}} \cdots e^{-\frac{x_n^2}{4t}} = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

对方程 $\begin{cases} \partial_t u - \Delta u = 0 \\ u(x,0) = u_0(x) \end{cases}$ 两边作Fourier变换，可得

$$\begin{cases} \partial_t \hat{u} + (\xi_1^2 + \cdots + \xi_n^2)\hat{u} = 0 \\ \hat{u}(\xi,0) = \hat{u}_0(\xi) \end{cases} \quad 即 \quad \begin{cases} \partial_t \hat{u} + |\xi|^2 \hat{u} = 0 \\ \hat{u}(\xi,0) = \hat{u}_0(\xi) \end{cases} \quad \to \hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0(\xi)$$

于是，取Fourier逆变换，可得 $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * u_0$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int e^{-\frac{|x-y|^2}{4t}} u_0(y)\,dy.$$

令 $K(x) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4}}$ (Heat kernel)，$K_t(x) = t^{-\frac{n}{2}} K(t^{-1}x)$

则 $u(x,t) = K_t * u_0$.

对于 $\{K_t\}_{t>0}$，有 (i) $\int_{R^n} K_t(x)dx = \int_{R^n} K(x)dx = 1$

(ii) $\int_{R^n} |K_t(x)|\,dx = 1$ (bdd).

(iii) $\forall \eta > 0$, $\int_{|x|>\eta} |K_t(x)|\,dx = \int_{|y|>\eta/t} |K(y)|\,dy \xrightarrow{t\to 0^+} 0$.

事实上，$\{K_t\}_{t>0}$ 是一族逼近恒等 (approximation to the identity).

Claim: $\lim_{t\to 0^+} u(x,t) = u_0(x)$, if $u_0$ continuous. bounded (只需 $|u_0(x)| \le e^{\delta |x|^2}$,)

$$u(x,t) - u_0(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int e^{-\frac{|y|^2}{4t}} u_0(x-y)\,dy - u_0(x)$$

$$= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int e^{-\frac{|y|^2}{4t}} \left(u_0(x-y) - u_0(x)\right) dy$$

$$\overset{y=2\sqrt{t}z}{=\!=} \frac{1}{(4\pi)^{\frac{n}{2}}} \int e^{-|z|^2} \left(u_0(x - 2\sqrt{t}z) - u_0(x)\right) dz$$

$$\Rightarrow |u(x,t) - u_0(x)| \le \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{|z|>R} e^{-|z|^2} |u_0(x-2\sqrt{t}z) - u_0(x)|\,dz \quad ①$$

$$+ \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{|z|<R} e^{-|z|^2} |u_0(x-2\sqrt{t}z) - u_0(x)|\,dz \quad ②$$

① $\le \frac{M}{(4\pi)^{\frac{n}{2}}}\int_{|z|>R} e^{-|z|^2} dz < \varepsilon$，只要R充分大.

② ：当 $t>0$ 充分小时，$|u_0(x-2\sqrt{t}z)-u_0(x)|<\varepsilon$，$\forall |z|<R$.

于是，② $\le \varepsilon \frac{1}{(4\pi)^{\frac{n}{2}}}\int_{|z|<R} e^{-|z|^2} dz \le C\varepsilon$. 均充分小.

因此，$\lim\limits_{t\to 0^+} u(x,t)=u_0(x)$.

Rmk：(1) $u(x,t)\in C^\infty$，$\forall t>0$

(2) $\sup\limits_{x\in\mathbb{R}^n}|u(x,t)| \le \sup\limits_{x\in\mathbb{R}^n}|u_0(x)|$.

(3) 沿时间不能反向演化

(4) 无限传播速度.

非齐次方程
$$\begin{cases} \partial_t u-\Delta u=f(x,t) \\ u(x,0)=u_0(x) \end{cases}$$

两边同时关于x取Fourier变换，则
$$\begin{cases} \partial_t\hat{u}+|\xi|^2\hat{u}=\hat{f}(\xi,t) \\ \hat{u}(\xi,0)=\hat{u}_0(\xi). \end{cases}$$

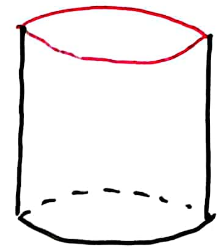
$$\Rightarrow \hat{u}(\xi,t)=e^{-t|\xi|^2}\hat{u}_0(\xi)+\int_0^t e^{-|\xi|^2(t-s)}\hat{f}(\xi,s)\,ds$$

$$\Rightarrow u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int e^{-\frac{|x-y|^2}{4t}}u_0(y)\,dy+\int_0^t\frac{1}{[4\pi(t-s)]^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)\,dy\,ds.$$

极大值原理.

考虑热传导方程
$$\begin{cases} \partial_t u-\Delta u=0 & \text{in } \Omega\times\{t>0\} \\ u(x,0)=u_0(x) & \text{on } \Omega\times\{t=0\} \\ u(x,t)|_{\partial\Omega}=h(x,t) & t\ge 0. \end{cases}$$

令 $Q_T = \Omega \times (0,T]$，定义抛物边界 $\Gamma_T = \overline{Q_T} \setminus Q_T$.



$\Gamma_T$ 侧面 $\cup$ 底面

定理 4.2.1.（极值原理）令 $u(x,t)$ 在矩形 $R_T = \{\alpha < x < \beta,\ 0 < t < T\}$ 上连续并且在矩形内部满足热传导方程 $\partial_t u - \partial_x^2 u = 0$. 则它在抛物边界达到最大值和最小值.

证明：只考虑最大值的情形. 令 $M = \max\limits_{Q_T} u$，$m = \max\limits_{\Gamma_T} u$. 若结论不成立，则 $M > m$. 于是存在 $(x^*, t^*) \in R_T$ 使 $u(x^*,t^*) = M$. 于是，$\partial_t u(x^*,t^*) \geq 0$，$\partial_x^2 u(x^*,t^*) \leq 0$. 故 $(\partial_t u - \partial_x^2 u)(x^*,t^*) \geq 0$

构造函数 $v(x,t) = u(x,t) + \frac{M-m}{4l^2}(x-x^*)^2$，其中 $l = \beta - \alpha$. 则

$$\max_{\Gamma_T} v(x,t) \leq \max_{\Gamma} u(x,t) + \frac{M-m}{4} = m + \frac{M-m}{4} < M$$

$$\max_{R_T} v(x,t) \geq \max_{Q_T} u(x,t) = M$$

故 $v$ 在内部达到最大值.（未必是 $(x^*,t^*)$）. 设 $v$ 在 $(x_1,t_1) \in R_T$ 取最大值，则 $\partial_t v(x_1,t_1) \geq 0$，$\partial_x^2 v(x_1,t_1) \leq 0$ 于是 $(\partial_t v - \partial_x^2 v)(x_1,t_1) \geq 0$.

但是，$\partial_t v = \partial_t u$，$\partial_x^2 v = \frac{\partial^2 u}{\partial x^2} + \frac{M-m}{2l^2}$. 于是，

$$\left(\partial_t u - \partial_x^2 u - \frac{M-m}{2l^2}\right)(x_1,t_1) \geq 0.\ 即\ (\partial_t u - \partial_x^2 u)\Big|_{(x_1,t_1)} \geq \frac{M-m}{2l} > 0.\ 矛盾!$$

Rmk：若 $\partial_t u - \partial_x^2 u \leq 0$，则在抛物边界达到最大值.

定理 4.2.2（唯一性和稳定性）热传导方程的边值问题
$$\begin{cases} u_t - u_{xx} = f & R_T \\ u(x,0) = \varphi(x) \\ u(\alpha,t) = \mu_1(t),\ u(\beta,t) = \mu_2(t) \end{cases}$$
在 $R_T$ 上的解是唯一的. 而且连续地依赖于边界 $\Gamma_T$ 上给定的初始条件和边界条件.

证明：设 $u_1, u_2$ 是两个解. 令 $\tilde{u} = u_1 - u_2$. 则 $\tilde{u}$ 满足 $\begin{cases} \partial_t \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } R_T \\ \tilde{u}(\alpha, t) = 0, \ \tilde{u}(\beta, t) = 0 \\ \tilde{u}(x, 0) = 0 \end{cases}$

由极值原理，$\tilde{u}$ 在 $R_T$ 上的最大值、最小值都在抛物边界取到. 故 $\tilde{u}$ 的最大值最小值都是 0. 故 $\tilde{u} \equiv 0$ in $R_T$.
唯一性得证.

下面证明稳定性. 令 $u_1$ 满足 $\begin{cases} \partial_t u_1 - \partial_x^2 u_1 = f \\ u_1(\alpha, t) = \mu_1^1, \ u_2(\beta, t) = \mu_1^2 \\ u_1(x, 0) = \varphi_1 \end{cases}$ $u_2$ 满足 $\begin{cases} \partial_t u_2 - \partial_x^2 u_2 = f \\ u_2(\alpha, t) = \mu_2^1, \ u_2(\beta, t) = \mu_2^2 \\ u_2(x, 0) = \varphi_2 \end{cases}$

令 $u = u_1 - u_2$. 则 $u$ 满足 $\begin{cases} \partial_t u - \partial_x^2 u = 0 \\ u(\alpha, t) = \mu_1^1 - \mu_2^1, \ u(\beta, t) = \mu_1^2 - \mu_2^2 \\ u(x, 0) = \varphi_1 - \varphi_2. \end{cases}$

由极值原理，$\max\limits_{R_T} u \le \max\limits_{\Gamma_T} u \le \max\limits_{0 \le t \le T} \left( |\mu_1^1 - \mu_2^1| + |\mu_1^2 - \mu_2^2| \right) + \max\limits_{\alpha \le x \le \beta} |\varphi_1 - \varphi_2|$.

且 $\min\limits_{R_T} u \ge \min \left\{ \min\limits_{0 \le t \le T} \left( \mu_1^1 - \mu_2^1 + \mu_1^2 - \mu_2^2 \right), \ \min\limits_{\alpha \le x \le \beta} (\varphi_1 - \varphi_2) \right\}$

$$\ge \min_{0 \le t \le T} \left( -|\mu_1^1 - \mu_2^1| - |\mu_1^2 - \mu_2^2| \right) + \min_{\alpha \le x \le \beta} -|\varphi_1 - \varphi_2|$$

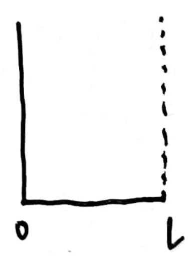
于是 $|u(x,t)| \le \max\limits_{0 \le t \le T} \left( |\mu_1^1 - \mu_2^1| + |\mu_1^2 - \mu_2^2| \right) + \max\limits_{\alpha \le x \le \beta} |\varphi_1 - \varphi_2|$.

第三类边值问题 $\begin{cases} \partial_t u - \partial_x^2 u = 0 \\ u(0, t) = \mu_1(t), \ (u_x + hu)(l, t) = \mu_2(t) \\ u(x, 0) = \varphi(x) \end{cases}$ $h > 0$

要证明唯一性，只需证明 (*) $\begin{cases} \partial_t u - \partial_x^2 u = 0 \\ u(0, t) = 0, \ (u_x + hu)(l, t) = 0 \\ u(x, 0) = 0 \end{cases}$ 只有零解.

否则，(*) 有非零解. 则此非零解 $u$ 在 $R_T$ 上有正的最大值或负的最小值.

由极大值原理，正的最大值只能在边界取到. 由于 $u(0,t) = 0$, $u(x,0) = 0$. 故正的最大值只能在 $x = l$ 上取到. 此时，$u_x(l, t^*) > 0$, 且 $u(l, t^*) = M > 0$. 故 $(u_x + hu)(l, t^*) > 0$. 矛盾!



类似可证 $u$ 在 $R_T$ 上没有负的最小值．因此 (*) 只有零解．　③

作业：证明稳定性：$\forall (x,t)\in R_T$,

$$|u_1(x,t)-u_2(x,t)| \le \max\left(\max_{0\le t\le T}|\mu_1^1(t)-\mu_2^1(t)|,\ \max_{0\le x\le l}|\varphi_1(x)-\varphi_2(x)|,\ \max_{0\le t\le T}|\mu_1^2(t)-\mu_2^2(t)|\right).$$

再考虑第二类边值问题

$$\begin{cases} \partial_t u-\partial_x^2 u=0 \\ u(0,t)=\mu_1(t),\quad u_x(l,t)=\mu_2(t) \\ u(x,0)=\varphi(x) \end{cases}$$

要证唯一性，只要证明

$$\begin{cases} \partial_t u-\partial_x^2 u=0 \\ u(0,t)=0,\ u_x(l,t)=0 \\ u(x,0)=0 \end{cases}$$

只有零解．仿照第二类边值问题的证明得不到矛盾．

令 $\tilde{u}=W(x)u$，则 $u=\frac{\tilde{u}}{W(x)}$，故 $u_t=\frac{\tilde{u}_t}{W(x)}$，$\partial_x u=-\frac{W_x}{W^2}\tilde{u}+\frac{1}{W}\tilde{u}_x$

$$\partial_x^2 u=\left(-\frac{W_{xx}}{W^2}+2\frac{(W_x)^2}{W^3}\right)\tilde{u}-2\frac{W_x}{W^2}\tilde{u}_x+\frac{1}{W}\tilde{u}_{xx}$$．于是，

$$0=u_t-u_{xx}=\frac{1}{W}(\tilde{u}_t-\tilde{u}_{xx})+2\frac{W_x}{W^2}\tilde{u}_x+\left(\frac{W_{xx}}{W^2}-2\frac{W_x^2}{W^3}\right)\tilde{u}$$

$$\Rightarrow\begin{cases} \tilde{u}_t-\tilde{u}_{xx}=-\left(\frac{W_{xx}}{W}-2\frac{W_x^2}{W^2}\right)\tilde{u}-2\frac{W_x}{W}\tilde{u}_x \\ \tilde{u}(0,t)=0,\quad \tilde{u}_x(l,t)=W_x'(l)u(l,t) \\ \tilde{u}(x,0)=0 \end{cases}$$

取 $W(x)=-x+l+1$，则 $W_x'=-1$．且 $W(x)>0,\ \forall x\in[0,l]$．于是，

$$\begin{cases} \tilde{u}_t-\tilde{u}_{xx}=2\frac{1}{(x-l-1)^2}\tilde{u}+\frac{2}{-x+l+1}\tilde{u}_x \\ \tilde{u}(0,t)=0,\quad (\tilde{u}_x+\tilde{u})(l,t)=0 \\ \tilde{u}(x,0)=0 \end{cases}$$

令 $V(x,t)=e^{-\lambda t}\tilde{u}(x,t)$，则 $\tilde{u}(x,t)=e^{\lambda t}V(x,t)$．故 $V(x,t)$ 满足

$$\begin{cases} V_t-V_{xx}+\left(\lambda-\frac{2}{(x-l-1)^2}\right)V+\frac{2}{x-l-1}V_x=0 \\ V(0,t)=0,\quad (V_x+V)(l,t)=0 \\ V(x,0)=0 \end{cases}$$

取 $\lambda > 2$，则 $\lambda - \frac{2}{(x-t-1)^2} > 0, \forall x \in [0,1]$.

Claim: 若 $v$ 在 $(x_*, t_*) \in R_T$ 达到正的最大值，则 $(x_*, t_*) \in \Gamma_T$.

否则，若 $(x_*, t_*) \in \mathring{R}_T$ 则 $v_x(x_*, t_*) = 0$, $v_t(x_*, t_*) \geq 0$, $\partial_x^2 v(x_*, t_*) \leq 0$.

于是，$\left(v_t - v_{xx} + \underbrace{\left(\lambda - \frac{2}{(x-t-1)^2}\right)}_{\text{严格正}} v + \frac{2}{x-t-1} v_x\right)(x_*, t_*) > 0$. 矛盾！

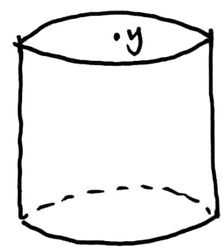
由于 $v(0,t) = 0$, $v(x,0) = 0$ 故 $v$ 只能在右边界的某处 $(\bar{x}, \bar{t})$ 取到.

此时，$v_x(\bar{x}, \bar{t}) \geq 0$. 于是，$(v + v_x)(\bar{x}, \bar{t}) \geq v(\bar{x}, \bar{t}) > 0$. 矛盾！

定理. 设 $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ 满足 $\begin{cases} \partial_t u - \Delta u = 0 \\ u(x,0) = \varphi(x) \end{cases}$ 且 $|u(x,t)| \leq A e^{a|x|^2}$, $\forall x \in \mathbb{R}^n$, $0 \leq t \leq T$, $A, a > 0$. 则有 $\sup\limits_{x,t \in \mathbb{R}^n \times [0,T]} u(x,t) \leq \max\limits_{x \in \mathbb{R}^n} \varphi(x)$.

证明：只要证明 $u(y,t) \leq \max\limits_{x \in \mathbb{R}^n} \varphi(x)$.

Step 1. 先设 $T$ 充分小. $(4aT < 1)$




~~$\forall r > 0$, 在 $B(y,r) \times [0,T]$ 上应用极值原理,~~

~~可得 $u(y,T) \leq \max\limits_{\Gamma_T} u$.~~

令 $|x-y| = r$, $0 \leq t \leq T$. ~~要估计 $u(x,t)$~~.

令 $v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}$，则 $v$ 也是热传导方程的解.

~~且 $v(x,0) \leq u(x,0) \leq \max\limits_{x} \varphi(x)$~~ 故 $\max\limits_{Q_T} v = \max\limits_{\Gamma_T} v$.

底面：$v(x,0) \leq u(x,0) \leq \max\limits_{x \in \mathbb{R}^n} \varphi(x)$.

例函: $v(x,t) \le Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon)}}$ 取 $T>0$, s.t. $\frac{1}{4(T+\varepsilon)} = a+\gamma$, $\gamma>0$.

$$\le Aa^{(|y|+r)^2} - \mu\left(4(a+\gamma)\right)^{\frac{n}{2}} e^{(a+\gamma)r^2} \xrightarrow{r\to+\infty} -\infty$$

故当 $r$ 充分大时, $v(x,t) \le \max\limits_{x\in\mathbb{R}^n} \varphi(x)$.

令 $\mu\to 0$, 则 $u(x,t) \le \max\limits_{x\in\mathbb{R}^n} \varphi(x)$.

Step 2. 若 $4aT\ge 1$. 选取 $T_1$, s.t. $4aT_1<1$. 在 $[0,T_1]$, $[T_1,2T_1]$, ⋯, $[kT_1,(k+1)T_1]$ 上不断重复上述过程, 直到 $(k+1)T_1>T$. 即可.

能量估计: 初值问题 $\begin{cases} u_t-\Delta u = f & \text{in } \Omega\times[0,T] \\ u(x,0)=\varphi(x) & \text{in } \Omega \\ u(x,t)|_{\partial\Omega\times[0,T]} = g(x,t) \end{cases}$ 只有一个解 $u\in C_1^2(Q_T)$.

证明: 只需证明 $f\equiv 0$, $\varphi\equiv 0$, $g\equiv 0$ 时只有零解.

$$\int_\Omega u(u_t-\Delta u)=0 \Rightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx - \int_\Omega \nabla\cdot(u\nabla u)\,dx + \int_\Omega |\nabla u|^2 dx = 0$$

$$\Rightarrow \frac{d}{dt}\int_\Omega |u|^2 dx - \int_{\partial\Omega} u\underbrace{\frac{\partial u}{\partial n}}_{=0} dS + \int_\Omega |\nabla u|^2 dx = 0$$

$$\Rightarrow \frac{d}{dt}\int_\Omega |u|^2 dx = -\int_\Omega |\nabla u|^2 dx \le 0$$

$$\Rightarrow \int_\Omega |u(x,t)|^2 dx \le \int_\Omega |u(x,0)|^2 dx = 0,$$

故 $u(x,t)\equiv 0$.

# 第五章 位势方程

## §5.1. 基本解与Green函数.

$\Delta u = f$ Poisson方程

当 $f=0$ 时, $\Delta u=0$ 调和方程. 称 $u$ 是调和函数.

如果 $u\in C^2(\Omega)$, 并在 $\Omega$ 上 $-\Delta u \leq 0$ $(\geq)$, 称 $u$ 是下调和的 (上)

若 $\Delta u=\delta$, 则称 $u$ 是基本解. 特别地, $\Delta u=0$, $\forall x\neq 0$.

若 $u$ 是径向函数. 即 $u(x)=u(|x|)$, 则 $\left(\partial_r^2+\frac{N-1}{r}\partial_r\right)u=0$.

令 $v=\partial_r u$. 则 $\partial_r v+\frac{N-1}{r}v=0$. $\Rightarrow v(r)=Cr^{-(N-1)}$

即 $\partial_r u=C_1 r^{-(N-1)} \Rightarrow u=-\frac{C_1}{N-2}r^{-(N-2)}+C_2 \quad N\neq 2$

$u=C_1\ln r+C_2 \quad N=2.$

定义 $k(x)=\begin{cases}\frac{1}{2\pi}\ln|x| & n=2\\ \frac{1}{n(2-n)w_n}|x|^{-(n-2)} & n\geq 3\end{cases}$ $w_n$ 是 $R^n$ 中单位球的体积.

$(n=3.\ -\frac{1}{4\pi}|x|^{-1}$

称 $k(x)$ 为调和方程的基本解.

Green公式: 若 $u,v\in C^2(\Omega)\cap C^1(\bar\Omega)$. 则

$$\int_\Omega u\Delta v\,dx=\int_\Omega \nabla\cdot(u\nabla v)\,dx-\int_\Omega \nabla u\cdot\nabla v\,dx$$

$$=\int_{\partial\Omega}u\frac{\partial v}{\partial n}\,dS-\int_\Omega \nabla u\cdot\nabla v\,dx$$

同样地, $\int_\Omega v\Delta u\,dx=\int_{\partial\Omega}v\frac{\partial u}{\partial n}\,dS-\int_\Omega\nabla u\cdot\nabla v\,dx$

$$\Rightarrow \int_\Omega(u\Delta v-v\Delta u)\,dx=\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS. \quad (\Delta\Delta)$$

Claim: $n=3$ 若 $u\in C^2(\Omega)\cap C^1(\bar{\Omega})$ 满足 $\Delta u=0$ in $\Omega$. 则 $\forall x_0\in\Omega$,

$$u(x_0)=\frac{1}{4\pi}\int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{|x-x_0|}\frac{\partial u}{\partial n}\right]dS \qquad (**)$$

证明: 由 $\Delta u=0$ 是平移不变的, 只须证明 $x_0=0$ 的情形. 即

$$u(0)=\frac{1}{4\pi}\int_{\partial(\Omega-x_0)}\left[-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial n}\right]dS. \qquad (*)$$

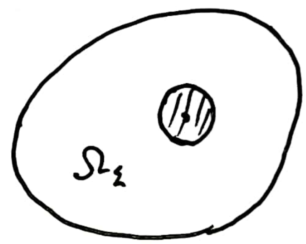
若 $(*)$ 成立. 对 $u(\cdot+x_0)$ 应用 $(*)$, 可得

$$u(x_0)=\frac{1}{4\pi}\int_{\partial(\Omega-x_0)}\left[-u(y+x_0)\frac{\partial}{\partial n}\left(\frac{1}{|y|}\right)+\frac{1}{|y|}\frac{\partial u}{\partial n}(y+x_0)\right]dS$$

$$\overset{y+x_0=x}{=}\frac{1}{4\pi}\int_{\partial\Omega}\left[-u(x)\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{|x-x_0|}\frac{\partial u}{\partial n}(x)\right]dS.$$

现在证明 $(*)$. 令 $\Omega_\varepsilon=\Omega\setminus\bar{B}_\varepsilon$. 则 $\partial\Omega_\varepsilon=\partial\Omega\cup\partial B_\varepsilon$.

且在 $\Omega_\varepsilon$ 上, $\Delta\left(\frac{1}{4\pi|x|}\right)=0$, $\Delta u=0$. 由Green第二公式




$$0=\frac{1}{4\pi}\int_{\partial\Omega}\left(-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial n}\right)dS$$

$$+\frac{1}{4\pi}\int_{\partial B_\varepsilon}\left(-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial n}\right)dS \qquad (1)$$

$$(1)=\frac{1}{4\pi}\int_{\partial B_\varepsilon}u\frac{\partial}{\partial r}\left(\frac{1}{r}\right)+\frac{1}{r}\left(-\frac{\partial u}{\partial r}\right)dS=\underbrace{-\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon}u\,dS}_{1a}\ \underbrace{-\frac{1}{4\pi\varepsilon}\int_{\partial B_\varepsilon}\frac{\partial u}{\partial r}dS}_{1b}$$

$$1a=-\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon}(u(x)-u(0))dS-u(0)$$

$$\left|\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon}(u(x)-u(0))dS\right|\le\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon}|u(x)-u(0)|dS\le\max_{\partial B_\varepsilon}|\nabla u|\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon}|x|dS$$

$$\le\varepsilon\max_{\bar{\Omega}}|\nabla u|\le C\varepsilon\longrightarrow 0,\ \text{as } \varepsilon\to 0.$$

$$1b=-\frac{1}{4\pi\varepsilon}\int_{\partial B_\varepsilon}\vec{n}\cdot\nabla u\,dS=-\frac{1}{4\pi\varepsilon}\int_{B_\varepsilon}\Delta u\,dx=0,\ \forall\varepsilon>0.$$

于是 (1) $\longrightarrow -u(0)$, as $\varepsilon \to 0$.

因此，$u(0) = \frac{1}{4\pi}\int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right) + \frac{1}{|x|}\frac{\partial u}{\partial n}\right]dS$.

Claim: $n=2$, $\forall x_0 \in \Omega$,

$$u(x_0) = \frac{1}{2\pi}\int_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\log|x-x_0|\right) - \log|x-x_0|\frac{\partial u}{\partial n}\right]dS.$$

证明留作业.

以上公式中的问题在于无论哪种边值，无法知道 $u, \frac{\partial u}{\partial n}$ 在边值的值！

解决办法：若 $g(x)$ 在 $\Omega$ 上调和，且 $g(x)|_{\partial\Omega} = \frac{1}{4\pi|x-x_0|}\Big|_{\partial\Omega}$，则由 Green 第二公式，$\int_{\partial\Omega}\left(-u\frac{\partial g}{\partial n} + g\frac{\partial u}{\partial n}\right)dS = 0$.

由 (**)，$u(x_0) = \int_{\partial\Omega} -u\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-x_0|} - g\right) + \left(\frac{1}{4\pi|x-x_0|} - g\right)\frac{\partial u}{\partial n}\, dS$

令 $G(x) = -\frac{1}{4\pi|x-x_0|} + g(x)$，则 $G|_{\partial\Omega} = 0$. 故

$$u(x_0) = \int_{\partial\Omega} u\frac{\partial G}{\partial n}dS. \qquad (\triangle)$$

定义 (Green 函数) $\Omega$ 上的算子 $-\Delta$ 的 Green 函数，满足

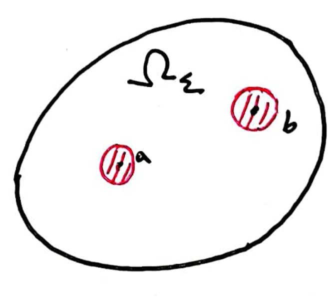
(1) $G(x) \in C^2(\Omega)$，且 $\Delta G = 0$，除了 $x = x_0$.

(2) $G(x) = 0$, $\forall x \in \partial\Omega$.

(3) $G(x) + \frac{1}{4\pi|x-x_0|}$ 在 $x_0$ 有限，处处二阶连续可导，且在 $x_0$ 调和.

性质：$G(x, x_0) = G(x_0, x)$, $\forall x \neq x_0$. (对易原理)

对 $u(x) = G(x,a)$, $v(x) = G(x,b)$ 在 $\Omega_\varepsilon$ 上应用 Green 第二公式，可得




$$0 = \int_{\Omega_\varepsilon}(u\Delta v - v\Delta u)dx = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$$

④

$$= \underbrace{\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS}_{\substack{\| \ (u,v)|_{\partial\Omega}=0 \\ 0}} + \underbrace{\int_{|x-a|=\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS}_{\overset{\Delta}{=} A_\varepsilon} + \underbrace{\int_{|x-b|=\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS}_{= B_\varepsilon}$$

故 $A_\varepsilon + B_\varepsilon = 0$.

$$A_\varepsilon = \underbrace{\int_{|x-a|=\varepsilon}\left(\left(u+\frac{1}{4\pi|x-a|}\right)\frac{\partial v}{\partial n}-v\frac{\partial}{\partial n}\left(u+\frac{1}{4\pi|x-a|}\right)\right)dS}_{=0} - \int_{|x-a|=\varepsilon}\left(\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n}-v\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-a|}\right)\right)dS$$

$$= -\frac{1}{4\pi\varepsilon}\int_{|x-a|=\varepsilon}\frac{\partial v}{\partial n}dS + \frac{1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon} v\,dS \xrightarrow{\varepsilon\to 0} v(a) = G(a,b)$$

同理 $\lim\limits_{\varepsilon\to 0} B_\varepsilon = -u(b) = -G(b,a)$.

故 $G(a,b) = G(b,a)$.

由 ($\Delta$) 知, $\begin{cases}\Delta u = 0 & \text{in } \Omega \\ u(x,0) = \varphi(x) & \text{on } \partial\Omega\end{cases}$ 的解为 $u(x) = \int_{\partial\Omega}\varphi(x)\frac{\partial G}{\partial n}dS$.

由 ($\Delta\Delta$), 取 $u$ 为 $\begin{cases}\Delta u = f & \text{in } \Omega \\ u(x,0) = \varphi & \text{on } \partial\Omega\end{cases}$ 的解, $v$ 为 Green 函数, 知

$$u(x) = \int_\Omega G f\,dx + \int_{\partial\Omega}\varphi\frac{\partial G}{\partial n}dS.$$

Green 函数的求法:

1. 半空间.


$$G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{1}{4\pi|x-x_0^*|},$$

$$x_0^* = (x_0^1, x_0^2, -x_0^3)$$

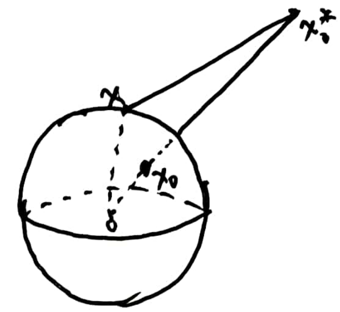
则 ① $G$ 在 $x \neq x_0$ 时二阶连续可微, 调和.

② $G(x,x_0) + \frac{1}{4\pi|x-x_0|}$ 在上半平面调和.

③ 当 $x_3 = 0$ 时, $G(x,x_0) = -\frac{1}{4\pi|(x_1-x_0^1, x_2-x_0^2, -x_0^3)|} + \frac{1}{4\pi|(x_1-x_0^1, x_2-x_0^2, x_0^3)|} = 0$

2. 球面. $\Omega = B_R(0) \subset R^3$

若 $x_0 \in B_R(0)$, 取 $0-x_0$ 延长线上一点. $x_0^*$



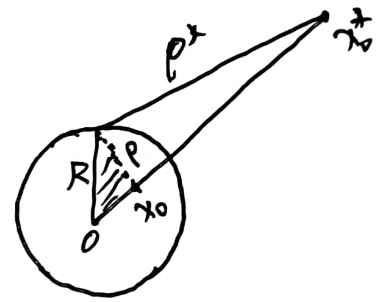

令 $G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{C}{4\pi|x-x_0^*|}$

则 (i) $G$ 在 $B_R(0)\setminus\{x_0\}$ 上二阶连续可微.

(ii) $G(x,x_0) + \frac{1}{4\pi|x-x_0|}$ 在 $B_R(0)$ 上二阶连续可微.

(iii). 选取 $x_0^*$, $C$, s.t. $G(x,x_0)=0$, $\forall |x|=1$.

令 $\rho = |x-x_0|$, $\rho^* = |x-x_0^*|$. s.t. $\forall |x|=1$, $\frac{\rho^*}{\rho} = C$.




若 $\Delta_小 \sim \Delta_大$. 则 $\frac{\rho_*}{\rho} = \frac{|x_0|}{R}$ (与 $x$ 无关).

选取 $x_0^*$, s.t. $\frac{|x_0^*|}{R} = \frac{R}{|x_0|}$ 即 $x_0^* = \frac{R^2}{|x_0|^2}x_0$

则 $\frac{\rho^*}{\rho} = \frac{R}{|x_0|}$ 故 $G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{R}{|x_0|}\frac{1}{4\pi|x-x_0^*|}$, $x_0^* = \frac{R^2}{|x_0|^2}x_0$.

$$\nabla G = \frac{x-x_0}{4\pi|x-x_0|^3} - \frac{R}{|x_0|}\frac{x-x_0^*}{4\pi|x-x_0^*|^3}$$

当 $|x|=1$, $\rho^* = \frac{R}{|x_0|}\rho$. 故 $\nabla G(x,x_0) = \frac{x-x_0}{4\pi|x-x_0|^3} - \frac{R}{|x_0|}\left(\frac{|x_0|}{R}\right)^3 \frac{x-x_0^*}{4\pi|x-x_0|^3}$

$$\Rightarrow \nabla G(x_0,x_0) = \left(1-\frac{|x_0|^2}{R^2}\right)\frac{x}{4\pi|x-x_0|^3} - \frac{1}{4\pi|x-x_0|^2}\left(x_0 - \frac{|x_0|^2}{R^2}x_0^*\right)$$

$$= \frac{R^2-|x_0|^2}{R^2}\frac{x}{4\pi|x-x_0|^3}$$

$$\Rightarrow \frac{\partial G}{\partial n} = \frac{x}{R}\cdot\nabla G = \frac{R^2-|x_0|^2}{R}\frac{1}{4\pi|x-x_0|^3}$$

$$\Rightarrow u(x_0) = \frac{R^2-|x_0|^2}{4\pi R}\int_{|x|=R}\frac{\varphi(x)}{|x-x_0|^3}dS$$

令 $K(x,y) = \frac{R^2-|x|^2}{4\pi R|x-y|^3}$, 则 $u(x) = \int_{|y|=R} K(x,y)\varphi(y)dS(y)$.

作业：分别用Green函数法和分离变量法求解二维圆盘上的Dirichlet问题 $\begin{cases}\Delta u=0 & \text{in } B_R(0)\\ u=\varphi & \text{on } \partial B_R(0)\end{cases}$

## §2. 调和函数的性质

### A. 平均值性质

极坐标公式：$\int_{\mathbb{R}^n} f(x)dx=\int_0^\infty\left(\int_{\partial B_r(x)} f(y)dS(y)\right)dr$

$$\frac{d}{dr}\left(\int_{B_r(x_0)} f\,dx\right)=\int_{\partial B(x_0,r)} f(y)dS(y).$$

定义：若 $u\in C(\Omega)$. (i) 称 $u$ 满足平均值性质，如果 $u(x)=\frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(y)dS_y$，$\forall B_r(x)\subset\Omega$. (ii) 称 $u$ 满足第二平均值性质，如果

$$u(x)=\frac{3}{4\pi r^3}\int_{B_r(x)} u(y)dy,\ \forall B_r(x)\subset\Omega.$$

注：两个定义等价. 若(i)成立，则 $\int_{B_r(x)} u(y)dy=\int_0^r\int_{\partial B_\rho(x)} u(y)dS(y)d\rho$

$$=\int_0^r 4\pi\rho^2 u(x)d\rho=\frac{4\pi}{3}r^3u(x).\quad \text{即 } u(x)=\frac{3}{4\pi r^3}\int_{B_r(x)} u(y)dy.$$

反之，若(ii)成立. $\frac{4\pi}{3}r^3u(x)=\int_0^r\int_{\partial B_\rho(x)} u(y)dS(y)d\rho$

对 $r$ 求导，可得 $4\pi r^2u(x)=\int_{\partial B_r(x)} u(y)dS(y)$. 故(i)成立.

定理2.1. 若 $u\in C^2(\Omega)$ 在 $\Omega$ 内调和，则 $u$ 满足平均值性质.

证明：若 $u\in C^2(\Omega)$，且 $\Delta u=0$. 则 $\forall B_r(x)\subset\Omega$，取 $0<\rho<r$. 由散度定理，

$$0=\int_{B_\rho(x)}\Delta u\,dx=\int_{\partial B_\rho(x)}\frac{\partial u}{\partial n}dS(y)=\int_{\partial B_\rho(x)}\frac{y-x}{\rho}\cdot(\nabla u)(y)\,dS(y)$$

$$\overset{\frac{y-x}{\rho}=w}{=\!=}\rho^2\int_{|w|=1} w\cdot(\nabla u)(x+\rho w)\,dw=\rho^2\int_{|w|=1}\frac{d}{d\rho}\big(u(x+\rho w)\big)\,dw$$

$$=\rho^2\frac{d}{d\rho}\int_{|w|=1} u(x+\rho w)\,dw$$

作业：分别用Green函数法和分离变量法求解二维圆盘上的Dirichlet问题 $\begin{cases}\Delta u=0 & \text{in } B_R(0)\\ u=\varphi & \text{on } \partial B_R(0)\end{cases}$

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$$\frac{d}{dr}\left(\int_{B_r(x_0)} f\,dx\right)=\int_{\partial B(x_0,r)} f(y)\,dS(y).$$

定义：若 $u\in C(\Omega)$. (i) 称 $u$ 满足平均值性质，如果 $u(x)=\frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(y)\,dS_y$, $\forall B_r(x)\subset\Omega$. (ii) 称 $u$ 满足第二平均值性质，如果

$u(x)=\frac{3}{4\pi r^3}\int_{B_r(x)} u(y)\,dy$, $\forall B_r(x)\subset\Omega$.

注：两个定义等价. 若(i)成立. 则 $\int_{B_r(x)} u(y)\,dy=\int_0^r\int_{\partial B_\rho(x)} u(y)\,dS(y)\,d\rho$

$=\int_0^r 4\pi\rho^2 u(x)\,d\rho=\frac{4\pi}{3}r^3 u(x)$. 即 $u(x)=\frac{3}{4\pi r^3}\int_{B_r(x)} u(y)\,dy$.

反之，若(ii)成立. $\frac{4\pi}{3}r^3 u(x)=\int_0^r\int_{\partial B_\rho(x)} u(y)\,dS(y)\,d\rho$

对 $r$ 求导，可得 $4\pi r^2 u(x)=\int_{\partial B_r(x)} u(y)\,dS(y)$. 故 (i) 成立.

定理2.1. 若 $u\in C^2(\Omega)$ 在 $\Omega$ 内调和，则 $u$ 满足平均值性质.

证明：若 $u\in C^2(\Omega)$，且 $\Delta u=0$. 则 $\forall B_r(x)\subset\Omega$，取 $0<\rho<r$. 由散度定理，

$$0=\int_{B_\rho(x)}\Delta u\,dx=\int_{\partial B_\rho(x)}\frac{\partial u}{\partial n}\,dS(y)=\int_{\partial B_\rho(x)}\frac{y-x}{\rho}\cdot(\nabla u)(y)\,dS(y)$$

$$\overset{\frac{y-x}{\rho}=w}{=\!=}\rho^2\int_{|w|=1} w\cdot(\nabla u)(x+\rho w)\,dw=\rho^2\int_{|w|=1}\frac{d}{d\rho}\big(u(x+\rho w)\big)\,dw$$

于是，$\frac{d}{d\rho}\int_{|w|=1} u(x+\rho w)dw = 0$. 单因此，

$$4\pi u(x) = \int_{|w|=1} u(x)dw = \int_{|w|=1} u(x+rw)dw$$

$$\Rightarrow u(x) = \frac{1}{4\pi r^2}\int_{|y-x|=r} u(y)dS(y)$$

定理2.2. 若$u\in C(\Omega)$在$\Omega$内具有平均值性质，则$u$在$\Omega$上光滑且调和.

证明：令$\varphi\in C_0^\infty(B_1(0))$，$\int\varphi=1$ 且$\varphi$ radial 即$\varphi(x)=\varphi(|x|)$. 于是，$\int\varphi(r)r^{n-1}dr=\frac{1}{w_n}$.

令$\varphi_\varepsilon(z)=\frac{1}{\varepsilon^n}\varphi(\frac{z}{\varepsilon})$，$\varepsilon>0$. 则$\forall x\in\Omega$，取$\varepsilon<dis(x,\partial\Omega)$，则

Claim: $u(x)=(u*\varphi_\varepsilon)(x)$，

In fact, $\int u(y)\varphi_\varepsilon(x-y)dy \xlongequal[\text{故 } y\in\Omega]{x\in\Omega,\ |x-y|<\varepsilon} \int_\Omega u(y)\varphi_\varepsilon(x-y)dy \overset{\varphi\text{ radial}}{=} \int_\Omega u(y)\varphi_\varepsilon(y-x)dy$

$$\xlongequal{z=y-x} \int_{\Omega-x} u(z+x)\varphi_\varepsilon(z)dz \xlongequal{\operatorname{supp}\varphi} \int_{B_\varepsilon(0)} u(z+x)\varphi_\varepsilon(z)dz$$

$$\xlongequal{z=\varepsilon y} \int_{B_1(0)} u(\varepsilon y+x)\varphi(y)dy$$

$$\xlongequal{y=rw} \int_0^1\int_{S^n} \varphi(r)u(r\varepsilon w+x)r^{n-1}dr\,dw$$

$$=\int_0^1\left(\int_{\partial B_1(0)} u(r\varepsilon w+x)dw\right)\varphi(r)r^{n-1}dr$$

$$\xlongequal{\text{平均值性质}} \int_0^1 u(x)\varphi(r)r^{n-1}dr|w_n| = u(x).$$

故$u\in C^\infty(\Omega)$. 且$\forall r>0$，$\forall x\in\Omega$

$$\int_{B_r(x)}\Delta u\,dy = \int_0^r r^{n-1}\frac{d}{dr}\int_{|w|=1}u(x+rw)dS(w)=0.$$

故$\Delta u=0$ in $\Omega$.（否则，$\exists x_0\in\Omega$, s.t. $\Delta u(x_0)>0$. 则$\exists r_0>0$, s.t. $\Delta u(x)>0$, $\forall x\in B_{r_0}(x_0)$ 故$\int_{B_{r_0}(x_0)}\Delta u(y)dy>0$. 矛盾!）.

命题2.3. 若 $u\in C(\bar{\Omega})$ 满足平均值性质，则 $u$ 只在 $\partial\Omega$ 上达到最大值和最小值，除非 $u$ 是常数.

证：只证明最大值的情形. 令 $\Sigma=\{x\in\Omega \mid u(x)=M=\max_{\bar{\Omega}} u\}\subset\Omega$.

(i) 若 $\{x_n\}\subset\Sigma$，且 $x_n\to\bar{x}$. 则由 $u\in C(\bar{\Omega})$ 知 $u(\bar{x})=\lim_{n\to\infty}u(x_n)=M$.

故 $\bar{x}\in\Sigma$. 即 $\Sigma$ 是闭集.

(ii) 对 $\forall x_0\in\Sigma$. 则 $\exists r_0>0$, s.t. $B_{r_0}(x_0)\subset\Omega$. 由平均值定性，

$$M=u(x_0)=\frac{1}{|B_{r_0}(x_0)|}\int_{B_{r_0}(x_0)}u(y)\,dy\le M.\quad 故\ u(x_0)\equiv M\ \text{in}\ B_{r_0}(x_0).$$

即 $B_{r_0}(x_0)\subset\Sigma$. 故 $\Sigma$ 是开集.

从而，$\Sigma$ 在 $\Omega$ 中既开又闭. 故 $\Sigma=\phi$ 或 $\Omega$.

命题2.4 (梯度估计) 设 $u\in C(\bar{B}_R)$ 是调和的，($B_R=B_R(x_0)$)，则成立

$$|Du(x_0)|\le\frac{n}{R}\max_{\bar{B}_R}|u|.$$

证明：为简单起见，设 $u\in C^1(\bar{B}_R)$. 则由定理2.1，$u$ 满足平均值性质.

则由定理2.2，$u$ 在 $B_R$ 光滑. 因此 $\partial_{x_i}u$ 也是调和. 即 $\partial_{x_i}\Delta u=0$. 再由定理2.1，

$\partial_{x_i}u$ 满足平均值性质，即

$$\partial_{x_i}u(x_0)=\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}\partial_{x_i}u(y)\,dy \overset{散度定理}{=} \frac{1}{|B(x_0)|}\int_{\partial B_R(x_0)}u\nu_i\,dS$$

$$\Rightarrow |\partial_{x_i}u|\le\max|u|\frac{|\partial B(x_0)|}{|B(x_0)|}\le\frac{n}{R}\max_{x\in\bar{B}_R}|u|.\quad\Rightarrow |Du(x_0)|=\left|\frac{1}{|B(x_0)|}\int_{\partial B(x_0)}u\vec{\nu}\,dS\right|\le\frac{n}{R}\max_{x\in\bar{B}_R}|u|.$$

命题 设 $u\in C(\bar{B}_R)$ 是 $B_R=B_R(x_0)$ 上的非负调和函数. 则成立

$$|Du(x_0)|\le\frac{n}{R}u(x_0).$$

证明：$|\partial_{x_i} u(x_0)| \le \frac{1}{|B_R(x_0)|}\int_{\partial B_R(x_0)} |u(y)|\,dS = \frac{1}{|B_R(x_0)|}\int_{\partial B_R(x_0)} u(y)\,dS \le \frac{|\partial B_R(x_0)|}{|B_R(x_0)|} u(x_0)$

故 $|\partial_{x_i} u(x_0)| \le \frac{n}{R} u(x_0)$.

推论. $\mathbb{R}^n$上的上有界或下有界的调和函数是常数. (Liouville定理)

证：设 $u \le M$ 是$\mathbb{R}^n$上的调和函数. 则 $M-u \triangleq v \ge 0$ 仍是调和的.

$\forall x_0 \in \mathbb{R}^n$, $\forall R>0$. 应用上述命题, $|\nabla v(x_0)| \le \frac{n}{R} v(x_0) \le \frac{Kn}{R}$

令$R\to+\infty$. 则 $|\nabla v(x_0)|=0$. 故 $v$是常数.

(不讲) 推论 (表示公式) 令$f\in C_c^2(\mathbb{R}^3)$ 则 $\Delta u = f$ 在 $\mathbb{R}^3$ 的任意有界解具有以下形式:

$$u(x) = \frac{1}{4\pi}\int \frac{1}{|x-y|} f(y)\,dy + C.$$

令 $v(x) = \frac{1}{4\pi}\int \frac{1}{|x-y|} f(y)\,dy$. 则 $\Delta v = f$. 故 $\Delta(u-v)=0$.

由于$u$有界. 又 $f\in C_c^2(\mathbb{R}^3)$, $|v(x)| \le \frac{1}{4\pi}\int \frac{1}{|x-y|}|f(y)|\,dy \le \frac{1}{4\pi}\int_{|y|\le K} \frac{|f(y)|}{|x-y|}\,dy$

$\le \frac{1}{4\pi}\|f\|_{L^\infty}\int_{|y|\le K}\frac{1}{|x-y|}\,dy \le C_K \|f\|_{L^\infty}$

故$v$有界. 因此, $u-v$有上界.

因此, $u-v=C$.

命题 (高阶梯度估计) 设 $u\in C(\overline{B_R})$ 在 $B_R = B_R(x_0)$ 内是调和的, 则对任意的多重指标$\alpha$, $|\alpha|=m$, $|D^\alpha u(x_0)| \le \frac{n^m e^{m-1} m!}{R^m}\max_{\overline{B_R}}|u|$.

证明：对$m$用数学归纳法. 当$m=1$时 $|Du(x_0)| \le \frac{n}{R}\max_{B_R}|u|$ 已证.

设$m$时结论成立. 则当$|\alpha|=m+1$时, 对 $D^m u$ 在$B_r$上应用梯度估计,

可得 $|D^{m+1}u(x_0)| \le \frac{n}{r}\max_{B_r}|D^m u| \le \frac{n}{r}\frac{n^m e^{m-1} m!}{(R-r)^m}\max_{B_r}|u|$

令 $r=(1-\theta)R$，则 $|D^{m+1}u(x_0)| \le \dfrac{n^{m+1}e^{m-1}m!}{R^{m+1}\theta^m(1-\theta)} \max\limits_{\overline{B}_R}|u|$.

取 $\theta$ 使得 $\theta^m(1-\theta)$ 达到最大值，即 $\theta=\frac{m}{m+1}$，则

$$|(D^{m+1}u)(x_0)| \le \frac{n^{m+1}e^{m-1}m!}{R^{m+1}}\left(1+\frac{1}{m}\right)^m(m+1)\max_{\overline{B}_R}|u|$$

$$\le \frac{n^{m+1}(m+1)!\,e^m}{R^{m+1}}\max_{\overline{B}_R}|u|$$

定理（解析性）调和函数是解析的。

设 $u$ 在 $\Omega$ 内调和，$\forall x\in\Omega$，$\exists R>0$. s.t. $\overline{B}_{2R}(x)\subset\Omega$. $\forall h$，$|h|<R$，由 Taylor 展开，

$$u(x+h)=u(x)+\sum_{i=1}^{m-1}\frac{1}{i!}\left[\left(h_1\frac{\partial}{\partial x_1}+\cdots+h_n\frac{\partial}{\partial x_n}\right)^i u\right](x)+R_m(h)$$

$$R_m(h)=\frac{1}{m!}\left[\left(h_1\frac{\partial}{\partial x_1}+\cdots+h_n\frac{\partial}{\partial x_n}\right)^m u\right](x_1+h_1\theta,\cdots,x_n+\theta h_n) \quad \text{for some } 0<\theta<1.$$

由高阶梯度估计，$|R_m(h)| \le \frac{1}{m!}|h|^m\max\limits_{\overline{B}_R}|D^m u| \le \frac{1}{m!}|h|^m\dfrac{n^m e^{m-1}m!}{R^m}\max\limits_{\overline{B}_{2R}}|u|$

$$\le \frac{1}{e}\left(\frac{|h|ne}{R}\right)^m\max_{\overline{B}_{2R}}|u|$$

选取 $h$，s.t. $|h|<\frac{R}{ne}$，则 $|R_m(h)|\to 0$, as $m\to+\infty$.

故 $u$ 在 $\Omega$ 内解析.

B. Harnack 不等式及应用.

定理 设 $u$ 在 $\Omega$ 内调和，则对于 $\Omega$ 中的任意紧集 $K$，存在正常数 $C=C(\Omega,K)$ 使得若 $u\ge 0$ in $\Omega$，则 $\frac{1}{C}u(y)\le u(x)\le Cu(y)$，$\forall x,y\in K$.

证明：Step 1. 取 $R=\frac{1}{4}\mathrm{dist}(K,\partial\Omega)$，则 $\forall x\in K$，$B_{4R}(x)\subset\Omega$.

若 $|x-y|<R$，则由平均值定理及 $B_R(y)\subset B_{2R}(x)$，有

$$u(x)=\frac{1}{4\pi(2R)^3}\int_{B_{2R}(x)}u(y)\,dy \ge \frac{1}{4\pi(2R)^3}\int_{B_R(x)}u(y)\,dy=\frac{1}{2^3}u(y)$$

令 $r=(1-\theta)R$，则 $|D^{m+1}u(x_0)| \le \dfrac{n^{m+1}e^{m-1}m!}{R^{m+1}\theta^m(1-\theta)}\max\limits_{\overline{B}_R}|u|$. (10)

取 $\theta$ 使得 $\theta^m(1-\theta)$ 达到最大值. 即 $\theta=\frac{m}{m+1}$. 则

$$|(D^{m+1}u)(x_0)| \le \frac{n^{m+1}e^{m-1}m!}{R^{m+1}}\left(1+\frac{1}{m}\right)^m(m+1)\max_{\overline{B}_R}|u|$$

$$\le \frac{n^{m+1}(m+1)!\,e^m}{R^{m+1}}\max_{\overline{B}_R}|u|$$

定理 (解析性) 调和函数是解析的.

设 $u$ 在 $\Omega$ 内调和，$\forall x\in\Omega$，$\exists R>0$. s.t. $\overline{B}_{2R}(x)\subset\Omega$. $\forall h$, $|h|<R$, 由Taylor展开,

$$u(x+h)=u(x)+\sum_{i=1}^{m-1}\frac{1}{i!}\left[\left(h_1\frac{\partial}{\partial x_1}+\cdots+h_n\frac{\partial}{\partial x_n}\right)^i u\right](x)+R_m(h)$$

$$R_m(h)=\frac{1}{m!}\left[\left(h_1\frac{\partial}{\partial x_1}+\cdots+h_n\frac{\partial}{\partial x_n}\right)^m u\right](x_1+h_1\theta,\cdots,x_n+\theta h_n)\ \text{for some } 0<\theta<1.$$

由高阶梯度估计. $|R_m(h)| \le \frac{1}{m!}|h|^m\max\limits_{\overline{B}_R}|D^m u| \le \frac{1}{m!}|h|^m\frac{n^m e^{m-1}m!}{R^m}\max\limits_{\overline{B}_{2R}}|u|$

$$\le \frac{1}{e}\left(\frac{|h|ne}{R}\right)^m\max_{\overline{B}_{2R}}|u|$$

选取 $|h|$, s.t. $|h|<\frac{R}{ne}$, 则 $|R_m(h)|\to 0$, as $m\to+\infty$.

故 $u$ 在 $\Omega$ 内解析.

B. Harnack不等式及应用.

定理 设 $u$ 在 $\Omega$ 内调和，则对于 $\Omega$ 中的任意紧集 $K$，存在常数 $C=C(\Omega,K)$

使得若 $u\ge 0$ in $\Omega$，则 $\frac{1}{C}u(y)\le u(x)\le Cu(y)$, $\forall x,y\in K$.

证明: Step1. 取 $R=\frac{1}{4}\mathrm{dist}(K,\partial\Omega)$，则 $\forall x\in K$, $B_{4R}(x)\subset\Omega$.

若 $|x-y|<R$，则由平均值定理及 $B_R(y)\subset B_{2R}(x)$. 有

$$u(x)=\frac{1}{4\pi(2R)^3}\int_{B_{2R}(x)}u(y)\,dy \ge \frac{1}{4\pi(2R)^3}\int_{B_R(y)}u(y)\,dy=\frac{1}{2^3}u(y)$$

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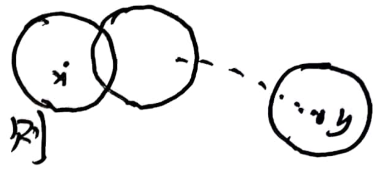
Step 2. 由于K紧，存在有限个半径为R的球$\{B_i: i=1,\dots,N\}$构成K的开覆盖.

不断重复Step 1，可得 $u(x) \ge \frac{1}{2^{7(N+1)}} u(y)$，$\forall x,y \in K$.



命题 (Harnack不等式) 设u在$B_R(x_0)$内调和且$u \ge 0$，则

$$\frac{R}{R+r}\frac{R-r}{R+r}u(x_0) \le u(x) \le \frac{R}{R-r}\frac{R+r}{R-r}u(x_0)，其中\ r=|x-x_0|<R.$$

证明：不妨设 $x_0=0$. 由Poisson公式，

$$u(x)=\frac{R^2-|x|^2}{4\pi R}\int_{\partial B_R}\frac{u(y)}{|x-y|^3}dy, \quad \forall x \in B_R.$$

由于$|y|=R$，故 $R-|x| \le |x-y| \le R+|x|$. 因此，

$$u(x) \le \frac{R^2-|x|^2}{4\pi R(R-|x|)^3}\int_{\partial B_R}u(y)dy = \frac{(R^2-|x|^2)4\pi R^2}{4\pi R(R-|x|)^3}\frac{1}{4\pi R^2}\int_{\partial B_R}u(y)dy$$

$$= \frac{(R+|x|)R}{(R-|x|)^2}u(x_0)$$

且
$$u(x) \ge \frac{R^2-|x|^2}{4\pi R(R+|x|)^3}\int_{\partial B_R}u(y)dy = \frac{R(R-|x|)}{(R+|x|)^2}u(x_0)$$

推论：若u是$\mathbb{R}^3$上的上有界或下有界调和函数，则u是常数.

证：设$u \ge 0$. $\forall x \in \mathbb{R}^3$，在$B_R(0)$ $(R>|x|)$上应用Harnack不等式，

则
$$\frac{(R+|x|)R}{(R-|x|)^2}u(0) \ge u(x) \ge \frac{(R-|x|)R}{(R+|x|)^2}u(0)$$

令$R \to +\infty$，则有 $u(x)=u(0)$.

定理（奇点可去定理）设u在$B_R\setminus\{0\}$内调和且满足 $u(x)=\begin{cases} o(\log|x|) & n=2 \\ o(|x|^{-1}) & n=3 \end{cases}$

as $|x| \to 0$，则可将u在原点定义~~数~~使得$u \in C^2(B_R)$且在$B_R$内调和.

证明：只证明 $n=3$ 的情形. 令 $\begin{cases}\Delta v=0 & \text{in } B_R(0)\\ v=u & \text{on } \partial B_R(0).\end{cases}$ 由Poisson

公式，$v$存在. 且 $v\in C^2(B_R)\cap C(\bar{B}_R)$. 只需证明 $v=u$ in $B_R\setminus\{0\}$.

则可重新定义 $u(0)=v(0)$ 使得 $u\in C^2(B_R)$ 在$B_R$调和.

令 $w=v-u$ in $B_R\setminus\{0\}$. $\forall 0<r<R$, 令 $M_r=\max\limits_{\partial B_r}|w|$. 则

$$|w(x)|\le M_r\frac{r}{|x|} \quad \text{on } \partial B_r.$$

由于 $M_r\frac{r}{|x|}\pm w(x)$ 在 $r<|x|<R$ 调和. ~~故~~ $W(x)\ge 0$, $|x|=r$,

$W(x)=M_r\frac{r}{R}>0$, $|x|=R$. 故由极大值原理，$W(x)>0$, $\forall r<|x|<R$.

故 $|w(x)|\le M_r\frac{r}{|x|}$, $\forall r<|x|<R$.

由于 ~~……~~ $v$是调和的，$\max\limits_{\bar{B}_R}|v|=\max\limits_{\partial B_R}|v|=\max\limits_{\partial B_R}|u|$. 故

$$\max_{\partial B_r}|v|\le\max_{\partial B_R}|u|\triangleq M.$$

于是，$M_r\le\max\limits_{\partial B_r}|v|+\max\limits_{\partial B_r}u\le M+\max\limits_{\partial B_r}|u(x)|$.

因此，$|w(x)|\le\frac{r}{|x|}M+\frac{r}{|x|}\max\limits_{\partial B_r}|u(x)|$, $\forall r<|x|<R$.

由已知条件，$\max\limits_{\partial B_r}|u(x)|\to 0$, as $\frac{r}{|x|}\to 0$. 因此，$r\max\limits_{\partial B_r}|u(x)|\to 0$. as $r\to 0$.

令 $r\to 0$. 则有 $w(x)=0$. in $B_R\setminus\{0\}$.

## §3. 极大值原理. 第二边值解的唯一性.

定理3.1 设 $u\in C^2(B_1)\cap C(\bar{B}_1)$ 是$B_1$上的次调和函数，即 $\Delta u\ge 0$. 则有 $\sup\limits_{B_1}u\le\sup\limits_{\partial B_1}u$.

证明：令 $u_\varepsilon(x)=u(x)+\varepsilon|x|^2$, $\varepsilon>0$. 则 $\Delta u_\varepsilon=\Delta u+\cancel{6\varepsilon}>0$

若$u_\varepsilon$在$B_1$的内部$x_0$达到最大值，则 $\Delta u_\varepsilon(x_0)\le 0$. 与 $\Delta u_\varepsilon>0$ 矛盾！

因此，$u_\varepsilon$ 的最大值只能在边界取到。即 $\sup\limits_{x\in B_1} u_\varepsilon = \sup\limits_{x\in\partial B_1} u_\varepsilon \le \sup\limits_{x\in\partial B_1} u + \varepsilon$

$\sup\limits_{B_1} u \le$

令 $\varepsilon\to 0^+$，则 $\sup\limits_{B_1} u \le \sup\limits_{\partial B_1} u$.

定理 3.2（内部梯度估计）设 $u$ 在 $B_1$ 内调和，则有 $\sup\limits_{B_{\frac12}} |Du| \le C\sup\limits_{\partial B_1}|u|$.

其中 $C=C(n)$ 是正常数。特别地，$\forall\,\alpha\in[0,1]$，

$$|u(x)-u(y)| \le C|x-y|^\alpha \sup_{\partial B_1}|u|,\ \forall\, x,y\in B_{\frac12},\ C=C(n,\alpha)>0.$$

证明：要估计 $|Du|$，只要考察 $|Du|^2$ 满足的方程

$$\Delta(|Du|^2) = \sum_{i,j=1}^n \partial_i^2(|\partial_j u|^2) = \sum_{i,j}\partial_i(2\partial_j u\,\partial_i\partial_j u) = \sum_{i,j}\left(2|\partial_i\partial_j u|^2 + 2\partial_j u\,\partial_j\partial_i^2 u\right)$$

$$= 2\sum_{i,j=1}^n |\partial_j\partial_i u|^2 \ge 0.$$

但是，由最大值原理，

$\max\limits_{B_1}|Du| \le \max\limits_{\partial B_1}|Du|$.

令 $\varphi\in C_0^\infty(B_1)$，则

$$\Delta(\varphi|Du|^2) = \Delta\varphi\,|Du|^2 + 2\nabla\varphi\cdot\nabla(|Du|^2) + \varphi\,\Delta(|Du|^2)$$

$$= \Delta\varphi\,|Du|^2 + 4\partial_j\varphi\,\partial_i u\,\partial_j\partial_i u + 2\varphi\,|\partial_i\partial_j u|^2 \quad ? \ -C|Du|^2$$

令 $\eta\in C_0^\infty(B_1)$，$\eta=1$ on $B_{\frac12}$。令 $\varphi=\eta^2$，则

$$\Delta(\eta^2|Du|^2) = \Delta(\eta^2)|Du|^2 + 8\eta\,\partial_j\eta\,\partial_i u\,\partial_i\partial_j u + 2\eta^2|\partial_i\partial_j u|^2$$

$$\ge \Delta(\eta^2)|Du|^2 - \frac{|2\eta\,\partial_i\partial_j u|^2 + |4\partial_j\eta\,\partial_i u|^2}{2} + 2\eta^2|\partial_i\partial_j u|^2$$

$$\ge \Delta(\eta^2)|Du|^2 - 8|D\eta|^2|Du|^2 \ge -C|Du|^2$$

$$\Delta(u^2) = 2u\Delta u + 4|Du|^2 = 4|Du|^2$$

于是，$\Delta(\eta^2|Du|^2 + \frac{C}{4}u^2) \ge 0$. 由极大值原理，

$$\max_{B_1}\left(\eta^2|Du|^2 + \frac{C}{4}u^2\right) = \max_{\partial B_1}\left(\eta^2|Du|^2 + \frac{C}{4}u^2\right) = \frac{C}{4}\left(\max_{\partial B_1}|u|\right)^2$$

$$\vee\!\!\!\vee$$

$$\left(\max_{B_{\frac12}}|Du|\right)^2 \quad\Rightarrow\quad \max_{B_{\frac12}}|Du| \le C\max_{\partial B_1}|u|.$$

引理3.3. 设$u$是$B_1$上的非负调和函数. 则存在$C=C(n)$, 使得

$$\sup_{B_{\frac{1}{2}}} |D\log u| \le C.$$

证明: (不讲).

推论3.4. 设$u$是$B_1$上的非负调和函数. 则存在$C=C(n)$, s.t.

$$u(x_1) \le C u(x_2), \quad \forall x_1, x_2 \in B_{\frac{1}{2}}.$$

若$u$在内部一点为零, 则$u\equiv 0$.

证明: 设$u>0$ in $B_1$. 由引理3.3, $\forall x_1, x_2 \in B_{\frac{1}{2}}$,

$$|\log u(x_1) - \log u(x_2)| = \left|\int_0^1 D\log u(tx_2+(1-t)x_1)\, dt\, (x_2-x_1)\right|$$

$$\le C|x_1-x_2| \le C$$

故 $u(x_1) \le C\, u(x_2)$.

命题3.5 (Hopf 引理) 设$u\in C(\bar{B}_1)$是$B_1=B_1(0)$上的调和函数. 若存在$x_0\in\partial B_1$, 使得$u(x)<u(x_0)$, $\forall x\in B_1$, 则存在$C=C(n)$使得$\frac{\partial u}{\partial n}(x_0) \ge C(u(x_0)-u(0))>0$.

证明: Step 1. 令$v(x) = e^{-\alpha|x|^2} - e^{-\alpha}$, 则 $v(x)>0$. $\forall x\in B_1$

且 $\partial_i v = -2\alpha x_i e^{-\alpha|x|^2}$, $\partial_i^2 v = -2\alpha e^{-\alpha|x|^2} + 4\alpha^2 x_i^2 e^{-\alpha|x|^2}$

$$\Rightarrow \Delta v = (-2\alpha n + 4\alpha^2|x|^2)\, e^{-\alpha|x|^2}$$

当$\frac{1}{2}<|x|<1$时, $-2\alpha n + 4\alpha^2|x|^2 > -2\alpha n + \alpha^2 > 0$ 若$\alpha = 2n+1$.

故当$\alpha = 2n+1$时, $\Delta v > 0$.

Step 2. $\forall \varepsilon>0$, 定义$h_\varepsilon(x) = u(x) - u(x_0) + \varepsilon v(x)$. 则 $\Delta h_\varepsilon = \Delta u + \varepsilon\Delta v = \varepsilon \Delta v > 0$, $\forall \frac{1}{2}<|x|<1$. 且 $h_\varepsilon(x) \le \max_{|x|=\frac{1}{2}} u(x) - u(x_0) + \varepsilon\left(e^{-\frac{\alpha}{4}} - e^{-\alpha}\right) \underset{\varepsilon\text{充分小}}{<} 0$, 当$|x|=\frac{1}{2}$

当$|x|=1$，$h_\varepsilon(x) = \max u|_{\partial B_1} - u(x_0) \le 0$. 由极大值原理，$h_\varepsilon$在$\{|x|=\frac{1}{2}\}\cup\{|x|=1\}$达到最大值. 又$h_\varepsilon(x_0)=0$，故$h_\varepsilon$在$x_0$处达到最大值. 于是，

$\frac{\partial h_\varepsilon}{\partial n}(x_0)\ge 0$. 即 $\frac{\partial u}{\partial n}(x_0) - 2\alpha x e^{-\alpha}\cdot\frac{x}{|x|}\varepsilon = \frac{\partial u}{\partial n}(x_0) - 2\alpha\varepsilon e^{-\alpha} \ge 0$.

故 $\frac{\partial u}{\partial n}(x_0) \ge 2\alpha\varepsilon e^{-\alpha} > 0$.

Step 3 估计$\varepsilon$. 令$w(x) = u(x_0) - u(x) > 0$，则$\Delta w = 0$ in $B_1$. 由Harnack不等式，

$$w(0) \le C \inf_{B_{\frac{1}{2}}} w(x) \iff u(x_0) - u(0) \le C \inf (u(x_0) - u(x)) = C\left(u(x_0) - \max_{B_{\frac{1}{2}}} u(x)\right)$$

即 $\max_{B_{\frac{1}{2}}} u(x) \le u(x_0) - \frac{1}{C}(u(x_0) - u(0))$

取$\delta$充分小，令$\varepsilon = \frac{\delta}{C}(u(x_0) - u(0))$. 即可. $\boxed{}$

称$\Omega$满足内切球条件，如果在它边界$\Gamma$上的每一点都可作一完全落在$\Omega$的球且与$\Gamma$在$x_0$相切.

定理3.6. 设$\Omega$具有内切球性质. 若$u\in C(\bar\Omega)$，$u$不恒为常数且调和，它在$x_0\in\partial\Omega$取最大值，则$\frac{\partial u}{\partial n}(x_0) > 0$.

证明：由于$u$不是常数，由极大值原理，$u$只在边界取到最大值.

由于$\Omega$具有内切球性质，在$x_0\in\partial\Omega$处，存在球$B\subset\Omega$且$B$在$x_0$处与$\partial\Omega$相切，且$\forall x\in B$，$u(x) < u(x_0)$. 由Hopf引理，$\frac{\partial u}{\partial n}(x_0) > 0$.

定理3.11（第二边值问题解的唯一性）若 $\Omega$ 满足内切球条件，则第二边值问题的解 $u\in C(\bar\Omega)\cap C^2(\Omega)$（在相差一个常数意义下）是唯一的。

证明：设 $u_1, u_2$ 是两个不同的解，则 $u=u_1-u_2$ 满足 $\begin{cases}\Delta u=0 & \text{in } \Omega\\ \frac{\partial u}{\partial n}=0 & \text{on } \partial\Omega.\end{cases}$

由极大值原理，若 $u$ 不是常数，则 $u$ 的最大值只能在边界取到。设 $u$ 在 $x_0\in\partial\Omega$ 达到最大值。由于 $\Omega$ 满足内切球性质，由Hopf引理，$\frac{\partial u}{\partial n}(x_0)>0$。这与 $\frac{\partial u}{\partial n}(x_0)=0$ 矛盾！故 $u$ 只能是常数。

定理3.12（最大模估计）设 $u\in C^2(\Omega)\cap C(\bar\Omega)$ 是 $\begin{cases}\Delta u=f & \text{in } \Omega\\ u=\varphi & \text{on } \partial\Omega\end{cases}$ 的解

则存在 $C=C(n,\Omega)>0$ 使得 $\max\limits_{\bar\Omega}|u|\le\Phi+CF$。

其中 $\Phi=\max\limits_{\partial\Omega}|\varphi|$，$F=\sup\limits_{\Omega}|f|$。

证明：不妨设 $\Omega$ 在条件区域 $0<x_1<d$ 中。作辅助函数 $v(x)=\Phi+(e^{ad}-e^{ax_1})F$，$a>1$

且在 $\Omega$ 中 $\begin{cases}\Delta(u-v)=f+a^2e^{ax_1}F\ge F-|f|\ge 0.\\ (u-v)|_{\partial\Omega}=\varphi-\Phi-(e^{ad}-e^{ax_1})F\le 0\end{cases}$

于是，由极大值原理，$u-v\le 0$ in $\Omega$.

因此，$u(x)\le v(x)=\Phi+(e^{ad}-e^{ax_1})F\le\Phi+(e^{ad}-1)F$，$\forall x\in\Omega$.

对 $-u$ 可类似证明。因此有 $\max\limits_{\bar\Omega}|u(x)|\le\Phi+(e^{ad}-1)F$. ∎

Rmk：最大模估计意味着Dirichlet问题解的唯一性和稳定性。

能量法证明Dirichlet问题解的唯一性：$\begin{cases}\Delta u=0 & \text{in } \Omega\\ u|_{\partial\Omega}=0\end{cases}$，要证 $u\equiv 0$ in $\Omega$.

$$0=\int_\Omega u\Delta u=\int_{\partial\Omega}u\frac{\partial u}{\partial n}dS-\int_\Omega|\nabla u|^2=-\int_\Omega|\nabla u|^2dx.$$

故 $\nabla u\equiv 0$。从而 $u\equiv\text{const}$。又 $u=0$ on $\partial\Omega$。故 $u\equiv 0$，in $\Omega$.