5 (a) $\begin{cases} u_{tt} - a^2 u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u|_{x-at=0} = \varphi(x),\quad u|_{x+at=0} = \psi(x),\quad \varphi(0) = \psi(0). & (a=1) \end{cases}$

(a):

换元，令 $x-at = \xi$，$x+at = \eta$，则有

$$x = \frac{\xi+\eta}{2},\quad t = \frac{\eta-\xi}{2}.$$

于是我们有 $\tilde{u}(\xi,\eta) = u(x,t) = u\left(\frac{\eta+\xi}{2}, \frac{\eta-\xi}{2}\right)$.

进而有 $u_t = -\tilde{u}_\xi + \tilde{u}_\eta \qquad u_{tt} = \tilde{u}_{\xi\xi} - 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}$

$u_x = \tilde{u}_\xi + \tilde{u}_\eta \qquad u_{xx} = \tilde{u}_{\xi\xi} + 2\tilde{u}_{\xi\eta} + \tilde{u}_{\eta\eta}$

进而，由于 $u_{tt} - u_{xx} = 0 \Rightarrow \tilde{u}_{\xi\eta} = 0$.

$\Rightarrow \tilde{u}(\xi,\eta) = F(\xi) + G(\eta)$.

由初值条件 $\tilde{u}(0,\eta) = u\left(\frac{\eta}{2}, \frac{\eta}{2}\right) = \varphi\left(\frac{\eta}{2}\right)$

$\qquad\qquad\qquad\ \ \| $

$\qquad\qquad F(0) + G(\eta)$.

$\tilde{u}(\xi,0) = u\left(\frac{\xi}{2}, -\frac{\xi}{2}\right) = \psi\left(\frac{\xi}{2}\right)$

$\quad \|$

$F(\xi) + G(0)$.

令 $\eta = 0 \Rightarrow F(0) + G(0) = \varphi(0) = \psi(0)$

$\Rightarrow \tilde{u}(\xi,\eta) = F(\xi) + G(\eta) = \varphi\left(\frac{\eta}{2}\right) + \psi\left(\frac{\xi}{2}\right) - \left(F(0) + G(0)\right)$

$$= \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0).$$

#

(b) $\begin{cases} u_{tt} - u_{xx} = 0 & x \in \mathbb{R},\ t > 0. \\ u|_{t=0} = \varphi(x),\quad u|_{x-t=0} = \psi(x),\quad \varphi(0) = \psi(0) \end{cases}$

(b). 同样考虑上述换元. 我们同理可以得到

$\tilde{u}(\xi,\eta) = F(\xi) + G(\eta)$.

由初值条件，$t=0 \Leftrightarrow \eta = \xi$.

$\Rightarrow F(\eta) + G(\eta) = \tilde{u}(\eta,\eta) = u(\eta,0) = \varphi(\eta)$.

$F(0) + G(\eta) = \tilde{u}(0,\eta) = u\left(\frac{\eta}{2}, \frac{\eta}{2}\right) = \psi\left(\frac{\eta}{2}\right)$.

可知 $\tilde{u}(\xi,\eta) = F(\xi) + G(\eta)$

$$= \varphi(\xi) - G(\xi) + G(\eta) = \varphi(\xi) - (G(\xi) + F(0)) + G(\eta) + F(0).$$

$$= \varphi(\xi) - \psi\left(\frac{\xi}{2}\right) + \psi\left(\frac{\eta}{2}\right).$$

$$\Rightarrow u(t,x) = \tilde{u}(\xi,\eta) = \varphi(x-t) - \psi\left(\frac{x-t}{2}\right) + \psi\left(\frac{x+t}{2}\right)$$

12. $u_{tt} - a^2 u_{xx} = 0 \quad 0 < x < l, \quad t > 0 \quad (a=1)$.

(a) $\begin{cases} u(x,0) = x^2 - 2lx, \quad u_t(x,0) = 0 \\ u(0,t) = u_x(l,t) = 0; \end{cases}$

(b) $\begin{cases} u(x,0) = \begin{cases} \frac{hx}{c} & \text{if } 0 < x \leq c \\ h\frac{l-x}{l-c} & \text{if } c < x \leq l \end{cases} \\ u_t(x,0) = 0 \\ u(0,t) = u(l,t) = 0 \quad t \geq 0. \end{cases}$

(a): 令 $u = X(x)T(t)$，代入原方程可得 $\frac{X''}{X} = \frac{T''}{T} = -\lambda$.

下面考察 $\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad X'(l) = 0 \end{cases}$

则若 $\lambda \leq 0$，类似于书上的讨论，我们知道上述方程只有零解，不符.

若 $\lambda > 0$，则方程的通解为 $X(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$.

则由于 $X(0) = 0 \Rightarrow C_2 = 0 \Rightarrow X(x) = C\sin(\sqrt{\lambda} x)$ 由 $X'(l) = 0 \Rightarrow \cos(\sqrt{\lambda} l) = 0$

$\Rightarrow \sqrt{\lambda} l = \frac{2k-1}{2}\pi \quad k = 1, 2 \ldots, \quad \Rightarrow \lambda_k = \left(\left(\frac{2k-1}{2}\right)\frac{\pi}{l}\right)^2 \quad k = 1, 2, \ldots$

进而 $T'' + \lambda_k T = 0 \Rightarrow T_k(t) = C_1 \sin\left(\frac{2k-1}{2}\frac{\pi}{l} t\right) + C_2 \cos\left(\frac{2k-1}{2}\frac{\pi}{l} t\right)$.

since $T_k'(0) = 0 \Rightarrow C_1 = 0 \Rightarrow T_k(t) = \cos\left(\frac{2k-1}{2}\frac{\pi}{l} t\right)$.

于是将 $u_k(t,x) = \cos\left(\frac{2k-1}{2}\frac{\pi}{l} t\right) \sin\left(\frac{2k-1}{2}\frac{\pi}{l} x\right)$ 叠加.

考察 $u(t,x) = \sum_{k=1}^{\infty} C_k \cos\left(\frac{2k-1}{2}\frac{\pi}{l} t\right) \sin\left(\frac{2k-1}{2}\frac{\pi}{l} x\right)$.

则令 $t = 0 \Rightarrow u(0,x) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{2k-1}{2}\frac{\pi}{l} x\right) = x^2 - 2lx$

$$\Rightarrow C_k = \frac{2}{l}\int_0^l \sin\left(\frac{2k-1}{2}\frac{\pi}{l} x\right)(x^2 - 2lx)\,dx = -\frac{32l^2}{(2k-1)^3\pi^3}$$

于是方程的解为

$$u(t,x)=\sum_{k=1}^{\infty}\frac{-32l^2}{(2k-1)^3\pi^3}\cos\left(\left(\frac{2k-1}{2}\right)\frac{\pi}{l}t\right)\sin\left(\frac{2k-1}{2}\cdot\frac{\pi}{l}x\right)$$

(b) 同样，我们令 $u=X(x)T(t)$. $\Rightarrow$ $\frac{X''}{X}=\frac{T''}{T}=-\lambda$.

则考察 $\begin{cases}X''(x)+\lambda X(x)=0\\ X(0)=0,\ X(l)=0.\end{cases}$

若 $\lambda\le 0$. 类似于书上讨论易知此时只有零解.

若 $\lambda>0$, 则方程的通解为 $X(x)=C_1\sin(\sqrt{\lambda}x)+C_2\cos(\sqrt{\lambda}x)$.

由于 $X(0)=0$ $\Rightarrow$ $C_2=0$ $\Rightarrow$ $X(x)=C_1\sin(\sqrt{\lambda}x)$.

由 $X(l)=0$ $\Rightarrow$ $\sin(\sqrt{\lambda}l)=0$ $\Rightarrow$ $\sqrt{\lambda}l=k\pi$ $\Rightarrow$ $\lambda_k=\left(\frac{k\pi}{l}\right)^2$ $k=1,2,\cdots$

代入可知. $T_k''+\lambda_kT_k=0$ $\Rightarrow$ $T_k(t)=C_1\sin\left(\frac{k\pi}{l}t\right)+C_2\cos\left(\frac{k\pi}{l}t\right)$.

$T_k'(0)=0$ $\Rightarrow$ $C_1=0$ $\Rightarrow$ $T_k(t)=\cos\left(\frac{k\pi}{l}t\right)$.

将 $u_k(t,x)=\sin\left(\frac{k\pi}{l}x\right)\cos\left(\frac{k\pi}{l}t\right)$, 叠加可知.

$$u(t,x)=\sum_{k=1}^{\infty}C_k\sin\left(\frac{k\pi}{l}x\right)\cos\left(\frac{k\pi}{l}t\right).$$

令 $t=0$ $\Rightarrow$ $\sum_{k=1}^{\infty}C_k\sin\left(\frac{k\pi}{l}x\right)=u(x,0)$.

$$\Rightarrow C_k=\frac{2}{l}\int_0^l\left(\sin\frac{k\pi}{l}x\right)u(x,0)\,dx=\frac{2}{l}\left(\int_0^c\frac{hx}{c}\sin\left(\frac{k\pi}{l}x\right)dx+\int_c^l h\frac{l-x}{l-c}\left(\sin\frac{k\pi}{l}x\right)dx\right)$$

$$=\frac{2hl^2}{c(l-c)(k\pi)^2}\sin\left(\frac{k\pi c}{l}\right).$$

于是便有 $u(t,x)=\sum_{k=1}^{\infty}\frac{2hl^2}{c(l-c)(k\pi)^2}\sin\frac{k\pi c}{l}\sin\left(\frac{k\pi}{l}x\right)\cos\left(\frac{k\pi}{l}t\right)$.

17. $\begin{cases}u_t=a^2(u_{xx}+u_{yy})\\ u(x,y,0)=A\\ u(0,y,t)=u_x(a,y,t)=0\\ u_y(x,0,t)=u(x,b,t)=0\end{cases}$ 其中 $a=1$.

令 $u(t,x,y)=T(t)X(x)Y(y)$.

于是代入原方程，我们有 $T'(t)X(x)Y(y)=(X''Y+Y''X)T$.

$$\Rightarrow \frac{T'}{T}=\frac{X''Y+Y''X}{XY}=\frac{X''}{X}+\frac{Y''}{Y}$$

令 $\frac{X''}{X} = -\lambda$

则考察边值问题 $\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0,\ X'(a) = 0 \end{cases}$

易知若 $\lambda \le 0$，上述方程只有零解

若 $\lambda > 0$，则方程通解为 $X(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$.

则由于 $X(0) = 0 \Rightarrow C_2 = 0 \Rightarrow X(x) = C_1 \sin \sqrt{\lambda} x$

$X'(a) = 0 \Rightarrow \cos(\sqrt{\lambda} a) = 0 \Rightarrow \sqrt{\lambda} a = \left(\frac{2k-1}{2}\right)\pi \Rightarrow \lambda_k = \left(\frac{(2k-1)\pi}{2a}\right)^2$

则此时 $X_k = \sin \frac{(2k-1)\pi}{2a} x \qquad k = 1, 2 \cdots$

同样地，我们令 $\frac{Y''}{Y} = -\mu$，考察边值问题

$$\begin{cases} Y'' + \mu Y = 0 \\ Y'(0) = 0,\ Y(b) = 0 \end{cases}$$

同样地，若 $\lambda \le 0$，上述方程只有零解

若 $\lambda > 0$，则方程通解为 $Y(y) = C_1 \sin(\sqrt{\mu} y) + C_2 \cos(\sqrt{\mu} y)$.

则 $Y'(0) = 0 \Rightarrow C_1 = 0 \Rightarrow Y(y) = C \cos(\sqrt{\mu} y)$

$Y(b) = 0 \Rightarrow \cos(\sqrt{\mu} b) = 0 \Rightarrow \sqrt{\mu} b = \left(\frac{2k-1}{2}\right)\pi \Rightarrow \mu_k = \left(\frac{(2k-1)\pi}{2b}\right)^2$

此时 $Y_k = \cos \frac{(2k-1)\pi}{2b} x$

则若 $\lambda = \lambda_k$，$\mu = \mu_m$ 时，则有

$$\frac{T'}{T} = -(\lambda_k + \mu_m) = -\left(\frac{(2k-1)\pi}{2a}\right)^2 - \left(\frac{(2m-1)\pi}{2b}\right)^2$$

$$\Rightarrow T = e^{-\left(\left(\frac{(2k-1)\pi}{2a}\right)^2 + \left(\frac{(2m-1)\pi}{2b}\right)^2\right)t}$$

进而 $u_{k,m} = e^{-\left(\left(\frac{(2k-1)\pi}{2a}\right)^2 + \left(\frac{(2m-1)\pi}{2b}\right)^2\right)t} \sin \frac{(2k-1)\pi}{2a} x \cos \frac{(2m-1)\pi}{2b} y$

则方程的解

$$u = \sum_{k,m=1}^{\infty} C_{k,m} e^{-\left(\left(\frac{(2k-1)\pi}{2a}\right)^2 + \left(\frac{(2m-1)\pi}{2b}\right)^2\right)t} \sin\left(\frac{(2k-1)\pi}{2a} x\right) \cos\left(\frac{(2m-1)\pi}{2b} y\right).$$

进而令 $t = 0 \Rightarrow u(0, x, y) = \sum_{k,m=1}^{\infty} C_{k,m} \sin\left(\frac{(2k-1)\pi}{2a} x\right) \cos\left(\frac{(2m-1)\pi}{2b} y\right) = A.$

则有 $C_{k,m} = \frac{2}{a}\int_0^a \frac{2}{b}\int_0^b A(x,y)\sin\left(\frac{(2k-1)\pi}{2a}x\right)\cos\left(\frac{(2m-1)\pi}{2b}y\right)dxdy$.

$$= \frac{4}{ab}\int_0^a\int_0^b A(x,y)\sin\left(\frac{(2k-1)\pi}{2a}x\right)\cos\left(\frac{(2m-1)\pi}{2b}y\right)dxdy$$

24、 $\vec{E} = (E_1, E_2, E_3)$ and $\vec{B} = (B_1, B_2, B_3)$ satisfies

$$\begin{cases} \vec{E}_t = \mathrm{rot}\,\vec{B} \\ \vec{B}_t = -\mathrm{rot}\,\vec{E} \\ \mathrm{div}\,\vec{B} = \mathrm{div}\,\vec{E} = 0. \end{cases}$$

⇒

Th. Then we have $u_{tt} - \Delta u = 0$ (Here $u = E_i$ or B_i).

Proof: 只对 $\vec{E}$ 作证明，$\vec{B}$ 由对称性类似

事实上， $\vec{E}_{tt} = \nabla\times\vec{B}_t = -(\nabla\times(\nabla\times\vec{E}))$

$$= -(\nabla(\mathrm{div}\,\vec{E}) - \Delta\vec{E})$$

$$= \Delta\vec{E}.$$

Rmk: 这里我们用到了 $\nabla\times(\nabla\times\vec{F}) = \nabla(\mathrm{div}\,\vec{F}) - \Delta\vec{F}$. 证明直接作展开即可

Recall：

由有限传播速度易知，若 $u(t,x)$ 是方程 $\begin{cases}\partial_t^2u-\Delta u=0\\(u(0),\partial_t u(0))=(\phi,\psi)\end{cases}$

的光滑解，则若 ϕ,ψ 具有紧支集，则 $u(t,x)$ 对于任意时刻 $t>0$ 也具有紧支集

（即，$\overline{\{x:\phi\neq 0\}}$，$\overline{\{x:\psi\neq 0\}}$ 是 $\mathbb{R}^n$ 中紧集）.

下面我们考察一般的非齐次方程.

$$\begin{cases}\partial_t^2u-\Delta u=F(t,x,u,Du,u_t)\\(u(0),\partial_t u(0))=(\phi,\psi)\end{cases}$$

此处 F 充分光滑.

对于给定的 F，是否上述性质依旧保持？

若 $F=f(t,x)$，$f(t,x)\neq 0$，则对于 $(\phi,\psi)=(0,0)$，$n=3$.

此时方程的解为 $u(t,x)=\dfrac{1}{4\pi}\displaystyle\int_{B(x,t)}\frac{f(y,t-|y-x|)}{|y-x|}dy$

显然对于不一定具有紧支集

下面我们考察另一种情形

$F=F(u,u_t,Du)$，满足 $F(0,0,0)=0$，则此时方程

$$\begin{cases}\partial_t^2u-\Delta u=F(u,u_t,Du)\\(u(0),\partial_t u(0))=(\phi,\psi)\end{cases}$$ 依旧满足上述性质.

即有如下结论.

若 F 是光滑函数（或者 Local Lip 即可），$F(0,0,0)=0$，u 是方程

$$\begin{cases}\partial_t^2u-\Delta u=F(u,\partial_t u,Du)\\(u(0),\partial_t u(0))=(\phi,\psi)\end{cases}$$ 的光滑解，若 ϕ,ψ 是具有紧支集的光滑函数，

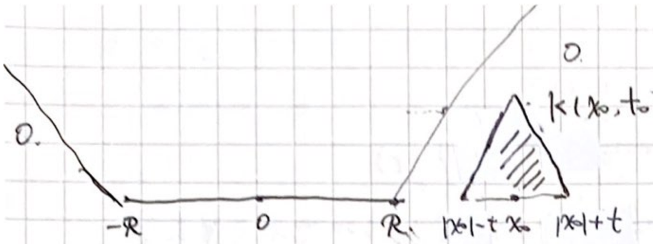
则，$\forall\, t>0$，$u(t,x)$ 也是关于 x 的紧支光滑函数

proof：事实对于上述方程，我们依旧可以证明有限传播速度的性质

由于 ϕ,ψ 的 (ϕ,ψ) 具有紧支集，则我们假设支集包含在 $B(0,R)$ 中

则任取 $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}_+$，满足 $|x_0|>R$，$B(x_0,t_0)\subseteq\{|x|>R\}$，则

由初始假设，我们知道，$(u(0),\partial_t u(0))=(0,0)$ 在 $B(x_0,t_0)$ 中



我们下面证明：在 $K(x_0,t_0)=\{(x,t)\in\mathbb{R}\times\mathbb{R}_+:\ |x-x_0|\le t_0-t,\ 0\le t\le t_0\}$ 中，都有 $u(x,t)=0$.

进而由此证明知 $\forall |x|>R+|t|$，我们有 $u(x,t)\equiv 0$，也即 $\forall t>0$，$u(t,x)$ 关于 x 具有紧支集

定义能量 $E(t)=\frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2+|\nabla u|^2+|u|^2\,dx$ → 这里比上课多了一项 $|u|^2$，在后面估计我们会看到多一项的必要性

则 $\frac{dE}{dt}=\frac{1}{2}\int_{B(x_0,t_0-t)} 2(u_tu_{tt}+\nabla u_t\cdot\nabla u+uu_t)\,dx$

$-\frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2+|\nabla u|^2+|u|^2dx$

$$=\int_{B(x_0,t_0-t)} u_tu_{tt}+uu_t\,dx+\int_{B(x_0,t_0-t)}\left(\operatorname{div}(u_t\nabla u)-u_t\Delta u\right)dx-\frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2+|\nabla u|^2+u^2dx$$

$$\overset{\text{散度定理}}{=}\int_{B(x_0,t_0-t)} u_t(u_{tt}-\Delta u)+uu_t\,dx+\underbrace{\int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial n}dx-\frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2+|\nabla u|^2+u^2dx}_{①}$$

由 Cauchy 不等式 $\left|u_t\frac{\partial u}{\partial n}\right|\le\frac{1}{2}(u_t^2+|\nabla u|^2)$ $\Rightarrow$ ① ≤ 0

$$\Rightarrow \frac{dE}{dt}\le\int_{B(x_0,t_0-t)} u_t(u_{tt}-\Delta u)+uu_t\,dx=\int_{B(x_0,t_0-t)} u_tF(u,u_t,\nabla u)+uu_t\,dx$$

由中值定理 $|F(u,u_t,\nabla u)|=|F(u,u_t,\nabla u)-F(0,0,0)|\le C(|u|+|u_t|+|\nabla u|)$

注意我们假定了 u 光滑，则在任意固定紧集上，都有 $|u|+|u_t|+|\nabla u|\le M$ 对某个固定的 $M>0$，这里我们任意固定一个包含 $K(x_0,t_0)$ 的紧集

则 $\frac{dE}{dt}\le C\int_{B(x_0,t_0-t)}|u_t|(|u|+|u_t|+|\nabla u|)+|u||u_t|\,dx$ ← 这时回到前面能量的定义，因为期待将右侧放成能量，而右侧又多了变量 u，所以 $E(t)$ 定义多个 u^2

由 Cauchy 不等式，上式右侧 $\le C\int_{B(x_0,t_0-t)}|u_t|^2+|\nabla u|^2+|u|^2dx=CE(t)$.

$\Rightarrow \frac{dE}{dt}\le CE(t)$.

结合 $E(0)=\frac{1}{2}\int_{B(x_0,t_0)}\psi^2+|\nabla\phi|^2+\phi^2=0$，由 Gronwall 不等式

$\Rightarrow E(t)\equiv 0$ $\Rightarrow$ 在 $K(x_0,t_0)$ 中，$u(t,x)\equiv 0$，进而结论成立

Rmk：① 特别地，上述结论对于 $F(u)=|u|^{p-1}u\ (p\ge 1)$ 成立，这在我们的实际运用中研究的非常多

② 利用上述方法，我们实际上还可以证明上述方程光滑解（或者 C^2 解）的存在唯一性，此时不需要 $F(0,0,0)=0$ 的假设。

事实上，假设 u, v 都是方程 $\begin{cases} \partial_t^2 u - \Delta u = F(u, Du, u_t) \\ (u(0), \partial_t u(0)) = (\phi, \psi) \end{cases}$ 的光滑解

则令 $w = u - v$，则 w 满足方程 $\begin{cases} \partial_t^2 w - \Delta w = F(u, Du, u_t) - F(v, Dv, v_t) \\ (w(0), \partial_t w(0)) = (0, 0). \end{cases}$

则同样 $\forall (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$，$k(x_0, t_0) = \{ |x - x_0| \le t_0 - t,\ 0 \le t \le t_0 \}$.

定义能量 $E(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} w_t^2 + |\nabla w|^2 + |w|^2 \, dx$

类似于前面的计算（前面和之前的计算一样）.

$$\frac{dE}{dt} \le \int_{B(x_0, t_0 - t)} w_t \big(F(u, Du, u_t) - F(v, Dv, v_t) \big) + w w_t \, dx$$

$$\overset{\text{中值定理}}{\le} C \int_{B(x_0, t_0 - t)} |w_t| \big(|u - v| + |D(u - v)| + |v_t - u_t| \big) + |w w_t| \, dx$$

这里同样用到了固定紧集 $|w|, |w_t|, |Dw|$ 均有界

$$= C \int |w_t| \big(|w| + |Dw| + |w_t| \big) + |w w_t| \, dx$$

$$\overset{\text{Cauchy}}{\le} C \int (w_t)^2 + |Dw|^2 + w^2 \, dx = C\, E(w)(t).$$

结合 $E(0) = 0$，再由 Gronwall's inequality，我们有

$$E(t) \equiv 0.$$

$\Rightarrow w(x_0, t_0) = 0$，由 (x_0, t_0) 的任意性 $\Rightarrow w \equiv 0 \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_+$

也即 $u \equiv v$，唯一性得证.

#

Suppose

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

definition: $\{x\in\mathbb{R}^n : g(x)\neq 0\}$ 是 $\mathbb{R}^n$ 中的紧集 就说 $g(x)$ 具有紧支集，用记号 supp g 代表集合 $\{x\in\mathbb{R}^n : g(x)\neq 0\}$ 称为 g 的支集

Here we assume g, h is smooth and have compact support. Then.

① if $n=3$, we have $|u(t,x)| \le C/\langle t\rangle$ $(x\in\mathbb{R}^3, t>0)$ $C\sim g,h$.

② if $n=2$, we have $|u(t,x)| \le C/\langle t\rangle^{\frac{1}{2}}$ $(x\in\mathbb{R}^2, t>0)$ $C\sim g,h$.

proof: ① By Kirchhoff's formula, we have

$$u(x,t) = ⨍_{\partial B(x,t)} t h(y) + g(y) + Dg(y)\cdot(y-x)\, dS_y \quad (x\in\mathbb{R}^3, t>0)$$

Then if $t<1$, then

$$|u(x,t)| = \left| \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t h(y) + g(y) + Dg(y)\cdot(y-x)\, dS(y) \right|$$

$$\lesssim \frac{1}{4\pi t^2} \int_{\partial B(x,t)} \left(\|h\|_{L^\infty} + \|g\|_{L^\infty} + \|Dg(y)\|_{L^\infty} \right) dS(y)$$

$$\lesssim \|h\|_{L^\infty} + \|g\|_{L^\infty} + \|Dg\|_{L^\infty}$$

If $t\ge 1$, then since g, h are compactly supported, then we may assume $(\text{supp } g \cup \text{supp } h) \subset B(0,R)$.

Then it's easy to know that $|\partial B(x,t) \cap B(0,R)| \le C$ → $|\cdot|$ 代表相交部分曲面面积 $\forall (x,t) \in \mathbb{R}^3\times\mathbb{R}$.

$$\Rightarrow |u(t,x)| = \frac{1}{4\pi t^2} \left| \int_{\partial B(x,t)} t h(y) + g(y) + Dg(y)\cdot(y-x)\, dS(y) \right|$$

$$\lesssim \frac{1}{4\pi t^2} \int_{\partial B(x,t)\cap B(0,R)} t\|h\|_{L^\infty} + \|g\|_{L^\infty} + \|Dg\|_{L^\infty} t \, dS(y)$$

$$\lesssim \frac{1}{4\pi t^2} C \left(t\|h\|_{L^\infty} + \|g\|_{L^\infty} + \|Dg\|_{L^\infty} t \right)$$

$$\lesssim \frac{C}{4\pi t} \quad \text{Here } C\sim g,h$$

As a result, $|u(t,x)| \le C/\langle t\rangle$

②. Firstly, By Poisson's formula, we have

$$u(t,x) = \frac{1}{2} ⨍_{B(x,t)} \frac{t g(y) + t^2 h(y) + t Dg(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}} dy \quad (x\in\mathbb{R}^2, t>0)$$

Firstly. If $t \le 1$, Just as before.

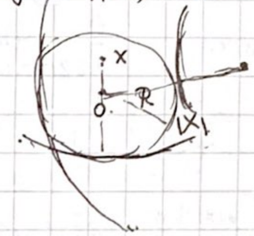
$$|u(t,x)| \le \frac{1}{2}\cdot\frac{1}{\pi t^2}\int_{B(x,t)} \frac{t\|g\|_{L^\infty} + t^2\|h\|_{L^\infty} + t^2\|Dg\|_{L^\infty}}{(t^2-|y-x|^2)^{1/2}}\,dy$$

$$\lesssim \frac{C}{t}\int_0^t dr \int_{\partial B(x,r)} \frac{dS_r(y)}{(t^2-r^2)^{\frac{1}{2}}}$$

$$\lesssim \frac{C}{t}\int_0^t \frac{2\pi r}{(t^2-r^2)^{\frac{1}{2}}}\,dr \lesssim C \sim g, h.$$

If $t \ge 1$, then Just as before, we assume that $(\operatorname{supp} g \cup \operatorname{supp} h) \subset B(0,R)$.

$\Rightarrow \forall\, x \in \mathbb{R}^2$, consider $\partial B(x,r) \cap (\operatorname{supp} g \cup \operatorname{supp} h) \ne \phi$.



Then if $|x| < R \Rightarrow 0 \le r \le |x| + R$.

if $|x| \ge R \Rightarrow |x| - R \le r \le |x| + R$.

Then

$$|u(t,x)| = \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{t g(y) + t^2 h(y) + t\,Dg(y)\cdot(y-x)}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy$$

$$\lesssim \frac{1}{2\pi t^2}\int_{B(x,t)\cap(\operatorname{supp} g \cup \operatorname{supp} h)} \frac{t^2(\|g\|_{L^\infty} + \|h\|_{L^\infty} + \|Dg(y)\|_{L^\infty})}{(t^2-|y-x|^2)^{\frac{1}{2}}}\,dy.$$

$$\lesssim C\int_{\max\{0,|x|-R\}}^{\min\{t,|x|+R\}} \int_{\partial B(x,r)\cap(\operatorname{supp} g \cup \operatorname{supp} h)} \frac{dS(y)}{(t^2-r^2)^{\frac{1}{2}}}$$

Just as before, we know that $|\partial B(x,r) \cap (\operatorname{supp} g \cup \operatorname{supp} h)| \lesssim C$.

$$\Rightarrow |u(t,x)| \lesssim C\int_{\max\{0,|x|-R\}}^{\min\{t,|x|+R\}} \frac{dr}{(t^2-r^2)^{\frac{1}{2}}} \lesssim C\frac{1}{\sqrt{t}}\int_{\max\{0,|x|-R\}}^{\min\{t,|x|+R\}} \frac{dr}{\sqrt{t-r}}$$

$$\lesssim \frac{C\sqrt{2R}}{\sqrt{t}} \lesssim \frac{C}{\sqrt{t}}.$$

As a result, we know that $|u(t,x)| \lesssim \dfrac{C}{\langle t\rangle^{\frac{1}{2}}}$ $\forall\, x \in \mathbb{R}^2,\ t > 0$.

Rmk: ① Here $\|g\|_{L^\infty} \hat{=} \sup_{x\in\mathbb{R}^n} |g|$ $\quad \langle t\rangle = (1+t^2)^{\frac{1}{2}}$

$$⨍_U f = \frac{1}{|U|}\int_U f.$$

② In general, $|u(t,x)| \lesssim C/\langle t\rangle^{\frac{n-1}{2}}$ $\quad (x \in \mathbb{R}^n,\ t > 0)$.

作业答案!

① 分离变量法一节作业答案见3班助教习题课讲义

② 能量法:，Kirchhoff公式

29. $u(x,t) = \frac{1}{4\pi t}\int_{S_t(x)} \psi(y)\,d\sigma + \left(\frac{1}{4\pi t}\int_{S_t(x)} \varphi(y)\,d\sigma\right)_t$

要证: $\begin{cases} u(x,0) = \varphi(x) \\ \partial_t u(x,0) = \psi(x) \\ \partial_t^2 u - \Delta u = 0 \end{cases}$

$$\partial_{tt} u = \left(\frac{1}{4\pi t}\int_{S_t(x)} \psi(y)\,d\sigma\right)_{tt} + \left(\frac{1}{4\pi t}\int_{S_t(x)} \varphi(y)\,d\sigma\right)_{ttt}$$

$$\partial_t u = \left(\frac{1}{4\pi t}\int_{S_t(x)} \psi(y)\,d\sigma\right)_t + \left(\frac{1}{4\pi t}\int_{S_t(x)} \varphi(y)\,d\sigma\right)_{tt}$$

核心: 处理 $\left(\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma\right)_{tt}$, $\left(\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma\right)_t$

Lemma1: 处理 $\left(\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma\right)_t$

$$\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma = \frac{1}{4\pi t}\int_{S_1(0)} g(x+ty)\,t^2\,d\sigma = \frac{t}{4\pi}\int_{S_1(0)} g(x+ty)\,d\sigma$$

$$\left(\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma\right)_t = \underset{①}{\frac{1}{4\pi}\int_{S_1(0)} g(x+ty)\,d\sigma} + \underset{②}{\frac{t}{4\pi}\int_{S_1(0)} \partial_t g(\vec{x}+t\vec{y})\,d\sigma}$$

$$\partial_t g(x_1+ty_1, \dots, x_n+ty_n) = (\partial_{x_1} g)\,y_1 + \dots + (\partial_{x_n} g)\,y_n = (\nabla_x g)\cdot\vec{y}$$

$\int_{S_1(0)} (\nabla_x g)(\vec{x}+t\vec{y})\,\vec{y}\,d\sigma$, 令 $h(\vec{y}) = g(\vec{x}+t\vec{y})$

$\nabla_y h = t(\nabla_x g)(\vec{x}+t\vec{y})$

$\Delta_y h = t^2 \Delta_x g$

$$② = \frac{t}{4\pi}\int_{S_1(0)} (\nabla_x g)\,\vec{y}\,d\sigma = \frac{1}{4\pi}\int_{S_1(0)} (\nabla_y h)\,\vec{y}\,d\sigma = \frac{1}{4\pi}\int_{B_1(0)} \Delta_y h\,d\vec{y}$$

$$= \frac{1}{4\pi}\int_{B_1(0)} t^2 (\Delta_x g)(\vec{x}+t\vec{y})\,d\vec{y}$$

$$= \frac{1}{4\pi t}\int_{B_t(x)} (\Delta_x g)(\vec{y})\,d\vec{y}$$

$$\Rightarrow \left(\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma\right)_t = \frac{1}{4\pi}\int_{S_1(0)} g(x+ty)\,d\sigma + \frac{1}{4\pi t}\int_{B_t(x)} (\Delta g)(\vec{y})\,d\vec{y}$$

Lemma 2: 处理 $\left(\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma\right)_{tt}$

$$\left(\frac{1}{4\pi t}\int_{S_t(x)} g(y)\,d\sigma\right)_{tt} = \frac{1}{t}\,\frac{t}{4\pi}\int_{S_1(0)} \partial_t g(x+ty)\,d\sigma + \frac{1}{4\pi t}\,\partial_t\int_{B_t(x)} (\Delta g)(\vec{y})\,dy$$

↑ 同上②处理　　⇓ 求导 $-\frac{1}{4\pi t^2}\int_{B_t(x)} (\Delta g)(\vec{y})\,dy$

$$= \frac{1}{4\pi t^2}\int_{B_t(x)} \Delta g(\vec{y})\,d\vec{y} + \frac{1}{4\pi t}\int_{\partial B_t(x)} (\Delta g)(\vec{y})\,dy - \frac{1}{4\pi t^2}\int_{B_t(x)} (\Delta g)(\vec{y})\,dy$$

$$= \frac{1}{4\pi t}\int_{\partial_t B(x)} (\Delta_x g)(\vec{y})\,d\sigma$$

开始证明:

Step 1: $u(x,0) = \lim_{t\to 0} \underbrace{\frac{1}{4\pi t}\int_{S_t(x)} \psi(y)\,d\sigma}_{①} + \lim_{t\to 0} \underbrace{\left(\frac{1}{4\pi t}\int_{S_t(x)} \varphi(y)\,d\sigma\right)_t}_{②}$

$$t\inf \psi(y) \le \frac{1}{4\pi t}\int_{S_t(x)} \psi(y)\,dy \le t\sup\psi(y)$$

$$\Rightarrow \lim_{t\to 0} ① = 0$$

$\lim_{t\to 0}$ ② 在Lemma 1 中令 $g=\varphi$ $\lim_{t\to 0}\left(\frac{1}{4\pi}\int_{S_1(0)} \varphi(x+ty)\,d\sigma + \frac{1}{4\pi t}\int_{B_t(x)} (\Delta g)(\vec{y})\,dy\right)$

$$= \lim_{t\to 0} \frac{1}{4\pi t^2}\int_{S_t(y)} \varphi(x+y)\,d\sigma$$

$$= \varphi(\vec{x})$$

$$\Rightarrow u(x,0) = \varphi(x)$$

Step 2: $\partial_t u(x,0) = \lim_{t\to 0}\left(\frac{1}{4\pi t}\int \psi(y)\,d\sigma\right)_t + \lim_{t\to 0}\left(\frac{1}{4\pi t}\int \varphi(y)\,d\sigma\right)$

①

②

①的处理同 step 1 中②的处理，不过是将 φ 换成了 ψ

$\lim_{t\to 0}$ ① $= \psi(x)$

利用 Lemma 2，将 $g=\varphi \Rightarrow$ ② $= \frac{1}{4\pi t}\int_{\partial B_t(x)} (\Delta g)(y)\,d\sigma$

$\Rightarrow \lim_{t\to 0}$ ② $=0$

$\Rightarrow \partial_t u(x,0) = \psi(x)$

step 3：证 $\partial_t^2 u - \Delta u = 0$

$$\partial_t^2 u = \left(\frac{1}{4\pi t}\int_{S_t(x)} \psi(y)\,d\sigma\right)_{tt} + \left(\frac{1}{4\pi t}\int_{S_t(x)} \varphi(y)\,d\sigma\right)_{ttt}$$

$$\Delta u = \Delta\left(\frac{1}{4\pi t}\int_{S_1(0)} \psi(x+ty)\,t^2\,d\sigma + \left(\frac{1}{4\pi t}\int_{S_1(0)} t^2\varphi(x+ty)\,d\sigma\right)_t\right)$$

$$= \frac{1}{4\pi t}\int_{S_1(0)} (\Delta_x\psi)(x+ty)\,t^2\,d\sigma + \left(\frac{t^2}{4\pi t}\int_{S_1(0)} (\Delta_x\varphi)(x+ty)\,d\sigma\right)$$

只需证 $\left(\frac{1}{4\pi t}\int_{S_t(x)} \psi(y)\,d\sigma\right)_{tt} - \frac{t^2}{4\pi t}\int_{S_1(0)} (\Delta_x\psi)(x+ty)\,d\sigma = 0$

同由 Lemma ② $\Rightarrow$ ok

32：令 $v(t,x,y,z) = e^z u(t,x,y)$

$$(\partial_t^2 v)(t,x,y,z) = e^z \partial_t^2 u(t,x,y)$$

$$\partial_x^2 v = e^z \partial_x^2 u(t,x,y)$$
$$\partial_y^2 v = e^z \partial_y^2 u(t,x,y)$$
$$\partial_z^2 v = e^z u(t,x,y)$$

$$\Rightarrow \begin{cases} \partial_t^2 v - \Delta v = 0 \\ v(0,x,y,z) = e^z\varphi(x,y), \quad \partial_t v(0,x,y,z) = e^z\psi(x,y) \end{cases}$$

$$\Rightarrow v(t,\vec{x}) = v(t,x_1,x_2,x_3) = \frac{1}{4\pi t}\int_{S_t(\vec{x})} e^{y_3}\psi(y_1,y_2)\,d\sigma$$

$$= e^{x_3} u(t, x_1, x_2)$$

$$+ \left(\frac{1}{4\pi t} \int_{S_t(\vec{x})} e^{y_3} \varphi(y_1, y_2)\, d\sigma \right)_t$$

$$u(t, x_1, x_2) = \frac{1}{4\pi t} \int_{S_t(\vec{x})} e^{y_3 - x_3} \psi(y_1, y_2)\, d\sigma + \left(\frac{1}{4\pi t} \int_{S_t(\vec{x})} e^{y_3 - x_3} \varphi(y_1, y_2)\, d\sigma \right)_t$$

34:
$$\begin{cases} u_{tt} - \Delta u = f(x, t) \\ u(x, 0) = 0, \quad \partial_t u(x, 0) = 0 \end{cases}$$

$$u(x, t) = \int_0^t w(x, t; \tau)\, d\tau, \qquad \begin{cases} \partial_t^2 w(x, t; \tau) - \Delta w(x, t; \tau) = 0 \\ w(x, \tau; \tau) = 0, \quad \partial_t w(x, \tau; \tau) = f(x, \tau) \end{cases}$$

$$w(x, t; \tau) = \frac{1}{2\pi} \int_{|x-y| \le t-\tau} \frac{f(y, \tau)}{\sqrt{(t-\tau)^2 - y_1^2 - y_2^2}}\, dy_1 dy_2$$

$$u(x, t) = \int_0^t \frac{1}{2\pi} \int_{|x-y| \le t-\tau} \frac{f(y, \tau)}{\sqrt{(t-\tau)^2 - y_1^2 - y_2^2}}\, dy_1 dy_2\, d\tau$$

$t - \tau = s$

$$u(x, t) = \int_0^t \frac{1}{2\pi} \int_{|x-y| \le s} \frac{f(y, t-s)}{\sqrt{s^2 - |y|^2}}\, d\vec{y}\, ds$$

38: $u_{tt} = u_{xx} - u_t$

$$\Rightarrow u_{tt} u_t = u_{xx} u_t - u_t^2$$

$$\Rightarrow \frac{1}{2} \frac{\partial}{\partial t} (u_t^2) = \frac{1}{2} \partial_x (u_x u_t) - \frac{1}{2} u_x u_{tx} - u_t^2$$

$$= \frac{1}{2} \partial_x (u_x u_t) - \frac{1}{2} \partial_t (u_x^2) - u_t^2$$

$$\Rightarrow \frac{1}{2} \partial_t (u_t^2 + u_x^2) - \frac{1}{2} \partial_x (u_x u_t) = -u_t^2$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \int_0^l u_t^2 + u_x^2\, dx - \frac{1}{2} \int_0^l \partial_x (u_x u_t)\, dx = -\int_0^l u_t^2$$

$$\left(\int_0^l u_t^2 + u_x^2\, dx \right)^{\frac{1}{2}}$$

$\frac{1}{2}E(t) = \left(\int_0 u_t \; u_x \, dx\right)$

$$\frac{1}{2}\frac{d}{dt}(E^2(t)) - \frac{1}{2}(u_x u_t)\Big|_0^l = -\int_0^l u_t^2$$

因为固定端点

$$\Rightarrow \quad E(t)\frac{d}{dt}E(t) = -\int_0^l u_t^2 \le 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}E(t) \le 0$$

$$E(t)\frac{d}{dt}E(t) \le E^2(t)$$

$$\Rightarrow \quad E(t) \le E(0) + \int_0^t E(s)\,ds$$

由 Gronwall $E(t) \le E(0)\,e^t \Rightarrow$ if $E(0)=0 \Rightarrow E(t)\equiv 0$

$\Rightarrow$ 解唯一

稳定性补在最后

40:

40. 设 $u(x,y,t)$ 是问题 (3.5.2) 的解，并记

$$E(t) = \iint_\Omega \left(\frac{\rho}{2}u_t^2 + \frac{T}{2}(u_x^2 + u_y^2)\right)\mathrm{d}x\mathrm{d}y + \int_{\partial\Omega}\frac{\sigma}{2}u^2\mathrm{d}s,$$

$$E_0(t) = \iint_\Omega u^2(x,y,t)\mathrm{d}x\mathrm{d}y.$$

试证成立能量不等式：

$$E_0(t) \leqslant \mathrm{e}^t E_0(0) + \frac{2}{\rho}(\mathrm{e}^t - 1)E(0).$$

$\rho = T = \sigma = 1$, $\begin{cases} u_{tt} - (u_{xx} + u_{yy}) = 0 \\ \left(\frac{\partial u}{\partial \nu} + u\right)\Big|_{\partial\Omega} = 0 \end{cases}$,

$$E(t) = \frac{1}{2}\iint_\Omega u_t^2 + u_x^2 + u_y^2 \, dxdy + \frac{1}{2}\int_{\partial\Omega} u^2 ds$$

$$E_0(t) = \iint_\Omega u^2(x,y,t)\,dxdy$$

proof:

$$u_{tt}u_t - (u_{xx} + u_{yy})u_t = 0$$

$$\frac{1}{2}\partial_t(u_t^2) - \partial_x(u_x u_t) - \partial_y(u_y u_t) + \frac{1}{2}\partial_t(u_x^2 + u_y^2) = 0$$

$$\Rightarrow \quad \frac{1}{2}\partial_t(u_t^2 + u_x^2 + u_y^2) = \partial_x(u_x u_t) + \partial_y(u_y u_t) = \nabla\vec{f}, \quad \vec{f} = (u_x u_t, u_y u_t)$$

$$\Rightarrow \quad \frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2 + u_x^2 + u_y^2 = \int_\Omega \nabla\vec{f} = \int_{\partial\Omega} \vec{f}\cdot\vec{\nu}$$

$$= \int_{\partial\Omega} u_t (\nabla u)\cdot \vec{\nu}$$

$$= -\int_{\partial\Omega} u_t u \, d\sigma$$

$$= -\frac{1}{2}\frac{d}{dt}\int_{\partial\Omega} u^2 d\sigma$$

$$\Rightarrow \frac{1}{2}\left(\frac{d}{dt}\int_{\Omega} (u_t^2+u_x^2+u_y^2) + \int_{\partial\Omega} u^2 d\sigma\right) = 0$$

$$\Rightarrow E(t) = E(0), \quad \frac{d}{dt}E(t) = 0$$

$$\frac{d}{dt}(E_0(t) + 2E(t)) = \frac{d}{dt}E_0(t) = 2\int_{\Omega} u\,\partial_t u \, dxdy$$

$$\leq \int_{\Omega} u^2 + (\partial_t u)^2 dxdy$$

$$\leq E_0(t) + 2E(t)$$

由 Gronwall $\Rightarrow$ ok

补充题：

分离变量法

(1) 设 $u(x,t)$ 是 $[0,1]\times\mathbb{R}_+$ 中初边值问题 $\begin{cases} u_{tt} = u_{xx} \\ u|_{x=0} = u|_{x=1} = 0 \\ u|_{t=0} = 0, \quad u_t|_{t=0} = x^2(1-x) \end{cases}$ 的解

求 $\lim\limits_{t\to\infty}\int_0^1 u_t^2(x,t) + u_x^2(x,t)\,dx$

(2) $u_{tt} - u_{xx} = x, \quad 0<x<l, \quad t>0$

$u(x,0) = \partial_t u(x,0) = 0, \quad 0\leq x\leq l$

$u(0,t) = 0, \quad u(l,t) = t, \quad t\geq 0$

能量法：principle：乘上一个量，利用求导 Lebnitz 法则

例：$\partial_t^2 u - \Delta u + \alpha_0(t,x)\partial_t u + \sum_{i=1}^{3}\alpha_i(t,x)\partial_i u = f, \quad 0\leq t\leq T, \ x\in\mathbb{R}^3$

α_0, α_i 连续

Let $A(t) = \max\limits_x |\alpha_0(x,t)| + \sum_i \max\limits_x |\alpha_i(x,t)|$

证明 $\max\limits_{0\leq s\leq t} (E_u(s))^{\frac{1}{2}} \leq \left[E_u(0)^{\frac{1}{2}} + \sqrt{2}\int_0^t \|f(\cdot,s)\|_{L^2} ds\right] \exp\left(2\int_0^t A(s)ds\right)$

2. Suppose that u is a smooth solution of the initial boundary value problem

$$\begin{aligned} &u_{tt} - u_{xx} = S'(x) \qquad 0 < x < 1,\ t > 0, \\ &u(x, t = 0) = f(x), \quad u_t(x, t = 0) = g(x), \\ &u(0, t) = u(1, t) = 0. \end{aligned}$$

a. Show that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(|u_t(x,t)|^2 + |u_x(x,t)|^2\right) dx = \int_0^1 S'(x)\, u_t(x,t)\, dx.$$

1

b. Conclude that there exists a constant C_0, depending only on f' and g, such that

$$\frac{1}{2}\int_0^1 \left(|u_t(x,t)|^2 + \frac{1}{2}|u_x(x,t)|^2\right) dx \le C_0 + 2\int_0^1 S^2(x)dx.$$

Hint: you may use the Cauchy-Schwarz $2\alpha\beta \ge -\frac{1}{2}\alpha^2 - 2\beta^2$.

Solution

1. 不要因为我写了分离变量法你们就拿分离变量做，要有自己的判断

当然算一算是有好处的

这题我们用能量法

$$u_{tt} = u_{xx} \Rightarrow u_{tt}u_t = u_t u_{xx}$$

$$\frac{1}{2}(u_t^2)_t = (u_t u_x)_x - \frac{1}{2}(u_x^2)_t$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 (u_t^2 + u_x^2)\, dx = \int_0^1 (u_t u_x)_x\, dx = (u_t u_x)\Big|_0^1$$

因为 $u|_{x=0} = u|_{x=1} \equiv 0$, $u_x(0,t) \equiv 0$, $u_x(1,t) \equiv 0$

$$\Rightarrow (u_t u_x)\Big|_0^1 \equiv 0$$

$$\Rightarrow \frac{d}{dt}\int_0^1 (u_t^2 + u_x^2) dx = 0 \Rightarrow \underbrace{\int_0^1 u_t^2 + u_x^2\, dx}_{= E(t)} \equiv \text{const}$$

$$E(t) = E(0) = \int_0^1 x^4(1-x)^2\, dx$$

$$= \int_0^1 x^4(x^2 - 2x + 1)\, dx$$

$$= \frac{1}{7} - \frac{2}{6} + \frac{1}{5}$$

$$= \frac{1}{7} - \frac{1}{3} + \frac{1}{5} = \frac{15-35+21}{105} = \frac{1}{105}$$

2、这题 提直的分 离变量

这题可用双算法，但不利于非齐次型的计算的掌握，我们就这样做

$$\begin{cases} u_{tt} - u_{xx} = x, \quad 0<x<l,\ t>0 \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0, \quad 0<x<l \\ u(0,t) = 0, \quad u(l,t) = t \end{cases}$$

我们只会处理齐次边界条件，设 $V = u - \frac{x}{l}t$

$$\Rightarrow \begin{cases} V_{tt} - V_{xx} = x \\ V(x,0) = 0, \quad \partial_t u(x,0) = -\frac{x}{l} \\ V(0,t) = 0, \quad V(l,t) = 0 \end{cases}$$

非齐次方程，解法：$V(t,x) = \sum_{k=0}^{+\infty} T_n(t) \sin\frac{n\pi x}{l}$

$$\Rightarrow \sum_{k=1}^{\infty} T_n''(t) \sin\frac{n\pi x}{l} + \sum_{k=1}^{\infty} \left(\frac{n\pi}{l}\right)^2 T_n(t) \sin\frac{n\pi x}{l} = x$$

考虑 x 的 fourier 展开，$x \in (0,l)$

设 $x = \sum_{n=1}^{+\infty} A_n \sin\frac{n\pi}{l}x$

$$\int_0^l x \sin\frac{m\pi}{l}x\,dx = \frac{l A_m}{2}$$

$$A_m = \frac{2}{l}\int_0^l x \sin\frac{m\pi}{l}x\,dx = \frac{2}{l}\int_0^l x\, d\frac{l}{m\pi}\cos\frac{m\pi}{l}x$$

$$= \frac{2}{l}\, l\, \frac{l}{m\pi}(-1)^m = \frac{2l}{m\pi}(-1)^{m+1}$$

$$T_n''(t) + \left(\frac{n\pi}{l}\right)^2 T_n(t) = (-1)^n \frac{2l}{n\pi}, \quad T_n(0) = 0$$

$$T_n(t) = (-1)^n 2\left(\frac{l}{n\pi}\right)^3 + A_n \cos\frac{n\pi}{l}t + B_n \sin\frac{n\pi}{l}t$$

$$\Rightarrow T_n(0) = 0 \Rightarrow T_n(t) = (-1)^n 2\left(\frac{l}{n\pi}\right)^3 + (-1)^{n+1} 2\left(\frac{l}{n\pi}\right)^3 \cos\frac{n\pi}{l}t + B_n \sin\frac{n\pi t}{l}$$

$$V(t,x) = \sum_{n=1}^{\infty} T_n(t) \frac{\sin n\pi x}{l}$$

$$\partial_t V(0,x) = \sum_{n=1}^{\infty} T_n'(0) \frac{\sin n\pi x}{l} = -\frac{x}{l}$$

$$\Rightarrow \quad T_n'(0) = -\frac{2}{l}\int_0^l \frac{x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$\Rightarrow \quad T_n'(0) = \frac{n\pi}{l}B_n = \frac{2(-1)^n}{n\pi} \Rightarrow B_n = (-1)^n\frac{2l}{(n\pi)^2}$$

$$\Rightarrow T_n(t) = (-1)^n 2\left(\frac{l}{n\pi}\right)^3 - (-1)^n 2\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi}{l}t + (-1)^n\frac{2l}{(n\pi)^2}\sin\frac{n\pi t}{l} \quad \text{-- ok}$$

例：

$$\partial_t^2 u - \Delta u + \alpha_0(t,x)\partial_t u + \sum_{i=1}^{3}\alpha_i(t,x)\partial_i u = f, \quad 0\le t\le T,\ x\in\mathbb{R}^3$$

α_0, α_i 连续

Let $A(t) = \max_x|\alpha_0(x,t)| + \sum_i \max_x|\alpha_i(x,t)|$

证明 $$\max_{0\le s\le t}(E_u(s))^{\frac12} \le \left[E_u(0)^{\frac12} + \sqrt{2}\int_0^t \|f(\cdot,s)\|_{L^2}ds\right]\exp\left(2\int_0^t A(s)ds\right)$$

proof: $$(\partial_t^2 u - \Delta u)\partial_t u + \alpha_0(t,x)(\partial_t u)^2 + \sum_{i=1}^{3}\alpha_i(t,x)(\partial_i u)(\partial_t u) = f\,\partial_t u$$

$$\Rightarrow \quad \frac12\partial_t\left((\partial_t u)^2 + \sum_{i=1}^{3}(\partial_i u)^2\right) - (u_t u_{x_i})_{x_i}$$

$$\Rightarrow \quad \frac12\frac{d}{dt}\int_{\mathbb{R}^3}(\partial_t u)^2 + \sum_{i=1}^{3}(\partial_i u)^2 = \int_{\mathbb{R}^3}\underbrace{(u_t u_{x_i})_{x_i}}_{=0} - \alpha_0(t,x)(\partial_t u)^2 - \sum_{i=1}^{3}\alpha_i(t,x)(\partial_i u)(\partial_t u) + f\partial_t u$$

$$\le C A(t)\int_{\mathbb{R}^3}\left((\partial_t u)^2 + \sum_{i=1}^{3}(\partial_i u)^2\right) + \left(\int f^2\right)^{\frac12}\left(\int(\partial_t u)^2\right)^{\frac12}$$

$$E_u(t) = \int_{\mathbb{R}^3}(\partial_t u)^2 + \sum_{i=1}^{3}(\partial_i u)^2$$

$$\Rightarrow \quad \frac12\frac{d}{dt}E_u(t) \le 2(A(t))E_u(t) + \left(\int_{\mathbb{R}^3}f^2dx\right)^{\frac12}E_u(t)^{\frac12}$$

$$\Rightarrow \quad 记\ e_u(t) = E_u(t)^{\frac12}$$

$$\Rightarrow \quad \frac12\times 2e_u(t)\frac{d}{dt}e_u(t) \le 2A(t)e_u^2(t) + \|f(\cdot,s)\|_{L^2}e_u(t)$$

$$\Rightarrow \quad \frac{d}{dt}e_u(t) \le 2A(t)e_u(t) + \|f(\cdot,s)\|_{L^2}$$

$$e_u(t) \le e_u(0) + \int_0^t\|f(\cdot,s)\|_{L^2}ds + 2\int_0^t A(t)e_u(t)$$

固定 $t_1 > t$，$\forall\, t\in[0,t_1]$

$$\Rightarrow \quad e_u(t) \le \left(e_u(0) + \int_0^{t}\|f(\cdot,s)\|_{L^2}ds\right)e^{2\int_0^t A(s)ds}$$

$$\le \left(e_\mu(0) + \int_0^{t_1} \|f(\cdot,s)\|_{L^2} ds \right) e^{2\int_0^{t_1} A(s)\,ds}$$

$$\Rightarrow \sup_{0\le t\le t_1} e_\mu(t) \le [\quad\quad]$$

#

3.

2. Suppose that u is a smooth solution of the initial boundary value problem

$$u_{tt} - u_{xx} = S'(x) \qquad 0 < x < 1,\ t > 0,$$
$$u(x, t = 0) = f(x), \quad u_t(x, t = 0) = g(x),$$
$$u(0, t) = u(1, t) = 0.$$

a. Show that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (|u_t(x,t)|^2 + |u_x(x,t)|^2)\,dx = \int_0^1 S'(x)\,u_t(x,t)\,dx.$$

1

b. Conclude that there exists a constant C_0, depending only on f' and g, such that

$$\frac{1}{2}\int_0^1 \left(|u_t(x,t)|^2 + \frac{1}{2}|u_x(x,t)|^2\right) dx \le C_0 + 2\int_0^1 S^2(x)dx.$$

Hint: you may use the Cauchy-Schwarz $2\alpha\beta \ge -\frac{1}{2}\alpha^2 - 2\beta^2$.

a. $(u_{tt} - u_{xx})u_t = S'(x)u_t$

$$\Rightarrow \frac{1}{2}\partial_t(u_t^2) - (u_t u_x)_x + \frac{1}{2}\partial_x(u_x^2) = S'(x)u_t$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 (u_t^2 + u_x^2)\,dx = \int_0^1 S'(x)u_t(t,x)\,dx$$

$\Downarrow$

b.
$$\int_0^1 S'(x)u_t(t,x)\,dx = S(x)u_t(t,x)\Big|_0^1 - \int_0^1 S(x)u_{tx}(t,x)\,dx$$

$$= -\frac{d}{dt}\int_0^1 S(x)u_x(t,x)\,dx$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 (u_t^2 + u_x^2)\,dx = -\frac{d}{dt}\int_0^1 S(x)u_x(t,x)\,dx$$

$$\Rightarrow \frac{1}{2}\int_0^1 u_t^2 + u_x^2\,dx + \int_0^1 S(x)u_x(t,x) = \text{const}$$

$$= \frac{1}{2}\int_0^1 (f')^2 + g^2\,dx + \int_0^1 S(x)f'(x)\,dx$$

$$左 = \frac{1}{2}\int_0^1 u_t^2+u_x^2\,dx - \int_0^1 \sin - \frac{1}{4}\int_0^1 u_x^2$$

$$= \frac{1}{2}\int_0^1 u_t^2+\frac{1}{2}u_x^2\,dx - \int_0^1 \sin^2$$

$$右 \le \frac{1}{2}\int_0^1 (f')^2+g^2\,dx + \int_0^1 s^2(x) + \frac{1}{4}\int_0^1 (f')^2(x)\,dx$$

$\Rightarrow$ ok

38 稳定性

$$\begin{cases} u_{tt} - u_{xx} = f - u_t \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \varphi, \quad \partial_t u = \psi \end{cases}$$

$$u_{tt}u_t - u_t u_{xx} = f u_t - u_t^2$$

$$\Rightarrow \frac{1}{2}\partial_t(u_t^2) - \partial_x(u_t u_x) + \frac{1}{2}\partial_t(u_x^2) = f u_t - u_t^2$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_0^l u_t^2+u_x^2 = \int_0^l f u_t - \int_0^l u_t^2$$

$$\le \int_0^l \frac{1}{4}f^2 + \int_0^l u_t^2 - \int_0^l u_t^2$$

$$= \int_0^l \frac{1}{4}f^2\,dx$$

$$\Rightarrow \quad E(t) - E(0) \le \int_0^t \int_0^l \frac{1}{2}f^2\,dx\,dt$$

若初值足够小，在 $[0,T]$ 上 $E(t)$ 足够小

$$\frac{d}{dt}\int_0^l u^2\,dx = 2\int_0^l u u_t\,dx \le \int_0^l u^2\,dx + E(t)$$

$$\Rightarrow \int_0^l u^2\,dx \le \int_0^l u^2(x,0)\,dx + \int_0^T E(t)\,dt + \int_0^t\int_0^l u^2\,dx$$

利用 Gronwall $\displaystyle\int_0^l u^2(x,t)\,dx \le \left(\int_0^l u^2(x,0)\,dx + \int_0^T E(t)\,dt\right)e^T$

$\Rightarrow$ 初值 + 非齐次项 的 L^2 这估计

$\Rightarrow$ u 的 L^2 这估计 $\Rightarrow$ 稳定性

12/18 习题课

$$\begin{cases} u_t = \Delta u + \vec{b}\cdot\nabla u & \mathbb{R}^n\times[0,\infty) \\ u(x,0) = u_0(x). \end{cases}$$

$\vec{b} = (1,1,\cdots,1)$

1. 解: $\vec{b}\cdot\nabla u = \sum_{i=1}^{n} \partial_i u$

$$\widehat{\partial_i u}(\xi) = \int_{\mathbb{R}^n} \partial_i u\, e^{-ix\cdot\xi}\,dx$$

$$= -\int_{\mathbb{R}^n} u\,\partial_i(e^{-ix\cdot\xi})\,dx$$

$$= i\xi_i\,\hat{u}(\xi).$$

$\widehat{\partial_i^2 u}(\xi) = -\xi_i^2\,\hat{u}(\xi)$. 则方程化为:

$$\begin{cases} \partial_t\hat{u} = -|\xi|^2\hat{u}(\xi) + i\sum_{i=1}^{n}\xi_i\,\hat{u}(\xi) & \text{in } \mathbb{R}^n\times[0,+\infty) \\ \hat{u}(\xi)|_{t=0} = \hat{u}_0(\xi). \end{cases}$$

则 $\hat{u}(\xi) = e^{t(-|\xi|^2 + i\sum_{i=1}^{n}\xi_i)}\,\hat{u}_0(\xi)$

$$u = \left(e^{t(-|\xi|^2 + i\sum_{i=1}^{n}\xi_i)}\right)^{\vee} * u_0(x).$$

$$\left(e^{t(-|\xi|^2 + i\sum_{i=1}^{n}\xi_i)}\right)^{\vee}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-t|\xi|^2 + it\vec{b}\cdot\xi + ix\cdot\xi}\,d\xi$$

令 $\tau = \sqrt{t}\,\xi$, 则 $= \frac{1}{(4\pi^2 t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|\tau|^2} e^{i(\sqrt{t}\vec{b} + \frac{1}{\sqrt{t}}x)\cdot\tau}\,d\tau$

$$= \frac{1}{(4\pi^2 t)^{\frac{n}{2}}}\left(e^{-|\tau|^2}\right)^{\wedge}\left(-(\sqrt{t}\vec{b} + \frac{1}{\sqrt{t}}x)\right) \quad (*)$$

而 $\mathbb{R}^n$ 中 $\left(e^{-|x|^2}\right)^{\wedge}(\xi) = \int_{\mathbb{R}^n} e^{-|x|^2} e^{-ix\cdot\xi}\,dx$

$$= \int e^{-x_1^2} e^{-ix_1\xi_1}\,dx_1 \cdots \int e^{-x_n^2} e^{-ix_n\xi_n}\,dx_n.$$

由 $\mathbb{R}$ 中 $\left(e^{-x^2}\right)^{\wedge}(\xi) = \sqrt{\pi}\,e^{-\frac{\xi^2}{4}}$ $= (\sqrt{\pi})^n\,e^{-\frac{|\xi|^2}{4}}$

$$\text{则 } (*) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\,e^{-\frac{1}{4t}|t\vec{b} + x|^2}.$$

$$\Rightarrow u = (*) * u_0 = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{1}{4t}|tb + x - y|^2}\,u_0(y)\,dy.$$

为所求解.

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2. $\begin{cases} \partial_t u = \partial_x^2 u \\ u(0,t) = \mu_1(t). \quad (u_x + hu)(l,t) = \mu_2(t) \quad h>0 \\ u(x,0) = \varphi(x). \end{cases}$

为第三类边值问题. 证明其稳定性:

证明: 对 u_1、u_2 分别满足方程

$$\begin{cases} \partial_t u_i = \partial_x^2 u_i \\ u_i(0,t) = \mu_1^i(t). \quad (u_{ix} + hu_i)(l,t) = \mu_2^i(t). \quad h>0 \\ u_i(x,0) = \varphi_i(x) \end{cases}$$

取 $u = u_1 - u_2$. 则 u 满足方程:

$$\begin{cases} \partial_t u = \partial_x^2 u \\ u(0,t) = \mu_1^1(t) - \mu_1^2(t) \quad (u_x + hu)(l,t) = \mu_2^1(t) - \mu_2^2(t) \\ u(x,0) = \varphi_1(x) - \varphi_2(x) \end{cases}$$

由极值原理. 可知 u 的极大,小值在边界取到

~~max~~ 若最大值 M 在 $\{0\} \times [0,\infty)$ 或 $[0,l] \times \{0\}$ 取到

则 $M \le \max\{\mu_1^1(t) - \mu_1^2(t)\} + \max\{~~\varphi(x)~~ \varphi_2(x)\}$

若在 $\{l\} \times [0,\infty)$ ~~上~~取到. 则 ~~ux~~ 设在 (l, t_0) 取到. 则

$u_x(l,t_0) \le 0 \Rightarrow M \le$ ~~max~~ $\frac{1}{h}(\mu_2^1(t_0) - \mu_2^2(t_0)) - u_x(l,t_0))$

$\le \frac{1}{h}(\mu_2^1(t_0) - \mu_2^2(t_0))$.

故 $\max u(x,t) \le C \max(\max|\mu_1^1(t) - \mu_1^2(t)| + \max|\varphi_1 - \varphi_2|, \max|\mu_2^1 - \mu_2^2|)$

则 u 稳定.

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$$\begin{cases} u_t - a^2 u_{xx} = f, & (x,t) \in Q_T = (0,l) \times (0,T] \\ u(x,0) = \varphi(x), & 0 \le x \le l \\ u(0,t) = g_1(t),\ u(l,t) = g_2(t) & 0 \le t \le T \end{cases}$$

试证 $\max_{Q_T} |u| \le FT + B$. $F = \sup_{Q_T} |f|$.

$B = \max\{\sup_{[0,l]} |\varphi|, \sup_{[0,T]} |g_1|, \sup_{[0,T]} |g_2|\}$.

证明: 取 $w(x,t) = \pm u(x,t) - Ft - B$. 则

$\partial_t w = u_t - F$. $\partial_x^2 w = u_{xx}$. 则

$$\begin{cases} w_t - a^2 w_{xx} = \pm f - F \le 0 & (x,t) \in Q_T \\ w(x,0) = \pm\varphi(x) - B \le 0 \\ w(0,t) = \pm g_1(t) - Ft - B \le 0 \quad u(l,t) \le 0 \end{cases}$$

则由极值原理, w 最大值在边界取到. $\max|u| \le Ft + B$

取 $u' = -u$, 则得另一部分. #

RMK: 回顾极值原理的证明: 证明极大值在边界取到时仅使用条件 $\partial_t u - \partial_x^2 u \le 0$, 因此对方程解做估计时, 构造的函数可以仅是方程下解.

若 $u_t - \Delta u \le 0$, 则称 u 为下解.

对高维情况 $Q_T = \Omega \times [0,\infty)$ 时. $\Omega \in \mathbb{R}^n$. 则原证明依旧可用. 取 $l = \operatorname{diam}\Omega = \max_{x,y \in \Omega} |x-y|$

记 $v(x,t) = u(x,y) + \frac{M-m}{4l^2} |x - x^*|^2$ $\max_{\Gamma_T} v \le M$

由于在内部处取到极值, 则 $\Delta u = \sum_{i=1}^n \partial_{ii} u$ $\max_{Q_T} \ge M$

每个 $\partial_{ii} u \le 0 \Rightarrow \Delta u \le 0$. $\Delta v \le 0$ $\partial_t u = 0$

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所以对 $\Omega\times[0,+\infty)$ 形区域、若 $\partial_t u-\Delta u\le 0$，则 u 的最大值在边界取到。

补充题一：u 为热方程下解，则 $v=|Du|^2+u_t^2$ 也为方程下解。

证：$\partial_t v=\partial_t\left(\sum_{i=1}^n \partial_i u\partial_i u\right)+2u_t u_{tt}$

$$=2\sum_{i=1}^n \partial_i u\partial_i u_t+2u_tu_{tt}.$$

$$\partial_i v=2\sum_{j=1}^n \partial_j u\,\partial_{ij}u+2u_t\partial_i u_t.$$

$$\partial_{ii}v=2\sum_{j=1}^n\partial_{ij}u\,\partial_{ij}u+2\sum_{j=1}^n\partial_j u\,\partial_{iij}u+2(\partial_i u_t)^2+2u_t\partial_{ii}u_t$$

$$\Delta v=2\sum_{i=1}^n\sum_{j=1}^n|\partial_{ij}u|^2+2\sum_{j=1}^n\partial_j u\,\partial_j(\Delta u)+\sum_{i=1}^n2(\partial_iu_t)^2+2u_t\partial_t(\Delta u)$$

$\Rightarrow$ ~~$\partial_t v=\Delta v$~~ 由 $\partial_t u\le\Delta u$，得

$$\partial_t v-\Delta v\le-2\sum_{i,j=1}^n|\partial_{ij}u|^2-2\sum_{i=1}^n(\partial_iu_t)^2\le 0.$$

故 $|Du|^2+u_t^2$ 最大值在边界取到。

补充题二：$u(x,t)$ 为方程：
$$\begin{cases}u_t=u_{xx} & (x,t)\in Q_\pi=(0,\pi)\times[0,\infty)\\ u|_{x=0}=u_x|_{x=\pi}=0\\ u|_{t=0}=\varphi(x)\end{cases}$$
的解，则有：

$$\sup_{Q_\pi}|u(t,x)|\le\sup_{(0,\pi)}|\varphi(x)|$$

证明：对 u 关于 $x=\pi$ 做偶延拓 $v(t,x)=\begin{cases}u(t,x) & x\in(0,\pi)\\ u(t,2\pi-x) & x\in[\pi,2\pi)\end{cases}$

下说明其合理性，由于为偶延拓，连续性显然，而对 $x>\pi$，$v_{xx}=u_{xx}(t,2\pi-x)$，$v_t=u_t$。

故 $v_t=v_{xx}$ on $[\pi,2\pi)\times[0,\infty)$.

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其一阶导 $v_x = 0$ on $x = \pi$. 故为良定义，则对 v 使用极值原理

由于 $v|_{x=0} = v|_{x=2\pi} = 0 \Rightarrow \sup|u| = \sup|v| \le \sup|\varphi|$

第二、三类边值的最大模问题

考虑混合边界问题：

$\alpha(t), \beta(t) \ge 0$.

$$\begin{cases} u_t - a^2 u_{xx} = f(x,t), & (x,t) \in Q_T \\ u|_{t=0} = \varphi(x), & x \in [0,l] \\ -u_x + \alpha(t) u|_{x=0} = g_1(t), & t \in [0,T] \\ u_x + \beta(t) u|_{x=l} = g_2(t), & t \in [0,T] \end{cases}$$

$u \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q}_T)$ 为方程的解，则

$$\max_{\overline{Q}_T} |u(x,t)| \le C(F+B).$$

$F = \sup_{Q_T} |f|$, $B = \max\{\max|\varphi|, \max|g_1|, \max|g_2|\}$

证明：首先证明引理：若满足

$$\begin{cases} u_t - a^2 u_{xx} \ge 0 \\ u|_{t=0} \ge 0 \\ [-u_x + \alpha(t) u]|_{x=0} \ge 0 \\ [u_x + \beta(t) u]|_{x=l} \ge 0 \end{cases}$$

则 $u \ge 0$ on $\overline{Q}_T$

由极值原理 $\Rightarrow$ u 在 Γ_T 上取最小值，仅须说明 u 在边界 Γ_T 上非负即可。若最小值在 $t=0$ 处取到，则命题成立

下说明 u 不可能在 $x=0$, $x=l$ 取最小值 负的。

① 若 $[-u_x + \alpha(t) u]|_{x=0} > 0$. $[u_x + \beta(t) u]|_{x=l} > 0$

且在 $x=0$、$x=l$ 取负的最小值，设在 t_0 处取到，

则 $-u_x(0,t_0) \le 0$ $\alpha(t_0) u(0,t_0) \le 0$ 矛盾。

$u_x(l,t_0) \le 0$, $\beta(t_0) u(l,t_0) \le 0$, 相同。

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② 对于一般情况 构造辅助函数，取 $\varepsilon>0$.

$$w(x,t) = 2a^2t + (x-\frac{l}{2})^2. \quad w_t = 2a^2. \quad w_x = 2x - l$$

$$w_{xx} = 2.$$

取 $v(x,t) = u(x,t) + \varepsilon w(x,t)$

而对 w 有 $\partial_t w - a^2 \partial_{xx} w = 0 \rightarrow v_t - a^2 v_{xx} = u_t - a^2 u_{xx} \geq 0$

而 $v|_{t=0} = u|_{t=0} + \varepsilon(x-\frac{l}{2})^2 \geq 0$

$[-v_x + \alpha(t) v]|_{x=0}$

$$= [-u_x + \alpha(t) u]|_{x=0} + \varepsilon[l + \alpha(t)(2a^2t + \frac{l^2}{4})]$$

> 0

$[v_x + \beta(t) v]|_{x=l}$

$$= [u_x + \beta(t) u]|_{x=l} + \varepsilon[l + \beta(t)(2a^2t + \frac{l^2}{4})]$$

> 0

故 $v(x,t) \geq 0$ on $\overline{Q}_T$. 令 $\varepsilon \to 0$. 则可得到

$$u(x,t) \geq 0.$$

下面证明定理，构造辅助函数：

$$v(x,t) = Ft + B\, z(x,t) \pm u(x,t).$$

$$z(x,t) = 1 + \frac{1}{l} w(x,t)\text{，则对 } z \text{ 有：}$$

$z_t - a^2 z_{xx} = 0. \quad z|_{t=0} = 1 + \frac{1}{l} w(x,0) \geq 1.$

$[-z_x + \alpha(t) z]|_{x=0} = 1 + \alpha(t)(1 + \frac{1}{l} w(0,t)) \geq 1.$

$[z_x + \beta(t) z]|_{x=l} = 1 + \beta(t)(1 + \frac{1}{l} w(l,t)) \geq 1$

则 $v_t - a^2 v_{xx} = F \pm f(x,t) \geq 0$

$v|_{t=0} \geq B \pm \varphi \geq 0$

$[-v_x + \alpha(t) v]|_{x=0} \geq B \pm g(t) \geq 0 \quad [v_x + \beta(t) v]|_{x=l} \geq 0$

则 $v \geq 0. \Rightarrow |u| \leq FT + B \cdot \max z \leq FT + (1 + \frac{2a^2T}{l} + \frac{l}{4}) B$

补充题: u 满足 $u_t - a^2\Delta u = 0$. ϕ 为[redacted]凸函数, 则 $v = \phi(u)$ 为下解.

$v_t = \phi'(u)\, u_t$.

$\partial_i v = \phi'(u)\,\partial_i u$. $\partial_i^2 v = \phi''(u)(\partial_i u)^2 + \phi'(u)\,\partial_i^2 u$.

$\Rightarrow v_t - a^2\Delta v = \phi'(u)(u_t - a^2\Delta u) - \phi''(u)(\partial_i u)^2 a^2$

≤ 0 为下解

关于 Green 函数的对称性

① 物理解：想象成电势　把点电荷放在A点，其在B点产生的电势 $\varphi_{A\to B}$
和　把点电荷放在B点，其在A点产生的电势 $\varphi_{B\to A}$

在物理上，显然有 $\varphi_{A\to B} = \varphi_{B\to A}$

即 直观上有 $G(A,B) = G(B,A)$

② 从对称性解（代数）

上课没有讲 Green 函数的存在性，所以我们（这里）假设其存在（在后续课程中会证明）

先来看一种对称性，若 $f'(0)=f'(1)=0$，$g'(0)=g'(1)=0$

则 $\int_0^1 f''g\,dx = \int_0^1 (f'g)' - \int_0^1 f'g'\,dx = (f'g)\Big|_0^1 - \int_0^1 f'g'\,dx = -\int_0^1 f'g'\,dx$

同理：$\int_0^1 fg''\,dx = -\int_0^1 f'g'\,dx$

$\Rightarrow \int_0^1 fg''\,dx = \int_0^1 f''g\,dx$　在 $f'(0)=f'(1)=g'(0)=g'(1)=0$ 的条件下！

利用上课的第二格林公式，更一般地，我们有若 $\frac{\partial f}{\partial n}\Big|_{\partial\Omega} = \frac{\partial g}{\partial n}\Big|_{\partial\Omega} = 0$

有 $\int_\Omega f\Delta g = \int_\Omega (\Delta f) g$　这是一种对称性

回顾 Green 函数定义：
$$\begin{cases} G(x) \in C^2(\Omega), & \Delta G = 0, \ \forall x \neq x_0 \\ G(x) = 0, & \forall x \in \partial\Omega \\ G + \frac{1}{4\pi|x-x_0|} & \text{在 } x_0 \text{ 处有限，处处二阶连续可导且在 } x_0 \text{ 处调和} \end{cases}$$

由 Green 函数，我们可以得到 $\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi & \text{on } \partial\Omega \end{cases}$ 的解

$$u(x) = \int_\Omega G(x,y) f(y)\,dy + \int_{\partial\Omega} \varphi(y) \frac{\partial G(x,y)}{\partial n(y)}\,d\sigma$$

我们利用上述对称性得出 $G(x,y) = G(y,x)$，我们看到 G 的定义与方程的边值条件无关，只与区域 Ω 有关，从而我们考虑如下方程的解

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta v = g & \text{in } \Omega \\ v|_{\partial\Omega} = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Rightarrow u(x) = \int_\Omega G(x,y) f(y)\, dy, \quad v(x) = \int_\Omega G(x,y) g(y)\, dy$$

$$\Rightarrow \int_\Omega u(x) g(x)\, dx = \int_\Omega u(x) (Lv)(x)\, dx \overset{\text{对称性}}{=\!=} \int_\Omega (Lu)(x) v(x)\, dx$$

$$= \int_\Omega f(x) v(x)\, dx$$

$$= \int_\Omega \int_\Omega G(x,y) f(x) g(y)\, dx\, dy$$

$$\int_\Omega u(x) g(x)\, dx = \int_\Omega \int_\Omega G(x,y) f(y) g(x)\, dx\, dy$$

换元 $x \to y$, $y \to x$:

$$\int_\Omega \int_\Omega G(x,y) f(y) g(x)\, dx\, dy = \int_\Omega \int_\Omega G(y,x) f(x) g(y)\, dx\, dy$$

$$\int_\Omega \int_\Omega G(y,x) f(x) g(y)\, dx\, dy = \int_\Omega \int_\Omega G(x,y) f(x) g(y)\, dx\, dy$$

由于 f, g 任意 $\Rightarrow G(y,x) = G(x,y)$

P118, 20.

$Lu = u_t - a^2 u_{xx} + b(x,t) u_x + c(x,t) u$, $c(x,t) \geq -C_0$, $C_0 > 0$ 是常数，若

$u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$，且 $Lu \leq 0$，则若 $\max\limits_{\Gamma_T} u(t,x) \leq 0$，则必有 $\max\limits_{\overline{Q_T}} u(t,x) \leq 0$

我们首先证明 19 的结论.

19. $Lu = u_t - a^2 u_{xx} + b(x,t) u_x + c(x,t) u = f(x,t)$.

设 $c \geq 0$，$u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$，且满足 $Lu \leq 0$，则 u 在 $\overline{Q_T}$ 上的非负最大值必在抛物边界上达到，i.e.

$$\max_{\overline{Q_T}} u(t,x) \leq \max_{\Gamma_T} u^+(t,x).$$

这里 $u^+(x,t) = \max\{u(t,x), 0\}$.

proof of 19:

我们首先考察 $Lu < 0$ 的情形，此时假设 u 在 Q_T 上取到正的最大值，

记 $u(x_0,t_0) = \max\limits_{\overline{Q_T}} u$，$(x_0,t_0) \in Q_T$ $u(x_0,t_0) > 0$.

则 $u_t(x_0,t_0) \geq 0$，$u_{xx} \leq 0$，$u_x = 0$，$c(t,x) u \geq 0$

$\Rightarrow Lu \geq 0$，这与 $Lu < 0$ 矛盾，于是此时有 $\max\limits_{\overline{Q_T}} u(t,x) \leq \max\limits_{\Gamma_T} u^+(t,x)$.

下面我们再考察 $Lu \leq 0$ 的情形，类似于书上的证明，令

$\Rightarrow$ $u^\varepsilon(t,x) = u(t,x) - \varepsilon t$ $(\varepsilon > 0)$

则 $Lu^\varepsilon = Lu - \varepsilon < 0$.

进而对 u^ε 用上述结论，我们可以得到 $\max\limits_{\overline{Q_T}} u^\varepsilon(t,x) \leq \max\limits_{\Gamma_T} (u^\varepsilon)^+(t,x)$.

令 $\varepsilon \to 0$，我们就有 $\max\limits_{\overline{Q_T}} u(t,x) \leq \max\limits_{\Gamma_T} u^+(t,x)$.

下面回到 20 的证明：

令 $v = u e^{-C_0 t}$，则 $v_t = u_t e^{-C_0 t} - C_0 u e^{-C_0 t} = u_t e^{-C_0 t} - C_0 v$.

$-a^2 v_{xx} + b(x,t) v_x + c(x,t) v = e^{-C_0 t}(-a^2 u_{xx} + b(x,t) u_x + c(t,x) u)$.

则 v 满足 $v_t - a^2 v_{xx} + b(t,x) v_x + (c(t,x) + C_0) v = e^{-C_0 t}(-a^2 u_{xx} + b(x,t) u_x + c(t,x) u + u_t)$

$= e^{-C_0 t} v \leq 0$

19的
则由于 $(t,x)+c_0\geq 0$，由上述结论，我们知道 $\max\limits_{\overline{Q_T}} v(t,x)\leq \max\limits_{\Gamma_T} v^+(t,x)$.

而 $v=ue^{-c_0t}$，则由此可知，若 $\max\limits_{\Gamma_T} u(t,x)\leq 0$，则必有 $\max\limits_{\overline{Q_T}} u(t,x)\leq 0$

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P119、22.

有界开集 $\Omega\subset\mathbb{R}^n$，$n\geq 2$，$Q=\Omega\times(0,+\infty)$，$\Gamma=\partial\Omega\times[0,+\infty)$

$$\begin{cases} u_t-\Delta u=0 & (x,t)\in Q \\ \left(\alpha\frac{\partial u}{\partial \nu}+\sigma u\right)\Big|_\Gamma=0 \\ u(x,0)=\varphi(x). \end{cases} \quad ①$$

其中 $\vec{\nu}$ 是 Γ 的外法向，α,σ 是不同时为0的非负常数，考虑①的 $C^{2,1}(Q)\cap C^{1,1}(\overline{Q})$ 解，记

$$E(t)=\frac{1}{2}\int_\Omega u^2(x,t)\,dx.$$

(a) $E_t(t)\leq 0$，$\forall\ t\geq 0$.

(b) $\|u(\cdot,t)\|_{L^2(\Omega)}\leq\|\varphi\|_{L^2(\Omega)}$. $\forall\ t>0$

(c) 初值问题解唯一

proof: (a). $E'(t)=\int_\Omega uu_t\,dx$

边界条件，这里不妨设 $\alpha>0$.

$$=\int_\Omega u\Delta u\,dx=\int_\Omega \operatorname{div}(uDu)\,dx-\int_\Omega|Du|^2dx$$

$$=\int_{\partial\Omega}u(Du\cdot\vec{n})\,dx-\int_\Omega|Du|^2dx$$

$$=-\int_{\partial\Omega}\frac{\sigma}{\alpha}u^2\,dx-\int_\Omega|Du|^2dx$$

$$<0$$

(b)，由(a)的结论，$E(t)$ 单调↓

$$\Rightarrow E(t)\leq E(0)=\frac{1}{2}\int_\Omega u^2(x,0)\,dx=\frac{1}{2}\int\varphi^2(x)\,dx$$

$$\Rightarrow \|u(\cdot,t)\|_{L^2(\Omega)}\leq\|\varphi\|_{L^2(\Omega)}$$

(c) 假设方程有另一解，设为 u_1，则令 $v=u_1-u$.

则易知 v 满足方程

$$\begin{cases} v_t - \Delta v = 0 & (x,t)\in Q. \\ \left(\alpha \frac{\partial v}{\partial \nu} + \sigma v\right)\Big|_{\Gamma} = 0 \\ u(x,0) = 0 \end{cases}$$

则由(b)的结论可知 $E(t)\equiv 0 \quad \forall\, t \iff \int_\Omega u^2(t,x)\,dx \equiv 0$

$\Rightarrow u(t,x)\equiv 0$ in Ω.

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补充:

若 $u\in C^2(\Omega)\cap C^1(\bar\Omega)$ 满足 $\Delta u=0$ in Ω, $n=2$, 则 $\forall\, x_0\in\Omega$, 有.

$$u(x_0) = \frac{1}{2\pi}\int_{\partial\Omega}\left(u\frac{\partial}{\partial n}\log|x-x_0| - \log|x-x_0|\frac{\partial u}{\partial n}\right)ds.$$

Proof: Recall Green 第二公式.

$$\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$$

令 $\Omega_\varepsilon = \Omega\setminus\overline{B_\varepsilon(x_0)}$, $\partial\Omega_\varepsilon = \partial\Omega\cup\partial B_\varepsilon$.

则由于 $\Delta(\log|x-x_0|)=0$ $\Delta u=0$, 再结合 Green 第二公式.

$$\Rightarrow 0 = \int_{\Omega_\varepsilon}\left(u\Delta(\log|x-x_0|) - \log|x-x_0|\Delta u\right) = \int_{\partial\Omega\cup\partial B_\varepsilon}\left(u\frac{\partial(\log|x-x_0|)}{\partial \vec{n}} - \log|x-x_0|\frac{\partial u}{\partial \vec{n}}\right)ds$$

$$= \int_{\partial\Omega}\left(u\frac{\partial(\log|x-x_0|)}{\partial \vec{n}} - \log|x-x_0|\frac{\partial u}{\partial \vec{n}}\right)ds.$$

$$+ \int_{\partial B_\varepsilon} u\frac{\partial(\log|x-x_0|)}{\partial \vec{n}} - \log|x-x_0|\frac{\partial u}{\partial \vec{n}}\,ds.$$

注意此时外法向向里

下面计算 $\int_{\partial B_\varepsilon} u\frac{\partial(\log|x-x_0|)}{\partial \vec{n}} - \log|x-x_0|\frac{\partial u}{\partial \vec{n}}\,ds$

$$= \underbrace{\int_{\partial B_\varepsilon} u\,\frac{\vec{x}-\vec{x_0}}{x-x_0}\cdot\vec{n}\cdot\frac{1}{|x-x_0|}}_{①} - \underbrace{\int_{\partial B_\varepsilon}\log|x-x_0|\frac{\partial u}{\partial \vec{n}}\,ds}_{②}.$$

$$① = \int_{\partial B_\varepsilon} u\,\frac{\vec{x}-\vec{x_0}}{|x_0-x|}\cdot\frac{-(\vec{x}-\vec{x_0})}{|x-x_0|}\cdot\frac{1}{|x-x_0|} = -\frac{1}{\varepsilon}\int_{\partial B_\varepsilon} u\,d\sigma = -\frac{2\pi}{2\pi\varepsilon}\int_{\partial B_\varepsilon} u\,d\sigma$$

$$\longrightarrow -2\pi\, u(x_0) \quad \text{as } \varepsilon\to 0$$

$|②| \le \int_{\partial B_\varepsilon} (|u| \log \varepsilon) ds \le \max_{\partial B_\varepsilon} |u| \varepsilon \log \varepsilon \to 0 \quad as \ \varepsilon \to 0.$

则

进而 $\int_{\partial B_\varepsilon} u \frac{\partial(\log|x-x_0|)}{\partial \vec{n}} - \log|x-x_0| \frac{\partial u}{\partial \vec{n}} d\sigma \to -2\pi u(x_0)$

于是，则有 $2\pi u(x_0) = \int_{\partial\Omega} \left(u \frac{\partial(\log|x-x_0|)}{\partial \vec{n}} - \log|x-x_0| \frac{\partial u}{\partial \vec{n}}\right) d\sigma$.

i.e $u(x_0) = \frac{1}{2\pi} \int_{\partial\Omega} \left(u \frac{\partial(\log|x-x_0|)}{\partial \vec{n}} - \log|x-x_0| \frac{\partial u}{\partial \vec{n}}\right) d\sigma$.

用分离变量法和 Green 函数法求解 2 维圆盘上的 Dirichlet 问题

$$\begin{cases} \Delta u = 0 & \text{in } B_R(0) \\ u = \varphi & \text{on } \partial B_R(0). \end{cases}$$

分离变量法：

令 $u(r,\theta) = R(r)\Theta(\theta)$.

Δu 在 2 维的极坐标表示 $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta}$

则 $r^2 R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0$

$$\Rightarrow \frac{r^2R''(r) + rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

令 $\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda$，由于 $\Theta(\theta)$ 以 2π 为周期，则 $\Theta(0) = \theta(2\pi)$.

进而 $\lambda_n = n^2$ 为特征值 $\Rightarrow$ 对应特征函数为 $\Theta_n(\theta) = a_n\cos n\theta + b_n \sin n\theta$

将特征值 $\lambda_n = n^2$ 代入 $\frac{r^2R'' + rR'(r)}{R(r)} = n^2$.

$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r) = 0$

上述方程即 Euler 方程 $\Rightarrow R_0 = c_0 + d_0 \ln r$, $n = 0$.

$R_n = c_n r^n + d_n r^{-n}$ $n \ge 1$.

由于在 $B_R(0)$ 内部连续，故在 0 处良定，则必有 $d_n = 0$, $d_0 = 0$ $\Rightarrow R_n = C_n r^n$. $n \ge 1$

$$\Rightarrow u(r,\theta) = \left(\sum_{n=1}^{\infty} A_n r^n \sin n\theta + B_n r^n \cos n\theta\right) + B_0.$$

由条件 $u(R,\theta) = \sum_{n=1}^{\infty}\left(A_n R^n \sin n\theta + B_n R^n \cos n\theta\right) + B_0 = \varphi(\theta)$.

若记 $\varphi(\theta)$ 的 Fourier 展开为 $a_0+\sum_{n=1}^{\infty} a_n\cos n\theta + b_n \sin n\theta$.

here $a_0=\frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)d\theta$.

$a_n=\frac{1}{\pi}\int_0^{2\pi}\varphi(\theta)\cos n\theta\, d\theta$

$b_n=\frac{1}{\pi}\int_0^{2\pi}\varphi(\theta)\sin n\theta\, d\theta$

进而对照系数可知 $B_0=a_0=\frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)d\theta$.

$A_n=\frac{1}{R^n}\frac{1}{\pi}\int_0^{2\pi}\varphi(\theta)\cos n\theta\, d\theta$.

$B_n=\frac{1}{R^n}\frac{1}{\pi}\int_0^{2\pi}\varphi(\theta)\sin n\theta\, d\theta$.

Green 函数法:

首先对于2维的情形, Green 函数应满足 (类比 $n=3$).

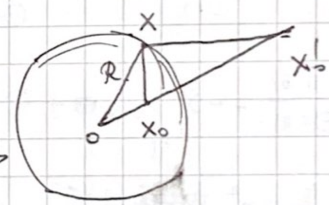
① $G(x)\in C^2(\Omega)$, 且 $\Delta G=0$, 除了 $x=x_0$

② $G(x)=0 \quad \forall x\in\partial\Omega$.

③ $G(x)+\frac{1}{2\pi}\log\frac{1}{|x-x_0|}$ 在 x_0 有限, 外处连续可导(二阶)且调和.

则方程 $\begin{cases}\Delta u=0\\ u|_{\partial B_R}=\varphi\end{cases}$ 的解为 $u(x)=\int_{\partial B_R}\varphi\frac{\partial G}{\partial n}ds$.

下求 Green 函数 G.



类似于3维的求法, 我们取 x_0' 为 x_0 关于圆周的对称点

由几何关系, 有 $|x_0||x_0'|=R^2 \Rightarrow |x_0'|=\frac{R^2}{|x_0|}$

$\Rightarrow x_0'=\frac{R^2}{|x_0|^2}x_0$

则令 $G(x,x_0)=\frac{1}{2\pi}\log|x-x_0|-\frac{1}{2\pi}\log(c|x-x_0'|)$.

由几何关系, $\frac{|x-x_0|}{|x-x_0'|}=\frac{|x_0|}{R}$, 则取 $c=\frac{R}{|x_0|}$, 则满足 $G|_{\partial B_R}=0$.

$\Rightarrow G(x,x_0)=\frac{1}{2\pi}\log|x-x_0|-\frac{1}{2\pi}\log\left(\frac{R}{|x_0|}|x-x_0'|\right)$.

进而 $\nabla G(x,x_0)=\frac{1}{2\pi}\frac{x-x_0}{|x-x_0|^2}-\frac{1}{2\pi}\frac{x-x_0'}{|x-x_0'|^2}$

注意到 $|x-x_0'|=\frac{R|x-x_0|}{|x_0|}$, 则 $\nabla G(x,x_0)=\frac{1}{2\pi}\frac{x-x_0}{|x-x_0|^2}-\frac{1}{2\pi}\frac{|x_0|^2}{R^2}\frac{x-x_0'}{|x-x_0|^2}$ ..

由于 $x_0' = \frac{R^2}{|x_0|^2} x_0$

则有 $\nabla G(x, x_0) = \frac{1}{2\pi} \frac{x}{|x-x_0|^2} - \frac{1}{2\pi} \frac{|x_0|^2}{R^2} \frac{x}{|x-x_0'|^2}$

$\Rightarrow \frac{\partial G}{\partial \vec{n}} = \nabla G \cdot \frac{x}{R} = \left(\frac{1}{2\pi} \frac{|x|^2}{|x-x_0|^2} - \frac{1}{2\pi} \frac{|x_0|^2}{R^2} \frac{|x|^2}{|x-x_0'|^2} \right) \frac{1}{R}$

在边界 ∂B_R 上，$|x| = R$

$\Rightarrow \frac{\partial G}{\partial \vec{n}} = \left(\frac{1}{2\pi} \frac{R^2}{|x-x_0|^2} - \frac{1}{2\pi} \frac{|x_0|^2}{|x-x_0|^2} \right) \frac{1}{R}$

$\Rightarrow u(x_0) = \frac{1}{2\pi R} \int_{\partial B_R} \varphi(x) \frac{R^2 - |x_0|^2}{|x-x_0|^2} dS_x$

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p167

26（调和函数极限定理）

设 $\{u_n\}$ 是 Ω 中的调和函数列，且 $u_n(x)$ 一致收敛到极限函数 $u(x)$，则 $u(x)$ 也是调和函数

proof:

method 1: 平均值公式

$\forall B_r(x) \subset \Omega$，对 $u_n(x)$，由平均值公式，我们首先有

$$u_n(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u_n(y) \, ds_y .$$

则由于 $u_n(x)$ 在 Ω 上一致收敛，则两边同时取极限，则有

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, dy$$

（这里后一项只需注意到 $\forall \varepsilon > 0$，$\frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} |u_n(y) - u(y)| \, ds_y < \varepsilon \cdot |\partial B_r(x)| \cdot \frac{1}{|\partial B_r(x)|}$

$= \varepsilon$（对于充分大的 n）即可）.

于是由此可知 $u(x)$ 是调和函数

method 2: Possion 公式

同样的，$\forall B_r(x_0) \subset \Omega$，对 $u_n(x)$，由 Possion 公式，我们有

$$u_n(x) = \frac{r^2 - |x-x_0|^2}{n\omega_n r} \int_{\partial B_r(x_0)} \frac{u_n(y)}{|x-y|^n} \, ds_y .$$

则同上，由一致收敛性，两边可同时取极限，得到

$$u(x) = \frac{r^2 - |x-x_0|^2}{n\omega_n r} \int_{\partial B_r(x_0)} \frac{u(y)}{|x-y|^n} \, ds_y .$$

则由此易知 $u(y)$ 也是 Ω 上的调和函数

Rmk:

① 由上述证明实际上可以发现只用内闭一致收敛即可满足上述结论.

② 注意球心在原点的Possion公式为

$$u(x) = \frac{R^2-|y|^2}{n\omega_n R}\int_{\partial B_R(0)}\frac{\varphi(y)}{|x-y|^n}\,ds_y$$

这里 $\begin{cases}\Delta u=0 & \text{on } B_R(0)\\ u|_{\partial B_R}=\varphi.\end{cases}$

则作平移 $u(x)=u(x_0+\tilde{x})$，对于以 x_0 为中心的球，我们可以得到

$$u(x) = \frac{R^2-|\tilde{x}|^2}{n\omega_n R}\int_{\partial B_R(0)}\frac{\varphi(x_0+\tilde{y})}{|\tilde{x}-\tilde{y}|^n}\,ds_{\tilde{y}}$$

$$= \frac{R^2-|x-x_0|^2}{n\omega_n R}\int_{\partial B_R(x_0)}\frac{\varphi(y)}{|x-y|^n}\,ds_y .$$

27. 二维调和函数奇点可去定理:

若 y 是调和函数 $u(x)$ 的孤立奇点，且 $u(x)=o\left(\frac{1}{|x-y|}\right)$ as $x\to y$

则奇点 y 可去。

proof: 不失一般性，我们假设 $y=0$ （否则作平移即可）

假设 u 在 $0<|x|\le R$ 上是连续的，不妨设 $R\le 1$，令 v 满足

$$\begin{cases}\Delta v=0 & \text{in } B_R\\ v=u & \text{on } \partial B_R\end{cases}$$

则易知 v 是整个 $\overline{B_R}$ 上的调和函数.

然后我们证明 $u\equiv v$ in $B_R\setminus\{0\}$

令 $w=v-u$ in $B_R\setminus\{0\}$ $\quad M_r=\max\limits_{\partial B_r}|w|$

则考察 $\pm w(x)+M_r\log\frac{1}{|x|}/\log\frac{1}{r}$ $\quad x\in B_R\setminus B_r$

则显然，在 ∂B_r 上，我们有 $\pm w(x)+M_r\log\frac{1}{|x|}/\log\frac{1}{r} = \pm w+M_r\ge 0$

在 ∂B_R 上，我们有 $\pm w(x)+M_r\log\frac{1}{R}/\log\frac{1}{r} = 0+M_r\log\frac{1}{R}/\log\frac{1}{r}\ge 0$.

(note that $w=u-v=0$ on ∂B_R)

⇒ 则由调和函数的极大值原理

$\pm w(x) + M_r \log\frac{1}{|x|} / \log\frac{1}{r} \geq 0 \quad \forall \ x \in B_R \setminus B_r$.

$\Rightarrow \ |w(x)| \leq M_r \dfrac{\log\frac{1}{|x|}}{\log\frac{1}{r}}$

记 $M = \sup\limits_{\partial B_R} |u|$

则 $M_r = \sup\limits_{\partial B_r} |w| = \sup\limits_{\partial B_r} |v-u| \leq \max\limits_{\partial B_r} |u| + \max\limits_{\partial B_r} |V| \leq \max\limits_{\partial B_r} |u| + \max\limits_{\partial B_R} |u|$ (这里 $\max\limits_{\partial B_R}|u| = M$.)

↑ Here note that $\begin{cases} \Delta v = 0 \\ v = u \ \text{ on } \partial B_R \end{cases}$ 由极大值原理. $|v| \leq \max\limits_{\partial B_R} |u|$.

进而我们就可以得到

$$|w(x)| \leq M \frac{\log\frac{1}{|x|}}{\log\frac{1}{r}} + \frac{\log\frac{1}{|x|}}{\log\frac{1}{r}} \max_{\partial B_r} |u|$$

对固定的 x，我们令 $r \to 0$

注意到 $u(x) = o(\log\frac{1}{|x|})$ as $|x| \to 0 \Rightarrow \dfrac{\max\limits_{\partial B_r}|u|}{\log\frac{1}{r}} \to 0$ as $r \to 0$.

而又显然有 $\dfrac{1}{\log\frac{1}{r}} \to 0$

进而有 $|w(x)| \to 0$ as $r \to 0$. $\Rightarrow w(x) = 0$.

由 x 的任意性 $(x \neq 0) \Rightarrow w(x) \equiv 0$ on $B_R \setminus \{0\}$

也即 $u \equiv v$ on $B_R \setminus \{0\}$，于是可以定义 $u(0) = v(0)$，使得 u 是 $B_R(0)$ 中的调和函数，

也即奇点可去.

28. 若三维调和函数 $u(x)$ 在奇点附近能表示成 $\dfrac{\varphi(x)}{|x-y|^\alpha}$，其中 $0 < \alpha \leq 1$，

$\varphi(x) \neq 0$ 在 $x = y$ 附近，且光滑，则必有 $\alpha = 1$

proof: 假设 $0 < \alpha < 1$，则易知 $|x-y| \left(\dfrac{\varphi(x)}{|x-y|^\alpha} \right) = |x-y|^{1-\alpha} \varphi(x) \to 0$ as $y \to x$

由奇点可去原理知，$\dfrac{\varphi(x)}{|x-y|^\alpha}$ 在 $x = y$ 处奇点可去，i.e 在 y 附近有界

而显然，由于 $\varphi(x)$ 在 y 附近不为0 $\Rightarrow \lim\limits_{x \to y} \dfrac{\varphi(x)}{|x-y|^\alpha} = \infty$.

这显然是个矛盾 $\Rightarrow 0 < \alpha < 1$ 不成立

则只能 $\alpha = 1$

Rmk ①: 事实上，基本解 $\dfrac{1}{|x-y|}$ 即满足上述情况.

② 对于只有一个奇点的调和函数，不妨设 u 在 $\mathbb{R}^n\setminus\{0\}$ $(n\geq 2)$ 上是调和的，且 u 上有界或下有界，则 $u(x)=a|x|^{2-n}+b$，其中 a, b 是固定常数

上述结论的证明可以 cf. Sheldon Axler. Harmonic Function theroy P50 ~ P54

29. $\Omega\subset\subset B_R(0)$，$u(r,\theta,\varphi)$ 在 Ω 中调和，设 $r_1=\frac{R^2}{r}$，则 $x_1=(r_1,\theta,\varphi)$ 是 $x=(r,\theta,\varphi)$ 关于球 $B_R(0)$ 的反演点

则对 u 作 Kelvin 变换，得到 $v(r_1,\theta,\varphi)=\frac{1}{r_1}u(\frac{R^2}{r_1},\theta,\varphi)$

则 $v(r_1,\theta,\varphi)$ 依旧是调和函数

proof: Recall Δu 的极坐标表示

$n=2$ $\quad \Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial^2\theta}$.

$n=3$ $\quad \Delta u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial^2\varphi}$

$$=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Delta_{S^2}u$$

$$\frac{\partial v}{\partial r_1}=-\frac{1}{r_1^2}u+\frac{1}{r_1}\frac{\partial u}{\partial r}\cdot\left(-\frac{R^2}{r_1^2}\right)$$

$$\frac{\partial^2 v}{\partial r_1^2}=\frac{2}{r_1^3}u-\frac{1}{r_1^2}\frac{\partial u}{\partial r}\left(-\frac{R^2}{r_1^2}\right)+\frac{3R^2}{r_1^4}\frac{\partial u}{\partial r}-\frac{R^2}{r_1^3}\times\left(-\frac{R^2}{r_1^2}\right)\frac{\partial^2 u}{\partial r^2}$$

$$=\frac{2}{r_1^3}u+\frac{4R^2}{r_1^4}\frac{\partial u}{\partial r}+\frac{R^4}{r_1^5}\frac{\partial^2 u}{\partial r^2}$$

（注意这里 $\frac{\partial u}{\partial r}$ 是对 $u(\frac{R^2}{r_1},\theta,\varphi)$ 中的 $\frac{R^2}{r_1}$ 整体求偏导）

于是代入，则有

$$\Delta v=\frac{2}{r_1}\frac{\partial v}{\partial r_1}+\frac{\partial^2 v}{\partial r_1^2}+\frac{1}{r_1^2}\Delta_{S^2}v$$

$$=\frac{2}{r_1}\left(-\frac{1}{r_1^2}u-\frac{R^2}{r_1^3}\frac{\partial u}{\partial r}\right)+\frac{1}{r_1^3}u+\frac{4R^2}{r_1^4}\frac{\partial u}{\partial r}+\frac{R^4}{r_1^5}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r_1^2}\cdot\frac{1}{r_1}\Delta_{S^2}u$$

$$=\frac{2R^2}{r_1^4}\frac{\partial u}{\partial r}+\frac{R^4}{r_1^5}\frac{\partial^2 u}{\partial r^2}+\frac{1}{r_1^3}\Delta_{S^2}u=\frac{R^4}{r_1^5}\left(\frac{2}{\frac{R^2}{r_1}}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{\left(\frac{R^2}{r_1}\right)^2}\Delta_{S^2}u\right)$$

注意此时 u 在 $(\frac{R^2}{r_1},\theta,\varphi)$ 取值，再结合 u 调和

$=0$

Then $v(r_1,\theta,\varphi)$ 也是调和的

30. 利用 kelvin 变换将3维空间有界域上的 Dirichlet 外问题化为内问题.

首先回顾 Dirichlet 外问题为:

$$\begin{cases} -\Delta u = f & x \in \mathbb{R}^n \setminus \Omega \\ u = \varphi(x) & x \in \partial\Omega \\ \lim\limits_{|x| \to +\infty} u = 0 \end{cases}$$

这里 Ω 是 $\mathbb{R}^n$ 中的一个有界区域.

proof: 不妨假设 $\Omega \subset B_R(0)$ 对充分大的 $R>0$.

则我们用极坐标表示上述方程, $u = u(r, \theta, \varphi)$

作 kelvin 变换, 则有 $v(r_1, \theta, \varphi) = \frac{1}{r_1} u(\frac{R^2}{r_1}, \theta, \varphi)$

原方程化为

$$-\Delta v\Big|_{(r_1,\theta,\varphi)} = -\frac{R^4}{r_1^5} \Delta u\Big|_{(\frac{R^2}{r_1}, \theta, \varphi)} = -\frac{R^4}{r_1^5} f(\frac{R^2}{r_1}, \theta, \varphi). \quad in\ \Omega' \quad ①$$

Ω' 为 $\overline{\mathbb{R}^n \setminus \Omega}$ 作 kelvin 变换后的有界域

$$v\Big|_{(r_1,\theta,\varphi)} = \frac{1}{r_1} \varphi(\frac{R^2}{r_1}, \theta, \varphi) \quad x \in \partial\Omega' \quad ②$$

而由于 $\lim\limits_{|x| \to +\infty} u = 0 \Rightarrow |r_1 v(r_1, \theta, \varphi)| = |u(\frac{R^2}{r_1}, \theta, \varphi)| \to 0$ as $r_1 \to 0$.

进而由奇点可去定理, v 在 0 处奇点可去

于是 v 在 Ω' 内每一点处都有定义, 且满足上述①、②.

这样我们就将 Dirichlet 外问题转化成了内问题.

31. 证明 $\mathbb{R}^3$ 中无界区域上的调和函数在无穷远处趋于0, 则趋于0的阶数至少为 $O(\frac{1}{r})$, 对于一般 $n \geq 3$, 则 $\to 0$ 的阶数至少为 $O(\frac{1}{r^{n-2}})$

proof: 我们直接考虑 $n \geq 3$ 的一般情形,

对一般高维, kelvin 变换则变为 $v(r_1, \omega) = \frac{1}{r_1^{2-n}} u(\frac{R^2}{r_1}, \omega)$

这里的 ω 为球面 S^{n-1} 上的一点

(特别地, 在3维情形 (θ, φ) 就对应球面上点 $(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$).

同样可以验证 $v(r, \omega)$ 也是调和函数 (注意 kelvin 变换只有径向反演, 切向不变)

则由于 u 在 $\mathbb{R}^n$ 中无穷远处取0, 不妨设 $u \subset \mathbb{R}^n \setminus B_R(0)$.

则对u作 kelvin 变换，我们得到

$$v(r_1,\omega)=\frac{1}{r_1^{n-2}}u\left(\frac{R^2}{r_1},\omega\right)$$

$\Rightarrow \left|r_1^{n-2}v(r_1,\omega)\right| = \left|u\left(\frac{R^2}{r_1},\omega\right)\right| \to 0$ as $r_1\to 0$.

则 0是 $v(r_1,\omega)$ 的可去奇点，则 $v(r,\omega)$ 在 0 附近有界，进而

由于kelvin 变换可逆，则

$$u(r,\omega)=\frac{1}{r^{n-2}}v\left(\frac{R^2}{r_1},\omega\right)$$

由于 $v(r_1,\omega)$ 在 0 附近有界，则 $\lim\limits_{r_1\to+\infty} v\left(\frac{R^2}{r_1},\omega\right)$ 是有界的.

于是在无穷远处 $|u(r,\omega)|\leq \frac{C}{r^{n-2}}$

也即趋于无穷远的阶数至少为 $O\left(\frac{1}{r^{n-2}}\right)$

Rmk：这门课的无界区域没有特别说明，我们总默认是有一个有界区域的补集.

4.1、用 Hopf 引理证明调和函数的最大最小值原理

proof: Recall 最大最小值原理：若调和函数在区域Ω内部取到最大值和最小值，则此函数必为常数.

我们只证明在内部取到最大值的情形，在内部取到最小值我们只用考察 $-u$. 即可.

令 $M=\max\limits_{\Omega} u$, $C=\{x\in\Omega \mid u(x)\equiv M\}$

若 $u\not\equiv M$，则易知 $V=\{x\in\Omega \mid u(x)<M\}$ 是开集

由几何关系易知，可以选取 $y\in V$, s.t $dist(y,C)<dist(y,\partial\Omega)$，令 B是以y为中心且B的内部都在V中的半径最大的球，则此时由y的选取，y必先到达C，再到达∂Ω，则∃一点 $x_0\in\partial B$，使得 $x_0\in C$, i.e $u(x_0)=M$

则 $u(x_0)>u(x)$ $\forall x\in B$，由 Hopf 引理，我们知道 $\frac{\partial u(x_0)}{\partial\vec{n}}>0$.

而由于 x_0 是Ω内部的最大值点，又有

$$Du(x_0)=0 \quad\Rightarrow\quad \frac{\partial u(x_0)}{\partial\vec{n}}=Du(x_0)\cdot\vec{n}=0$$

这和 $\frac{\partial u(x_0)}{\partial\vec{n}}>0$ 矛盾，于是假设不成立

$\Rightarrow u\equiv M$，即u是常数　　Rmk：上述证明可推广到一般二阶线性方程，那时不再有平均值公式

42、用 Hopf 最大最小值定理 及强极大极小值原理，证明：

若 Ω 的边界满足内球条件，则方程解 $\begin{cases}\Delta u=0\\ \left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial\Omega}=f\end{cases}$ $\sigma>0$

解唯一

proof: 假设存在两个解，记为 u,v，则令 $w=u-v$，w 满足

$$\begin{cases}\Delta w=0\\ \left(\frac{\partial w}{\partial n}+\sigma w\right)\Big|_{\partial\Omega}=0\end{cases}\quad \sigma>0$$

下面我们证明 $w\equiv 0$

method 1: Hopf 最大最小值原理

假设 $w\not\equiv 0$

不失一般性，我们可以假设 $\max\limits_{\Omega} w>0$，其余情况同理可以证明（事实上，可以考虑 $-w$）

则首先由调和函数的极大值原理，我们知道 $\exists x_0\in\partial\Omega$，$w(x_0)=\max\limits_{\Omega} w$

① 则若在内部也有一点 x，$w(x)=w(x_0)$，

则由强极大值原理 $w(x)\equiv M$，再结合边界处 $\frac{\partial w}{\partial n}+\sigma w\Big|_{\partial\Omega}=0$，$\sigma>0$

得到 $w\equiv 0$

② 若在内部取不到最大值，则 $\forall x\in\Omega$，我们有 $w(x_0)>w(x)$

且此时 Ω 满足内球条件，则由 Hopf 最大最小值原理，得到 $\frac{\partial w}{\partial n}(x_0)>0$

而由假设 $w(x_0)>0$，则 $\left(\frac{\partial w}{\partial n}+\sigma w\right)(x_0)>0$

这与 $\left(\frac{\partial w}{\partial n}+\sigma w\right)(x_0)=0$ 矛盾 $\Rightarrow$ 假设不成立 $\Rightarrow$ $w\equiv 0$

$\Rightarrow$ 解唯一

method 2: 能量法

原方程为 $\Delta w=0$，方程两边同乘 w，在 Ω 上积分，有

$$\int_\Omega w\Delta w\,dx=0$$

由散度定理 $0=\int_\Omega w\Delta w\,dx=\int_\Omega \operatorname{div}(w\nabla w)\,dx-\int_\Omega|\nabla w|^2dx$

$$=\int_{\partial\Omega} w\frac{\partial w}{\partial n}\,dx-\int_\Omega|\nabla w|^2dx$$

于是则有 $\int_\Omega|\nabla w|^2dx=\int_{\partial\Omega} w\frac{\partial w}{\partial n}d\sigma(x)\overset{\text{边界}}{=}-\int_{\partial\Omega}\sigma w^2d\sigma(x)\leq 0$

$\Rightarrow \int |\nabla w|^2 dx \equiv 0$

$\Rightarrow |\nabla w| \equiv 0 \quad \Rightarrow \quad w \equiv C$，再结合 $\frac{\partial w}{\partial n} + \sigma w \Big|_{\partial\Omega} = 0 \quad \sigma > 0$

则必有 $w \equiv 0 \quad \Rightarrow$ 解是存在唯一的

Rmk：在本课程中证明解的唯一性主要就两种方法

① 极大值原理 $\longrightarrow$ 位势方程，热方程

② 能量法 $\longrightarrow$ 波方程，热方程，位势方程皆可 (yyds！)

由此也可见能量法在存在唯一性的证明中更为普遍，也是大家最值得关注的方法

一些补充：

① u是全空间调和函数，若 $u \in L^2(\mathbb{R}^n)$，则 $u \equiv 0$

Proof：首先由平均值公式 $\forall\ x \in \mathbb{R}^n$，以及 $B_R(x)$，$R>0$，有.

$$u(x) = \frac{1}{|B_R(x)|}\int_{B_R(x)} u(y)\,dy$$

$$= \frac{1}{C(n)\,R^n}\int_{B_R(x)} u(y)\,dy$$

由 Cauchy 不等式

$$\left|\int_{B_R(x)} u(y)\,dy\right| = \left|\int_{B_R(x)} u(y)\cdot 1\,dy\right|$$

$$\le \left(\int_{B_R(x)} |u(y)|^2 dy\right)^{\frac{1}{2}} \left(\int_{B_R(x)} 1\right)^{\frac{1}{2}}$$

$$= \left(\int_{B_R(x)} |u(y)|^2 dy\right)^{\frac{1}{2}} (C(n)R^n)^{\frac{1}{2}}$$

$$\le \|u\|_{L^2(\mathbb{R}^n)} (C(n)R^n)^{\frac{1}{2}}$$

于是

$$|u(x)| \le \frac{1}{C(n)R^n}(C(n)R^n)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} \le \frac{C}{R^{\frac{n}{2}}}\|u\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as } R \to +\infty$$

$\Rightarrow u(x) = 0$

由 x 的任意性 $\Rightarrow u(x) \equiv 0$

Rmk：若 $u \in L^p(\mathbb{R}^n)$，$(p \ge 1)$，则上述结论依旧成立.

(这里 $L^p(\mathbb{R}^n) = \left\{ f : \left(\int |f|^p\right)^{\frac{1}{p}} < +\infty \right\}$)

证明要用到如下 Hölder 不等式 $\|fg\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \quad \frac{1}{p} + \frac{1}{q} = 1$

特别地 $p = q = 2$，就是我们上面讨论的特殊情况.

② $u\in C^2(\bar{U})$，则 $-\Delta u\le 0$ in U $\Leftrightarrow$ $u(x)\le ⨍_{B(x,r)}u\,dy$ $\forall$ $B(x,r)\subset U$

"⇒"

proof: 首先，若 $-\Delta u\le 0$，类似于书上的推导。 ☆ 很重要

令 $\phi(r):=⨍_{\partial B(x,r)}u(y)\,dS_y = ⨍_{\partial B(0,1)}u(x+rz)\,dS_z$ （$y=x+rz$ 替换到 $B(0,1)$ 上）

则 $\phi'(r)=⨍_{\partial B(0,1)}Du(x+rz)\cdot z\,dS_z$ （回代 $z=\frac{y-x}{r}$）

$= ⨍_{\partial B(x,r)}Du(y)\cdot\frac{y-x}{r}\,dS_y$ （$\frac{y-x}{r}$ 为外法向）

散度定理
$= \frac{r}{n}⨍_{B(x,r)}\Delta u(y)\,dy \ge 0$

$\Rightarrow$ $\phi(r)$ 单调 ↗ $\Rightarrow$ $\phi(r)=⨍_{\partial B(x,r)}u(y)\,dy \ge \lim_{r\to0}⨍_{\partial B(x,r)}u(y)\,dS_y$

$= u(x).$

$\Rightarrow u(x)\le ⨍_{\partial B(x,r)}u(y)\,dS_y = \frac{1}{\alpha(n)r^{n-1}}\int_{\partial B(x,r)}u(y)\,dS_y$.

$\Rightarrow n\,u(x)\,\alpha(n)\,r^{n-1}\le \int_{\partial B(x,r)}u(y)\,dS_y$. $\alpha(n)$ 为 $B(0,1)$ 体积，$|\partial B(0,1)|=n\alpha(n)$.

$\Rightarrow \int_0^r n\,u(x)\,\alpha(n)\,s^{n-1}\,ds \le \int_0^r\left(\int_{\partial B(x,s)}u(y)\,dS_y\right)ds$

$\Leftrightarrow n\frac{1}{n}r^n u(x)\,\alpha(n)\le \int_{B(x,r)}u(y)\,dy$

$\Rightarrow u(x)\le \frac{1}{|B(x,r)|}\int_{B(x,r)}u(y)\,dy$

"⇐"

反证：假设 $\exists\, x_0\in U$，$-\Delta u>0$，则 $\exists$ 邻域 $\overline{B(x,R)}\subset U$，s.t $-\Delta u>0$.

则同样，令 $\phi(r)=⨍_{\partial B(x,r)}u(y)\,dS_y$.

$\phi'(r)=\frac{r}{n}⨍_{B(x,r)}\Delta u(y)\,dy<0$ $\quad 0<r\le R$.

严格.

则 $\phi(r)$ 在 $0<r\le R$ 单调 ↓ $\Rightarrow$ $\phi(r)=⨍_{\partial B(x,r)}u(y)\,dy < u(x)$. $\forall$ $0<r\le R$.

$\Rightarrow n\,u(x)\,\alpha(n)\,r^{n-1} > \int_{\partial B(x,r)}u(y)\,dS_y$.

$\Rightarrow \int_0^R n\,u(x)\,\alpha(n)\,r^{n-1}\,dr > \int_0^R\left(\int_{\partial B(x,r)}u(y)\,dS_y\right)dr \Leftrightarrow u(x) > \frac{1}{|B(x,R)|}\int_{B(x,R)}u(y)\,dy$

这和条件假设矛盾 ⇒ 假设不成立，则 $-\Delta u\le 0$.

Rmk：1. 同样可以证明 $u\in C^2(\bar{U})$，则 $-\Delta u\geq 0$ 在 U $\Leftrightarrow$ $u(x)\geq ⨍_{B(x,r)} u\,dy$

2. $\frac{d}{dr}\left(\int_{B(x,r)} u(y)\,dy\right) = \frac{d}{dr}\left(\int_0^r \int_{\partial B(x,s)} u(y)\,dS_y\right) = \int_{\partial B(x,r)} u(y)\,dS_y$ $\forall B(x,r)\subset U$. ☆

3. 由上述不等式 $u(x)\leq ⨍_{B(x,r)} u\,dy$ $\forall x\in B(x,r)\subset U$；可以证明 ☆

$\max\limits_{\bar{U}} u = \max\limits_{\partial U} u$，且在内部取到最大值为常数
某点 x_0 $x_0\in B(x,r)$

事实上，假设 u 在内部取到最大值 M 记 $u(x_0)=M$，则有

$$M = u(x_0)\leq ⨍_{B(x,r)} u\,dy \leq M \Rightarrow \forall y\in B(x,r),\ 有\ u(y)=M.$$

则 $\{x\in U \mid u(x)\equiv M\}$ 是开集，而又由连续性，$\{x\in U \mid u(x)\equiv M\}$ 又是相对 U 的闭集，由 U 的连通性 $\Rightarrow u\equiv M$，
（限制拓扑意义下）

则也有 $\max\limits_{\bar{U}} u = \max\limits_{\partial U} u$

同样地 $u(x)\geq ⨍_{B(x,r)} u\,dy$，$\forall x\in B(x,r)\subset U$，可以证明

$\min\limits_{\bar{U}} u = \min\limits_{\partial U} u$.

4. 若 $\phi: \mathbb{R}\to\mathbb{R}$ 是光滑凸函数，则当 u 是调和函数时，$v=\phi(u)$ 满足，$\Delta v\leq 0$

In fact $v_i = \phi'(u)u_i$ $v_{ii} = \phi''(u)u_i^2 + \phi'(u)u_{ii}$

$$\Rightarrow \Delta v = \phi''(u)|\nabla u|^2 + \phi'(u)\Delta u = \phi''(u)|\nabla u|^2 \geq 0.$$

③（Hadamard 三圆定理）

设 D 是 $\mathbb{R}^2$ 中以原点为中心的环形区域，大圆和小圆的半径分别为 R_2, R_1，$-\Delta u\leq 0$

记 $M(r) = \max\limits_{x^2+y^2=r^2} u(x,y)$ $R_1 < r_1 < r < r_2 < R_2$

其中 $r=\sqrt{x^2+y^2}$

$$则有\ M(r)\leq \frac{M(r_1)\log\left(\frac{r_2}{r}\right) + M(r_2)\log\left(\frac{r}{r_1}\right)}{\log\left(\frac{r_2}{r_1}\right)}$$

proof: 令 $\varphi(r) = a + b\log r$ $r\neq 0$

代入 $\varphi(r_1)=M(r_1)$，$\varphi(r_2)=M(r_2)$，我们可以解得 $a = \frac{M(r_1)\log r_2 - M(r_2)\log r_1}{\log r_2 - \log r_1}$

$$b = \frac{M(r_2)-M(r_1)}{\log r_2 - \log r_1} \Rightarrow \varphi(r) = \frac{M(r_1)\log\left(\frac{r_2}{r}\right) + M(r_2)\log\left(\frac{r}{r_1}\right)}{\log\left(\frac{r_2}{r_1}\right)}$$

令 $v(x,y) = u(x,y) - \varphi(\sqrt{x^2+y^2})$

(note that 2维情形时 $\Delta\varphi = 0$ since $\log\sqrt{x^2+y^2}$ 是 $\Delta u = 0$ 基本解)

上述选取 $\varphi(r_1) = M(r_1)$ $\varphi(r_2) = M(r_2)$ 的原因是为了保证

$v|_{\partial B_{r_1}} = u(x,y)|_{\partial B_{r_1}} - M(r_1) \le 0$

$v|_{\partial B_2} = u(x,y)|_{\partial B_{r_2}} - M(r_2) \le 0$

而 $-\Delta v = -\Delta u + \Delta\varphi \le 0$

则由前面 ② 的 Rmk 3，我们知道 $\max\limits_{\bar{D}} v = \max\limits_{\partial D} v \le 0$

$\Rightarrow v \le 0 \quad \forall (x,y) \in D \Rightarrow u(x,y) \le \varphi(r) \quad \forall r_1 < r < r_2 \quad x^2+y^2 = r^2$

对左边取 sup $\Rightarrow M(r) = \sup\limits_{x^2+y^2=r^2} u(x,y) \le \varphi(r) \quad \forall r_1 < r < r_2$

$$\Rightarrow M(r) \le \frac{M(r_1)\log\frac{r_2}{r} + M(r_2)\log\frac{r}{r_1}}{\log\left(\frac{r_2}{r_1}\right)}$$

#

idea:

Rmk: 观察证明的不等式形式，右侧可写成 $\varphi(r) = a\log r + b$，

而 $\varphi(r)$ 在 2 维为调和函数 $\Delta\varphi = 0$ （a, b 待定）.

左侧对应 u，则 ⟺ 证明 $u(x,y) - (a\log r + b) \le 0 \quad \forall r_1 < r < r_2 \quad x^2+y^2 = r^2$

自然考虑辅助函数 $v(x,y) = u(x,y) - (a\log r + b)$

$\Delta v = \Delta u - \Delta\varphi \ge 0$，

则联想次调和函数的最大值在边界取得，则我们只须选取 a, b，使得

$v|_{\partial B_{r_1}} \le 0$，$v|_{\partial B_{r_2}} \le 0$，就可保证 $u(x,y) - (a\log r + b) \le 0 \quad \forall r_1 < r < r_2$ $x^2+y^2 = r^2$

显然，若取 a, b，使得 $\begin{cases} a\log r_1 + b = M(r_1) \\ a\log r_2 + b = M(r_2) \end{cases}$ 则可满足，这就得到了我们想要的结论.

④. 定义在 $\mathbb{R}^2$ 上的次调和函数 u (i.e 满足 $-\Delta u \le 0$)，且 u 上有界，则 u 是常数，

$n \ge 3$，上述假设不成立

proof: 假定 u 次调和且上有界，我们令

$$u^{\varepsilon}(x) = u - \varepsilon\log|x|$$

则先考察 $|x|>1$，则易知此时 $u^{\varepsilon}(x) = u - \varepsilon\log|x| < u$.

且由于 u 是上有界的，$\lim_{|x|\to+\infty}\log|x| = +\infty \Rightarrow \lim_{|x|\to+\infty} u - \varepsilon\log|x| = -\infty$

则在有限环 $1<|x|<R$，当 R 取的充分大时，显然由于 $\lim_{|x|\to+\infty} u-\varepsilon\log|x| = -\infty$，

我们有 $u - \varepsilon\log|x|\Big|_{\partial B_R} < u - \varepsilon\log|x|\Big|_{\partial B_1} = u\big|_{\partial B_1}$

进而再在环上，由次调和函数的最大值原理，(也即 ② 的 Rmk 3).

$$\Rightarrow \sup_{1<|x|<R} u^{\varepsilon} = \sup_{\partial B_1} u = \max_{\overline{B(0,1)}} u.$$

↑

(再在 $\overline{B(0,1)}$ 内部用一次极大值原理).

令 $R\to+\infty$，我们就有 $\sup_{|x|>1} u^{\varepsilon} = \max_{\overline{B(0,1)}} u$

则 $\forall x$ with $|x|>1$，我们有

$$u(x) = u^{\varepsilon}(x) + \varepsilon\log|x| \leq \max_{\overline{B(0,1)}} u + \varepsilon\log|x|$$

令 $\varepsilon\to 0 \Rightarrow u(x) \leq \max_{\overline{B(0,1)}} u. \quad \forall\ |x|>1$

则 $u(x)$ 有界，且在 $\overline{B(0,1)}$ 中取到最大值，则由 ② 的 Rmk 3 的推导，我们知道

$u\equiv M$ for some $M\in\mathbb{R}$.

$n\geq 3$ 时，此时会发现若类似于前面的作法，则 $u^{\varepsilon}(x) = u - \varepsilon|x|^{2-n}$.

However，$|x|\to+\infty \quad |x|^{2-n}\to 0$，则此时我们不可再像 $n=2$ 时用有限区域的界控制无穷远的界.

此时我们给出反例，任取 ρ 是支集在 $\{|x|<1\}$ 上的光滑函数，$0\leq\rho\leq 1$，且满足 $\int\rho\,dx = 1$，

则 $u(x) = \dfrac{-1}{n(n-2)\alpha(n)}\displaystyle\int_{\mathbb{R}^n}\frac{\rho(y)}{|x-y|^{n-2}}dy \quad (n\geq 3)$

是方程 $\Delta u = \rho$ 的一个解.（可以看 Evans p23）

显然 $\Delta u = \rho \geq 0$.

$$|u(x)| \leq C\int_{\mathbb{R}^n}\frac{\rho(y)}{|x-y|^{n-2}}dy \leq C\left(\int_{|x-y|\leq 1}\frac{\rho(y)}{|x-y|^{n-2}}dy + \int_{|x-y|>1}\frac{\rho(y)}{|x-y|^{n-2}}dy\right)$$

$$\leq C\left(\int_0^1 \frac{n\alpha(n)}{r^{n-2}} r^{n-1}\, dr + \int \rho(y)\, dy \right) \quad \text{note that } |\rho(y)| \leq 1.$$

显然有界，但显然 u 不是常数.

Rmk:

$$u(x) = \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|) f(y)\, dy & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy & (n\geq 3) \end{cases} \qquad f \in C_c^2(\mathbb{R}^n).$$

☆

是方程 $-\Delta u = f$ in $\mathbb{R}^n$ 的解.

我也不知道老师上课有没有具体推导过，不熟悉的同学可以参考 Evans P23 的证明.