

积分因子法

首先看上周课上主要介绍的方程 $\frac{dy}{dx} + p(x)y = q(x)$, 若 $q(x) \equiv 0$, 则为齐次线性方程. 若 $q(x) \neq 0$, 则为非齐次线性方程. 我们现在利用积分因子法来求解此类方程.

$$\text{我们对方程变形} \quad dy + p(x)y dx = q(x)dx$$

$$\Leftrightarrow dy + (p(x)y - q(x))dx = 0$$

注意到, 该微分形式并不是一个恰当形式, 但若我们在等号两边同时乘上 $\mu(x) = e^{\int p(x)dx}$

$$\Rightarrow \mu e^{\int p(x)dx} dy + p(x)y e^{\int p(x)dx} dx = q(x) e^{\int p(x)dx} dx$$

$$\Rightarrow d(e^{\int p(x)dx} y) = d \int_{x_0}^x q(s) e^{\int_{x_0}^s p(t)dt}$$

$$\Rightarrow e^{\int p(x)dx} y = \int q(x) e^{\int p(x)dx} dx + C$$

↑ 代表不定积分

如上: 将非恰当方程, 通过同时乘上因子 μ , 变成恰当方程的过程. 我们称为积分因子法

$\frac{dy}{dx} + p(x)y = q(x)$ 只是线性方程. 现在我们推广到一般的方程

$$P(x,y)dx + Q(x,y)dy = 0$$

判断该方程是否恰当: $d(P(x,y)dx + Q(x,y)dy) = 0 \Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

积分因子法: 在方程两边同时乘上一个因子, 使得这个方程是恰当方程

$$\mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy = 0$$

$$\Rightarrow \frac{\partial(\mu(x,y)P(x,y))}{\partial y} = \frac{\partial(\mu(x,y)Q(x,y))}{\partial x}$$

$$\Rightarrow \frac{\partial \mu}{\partial y} P(x,y) + \mu(x,y) \frac{\partial P}{\partial y} = \frac{\partial \mu}{\partial x} Q + \mu \frac{\partial Q}{\partial x}$$

$$\Rightarrow \mu(x,y) \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{\partial \mu}{\partial x} Q - \frac{\partial \mu}{\partial y} P$$

若 μ 是解 PDE 函数

但是若 $\mu(x,y) = \mu(x)$, 则有 $\mu(x,y) \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = -\frac{\partial \mu}{\partial x} Q$

$$\Rightarrow \frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = -\frac{1}{Q} \frac{d\mu}{dx}$$

只与 x 有关

书本定理 2.4

例 2

微分方程 (2.51) 有一个只依赖于 x 的积分因子的充要条件是：
表达式 $\mu(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$ 只与 x 有关，而与 y 无关

例： $(3x^2y)dx + (2x^2y - x)dy = 0$

step 1: 验证是否是恰当方程，即 $\frac{\partial(3x^2y)}{\partial y} \stackrel{?}{=} \frac{\partial(2x^2y - x)}{\partial x}$
 $\neq$

step 2: 用积分因子 $\frac{\partial(\mu(x,y)(3x^2y))}{\partial y} = \frac{\partial(2x^2y - x)\mu(x,y)}{\partial x}$

$\Rightarrow \frac{\partial \mu}{\partial y}(3x^2y) + \mu = \frac{\partial \mu}{\partial x}(2x^2y - x) + (4xy - 1)\mu$

$\Rightarrow (4xy - 2)\mu = \frac{\partial \mu}{\partial y}(3x^2y) - \frac{\partial \mu}{\partial x}(2x^2y - x)$

$= \frac{\partial \mu}{\partial y}(3x^2y) - \frac{\partial \mu}{\partial x}(2xy - 1)x \quad \mu = \mu(x)$

$\Rightarrow 2\mu = -x \frac{d\mu}{dx}$

$\Rightarrow -2 \frac{dx}{x} = \frac{d\mu}{\mu}$

$\Rightarrow d \ln \mu = -2 d \ln x$

$\ln \mu + 2 \ln x = C$ 一同的，找到一个满足条件的 μ 。

$\mu(x) = \frac{1}{x^2}$

$\Rightarrow (3 + \frac{y}{x})dx + (2y - \frac{1}{x})dy = 0$

$\Rightarrow \frac{d}{dx}(\frac{y}{x} + x^2) = 0$
 $\Phi(x,y) = \int \frac{3}{x} dx + \int (2y - \frac{1}{x}) dy$

$= 3x - \frac{y}{x} + y^2 - \frac{y}{x}$
 $\Phi(x,y) = 3x - \frac{y}{x} + y^2$

$\Phi(x,y) = 3x - \frac{y}{x} + y^2$

找特解 $x=0$

注意：找特解要回原方程去找

分因式法: $(3-x^2y)dx + (2xy - y)dy = 0$

$$\Rightarrow 3x^2dx + 2x^2ydy + (ydx - xdy) = 0$$

$$\Rightarrow 3x^2dx + 2x^2ydy + (ydx - xdy) = 0$$

↑ 有积因子 $x^2, y, \frac{1}{x+y}$

选取 $\frac{1}{x}$ 为积因子。

定理: 若 $\mu(x,y)$ 是方程 $P(x,y)dx + Q(x,y)dy = 0$ 的积因子, 则 $\mu P(x,y)dx + \mu Q(x,y)dy = d\phi(x,y)$

则 $\mu(x,y)g(\phi(x,y))$ 也是积因子。

$$\text{proof: } \frac{\partial(\mu(x,y)g(\phi(x,y))P(x,y))}{\partial y} = \frac{\partial(\mu(x,y)g(\phi(x,y))Q(x,y))}{\partial x}$$

$$\Leftrightarrow \frac{\partial g(\phi(x,y))}{\partial y} \mu(x,y)P(x,y) = \frac{\partial g(\phi(x,y))}{\partial x} \mu(x,y)Q(x,y)$$

$$\Leftrightarrow g' \frac{\partial \phi}{\partial y} \mu P = g' \frac{\partial \phi}{\partial x} \mu Q$$

$$\Leftrightarrow \mu Q \mu P = \mu Q \mu P \quad \text{---ok.}$$

例: $(x^3y - 2y^2)dx + x^4dy = 0$
 $(x^3y dx + x^4dy) - 2y^2dy = 0$

↑ 找积因子

$$\frac{\partial(\mu(x,y)x^3y)}{\partial y} = \frac{\partial(\mu(x,y)x^4)}{\partial x}$$

$$\Rightarrow \frac{\partial \mu}{\partial y} x^3y + \mu \cdot x^3 = \frac{\partial \mu}{\partial x} x^4 + 4x^3\mu$$

$$3x^3\mu = -\frac{\partial \mu}{\partial x} x^4 + x^3y \frac{\partial \mu}{\partial y}$$

$$\mu = -\frac{\partial \mu}{\partial x} \frac{x}{3} + \frac{y}{3} \frac{\partial \mu}{\partial y}$$

① $\mu = \mu(x), \quad \mu \frac{d\mu}{dx} = -\frac{3}{x} \Rightarrow \frac{d \ln \mu}{dx} = -\frac{3}{x} \Rightarrow \mu(x) = \frac{1}{x^3}, \quad \phi(x,y) = xy = C$

② $\mu = \mu(y), \quad -\frac{3}{y} \frac{d\mu}{dy} + \mu \frac{d\mu}{dy} = \frac{3}{y} \quad \mu(y) = \frac{1}{y^3} \quad \phi(x,y)$

$-2y^2dy$ 有积因子 $\frac{1}{y}$
 $\frac{y}{x=C} \cdot \phi(x,y) = x$

$$\frac{1}{x} g_1(x) = \frac{1}{x} g_2(x) \Rightarrow \frac{1}{x^5 y^2}$$

two examples.

例1: $(3x + \frac{6}{y})dx + (\frac{x^2}{y} + \frac{3y}{x})dy = 0$

step1: $\frac{\partial(3x + \frac{6}{y})}{\partial y} \stackrel{?}{=} \frac{\partial(\frac{x^2}{y} + \frac{3y}{x})}{\partial x} \neq$

step2: $\frac{\partial \mu(x,y)(3x + \frac{6}{y})}{\partial y} = \frac{\partial \mu(x,y)(\frac{x^2}{y} + \frac{3y}{x})}{\partial x}$

$$\frac{\partial \mu}{\partial y} (3x + \frac{6}{y}) + \mu(x,y) \frac{6}{y^2} = \frac{\partial \mu}{\partial x} (\frac{x^2}{y} + \frac{3y}{x}) + \mu(x,y) (\frac{2x}{y} - \frac{3y}{x^2})$$

$$\frac{\partial \mu}{\partial y} (3x + \frac{6}{y}) + \frac{\partial \mu}{\partial x} (\frac{x^2}{y} + \frac{3y}{x}) \stackrel{?}{=} \mu(x,y) (\frac{6}{y^2} + \frac{2x}{y} - \frac{3y}{x^2})$$

step3: $\frac{(3x dx + \frac{x^2}{y} dy)}{\frac{1}{x^2}} + \frac{(\frac{6}{y} dx + \frac{3y}{x} dy)}{\frac{1}{y}} = 0$

$$\mu(x) = \frac{1}{x^2} \quad \mu(y) = \frac{1}{y}$$

$$(3x^2y + 6x)dx + (x^2 + 3y^2)dy = 0 \quad \dots \text{ok}$$

例2: $y dx - (x^2 + y^2 + x) dy = 0$

step1: $\frac{\partial y}{\partial y} \neq -\frac{\partial(x^2 + y^2 + x)}{\partial x}$

step2: $\frac{\partial \mu(x,y)y}{\partial y} = -\frac{\partial \mu(x,y)(x^2 + y^2 + x)}{\partial x}$

$$\Rightarrow \frac{\partial \mu}{\partial y} y + \frac{\partial \mu}{\partial x} (x^2 + y^2 + x) = -2\mu(x+1)\mu$$

step3: $(y dx - x dy) \cdot \frac{1}{x^2 + y^2} = 0$

$$\frac{1}{x^2} \cdot \frac{1}{y^2} \cdot \frac{1}{x^2 + y^2} \quad \frac{1}{x^2 + y^2}$$

$$\frac{1}{x^2 + y^2} \quad \dots \text{ok}$$

P37/1 (3) 解: $x=0$ 时 $\lambda \Rightarrow y=0$.

$$x \neq 0 \Rightarrow 1y + \frac{2}{x}y dx = \frac{\sin x}{x} dx$$

积分因子: $e^{\int \frac{2}{x} dx} = x^2$. 乘到方程两边:

$$\frac{d}{dx}(x^2 y) = \frac{x \sin x}{x^2} \quad \text{积分:}$$

$$x^2 y = \sin x - x \cos x + C \quad \text{令 } x=\pi, \Rightarrow C=0$$

$$\Rightarrow y = \frac{1}{x^2}(\sin x - x \cos x)$$

(4) 解: 积分因子: $e^{\int \frac{1}{1-x^2} dx} = \sqrt{\frac{1+x}{1-x}}$. 乘到方程两边, 积分:

$$\Rightarrow y = \sqrt{\frac{1+x}{1-x}} \left(C + \int \sqrt{\frac{1-x}{1+x}} dx \right)$$

$$= \sqrt{\frac{1+x}{1-x}} \left(C + \begin{cases} \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \arcsin x & -1 \leq x \leq 1 \\ \frac{1}{2} (x \sqrt{x^2-1} + \ln(x + \sqrt{x^2-1})) & x > 1 \\ \frac{1}{2} (x \sqrt{x^2-1} + \ln(\sqrt{x^2-1} - x)) & x < -1 \end{cases} \right)$$

$$\text{由 } y(0)=1 \Rightarrow \text{划掉 } -1 \leq x \leq 1 \text{ 时 } C=1.$$

$$\text{划掉 } x < -1 \text{ 时 } C=1$$

P37/2 (2) $\frac{dy}{dx} = \frac{x}{y} + y$ 则化为了关于 y 的线性微分方程

P37/5 (1) 证明: $g \equiv 0$. v. 由通解公式:

$$y = C \cdot e^{-\int_0^x p(s) ds}.$$

y 以 ω 为周期 $\Leftrightarrow y(x+\omega) = y(x)$ for $\forall x$

$$\Leftrightarrow \int_x^{x+\omega} p(s) ds = 0 \text{ for } \forall x$$

由 $p(s)$ 以 ω 为周期 $\Leftrightarrow \bar{p} = \frac{1}{\omega} \int_0^\omega p(s) ds = 0$

$$(2) g \neq 0 \Rightarrow y = e^{-\int_0^x p(s) ds} \left[C + \int_0^x g(s) e^{\int_0^s p(t) dt} ds \right]$$

首先证明 $\exists C, s$ 使 $y(x)$ 为一个周期解. 记 $\bar{p} = \frac{1}{\omega} \int_0^\omega p(s) ds$

$$\text{则令 } y(x) = y(x+\omega).$$

$$\Rightarrow e^{-\int_{x_0}^x p(u) du} \left[c + \int_{x_0}^x q(s) e^{\int_{x_0}^s p(u) du} ds \right] = e^{-\int_{x_0}^{x+w} p(u) du} \left[e^{\int_{x_0}^{x+w} p(u) du} \int_{x_0}^{x+w} q(s) e^{\int_{x_0}^s p(u) du} ds \right]$$

$$\Rightarrow c = (e^{\bar{p}w} - 1)^{-1} \left(\int_{x_0}^{x+w} q(s) e^{\int_{x_0}^s p(u) du} ds - e^{\bar{p}w} \cdot \int_{x_0}^x q(s) e^{\int_{x_0}^s p(u) du} ds \right)$$

只须说明右侧是个常数. 两侧同时对 x 求导. 右侧

$$(RHS)' = (e^{\bar{p}w} - 1)^{-1} \left(q(x+w) e^{\int_{x_0}^{x+w} p(u) du} - e^{\bar{p}w} \cdot q(x) e^{\int_{x_0}^x p(u) du} \right) (*)$$

$$\text{由于 } q(x+w) = q(x), \int_{x_0}^{x+w} p(u) du = \int_{x_0}^x p(u) du + w\bar{p}.$$

若 $\bar{p} \neq 0$, 故 $(*) = 0 \Rightarrow \exists c$. 使得解为 w 为周期的解.

另一方面, 若周期解唯一, 则反证. 设 y_1, y_2 为周期解.

$y_1 - y_2$ 为 $-f(x) + p(x)y = 0$ 的 w 周期解. 若 $y_1 \neq y_2 \Rightarrow \bar{p} \neq 0$

矛盾. 故解唯一 $\Rightarrow \bar{p} \neq 0$

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P37/6. 解: 首先 $y_0(x) = \int_{-\infty}^x f(s) e^{s-x} ds$ 满足

$$|y_0(x)| \leq C \int_{-\infty}^x e^{s-x} ds = C \quad \text{其中 } C \text{ 为 } f \text{ 上界, 故 } y_0 \text{ 有界}$$

则 y_0 为满足条件的解 ($y_0' + y_0 = f(x)$). 若 $\exists y_1, s, t$.

$$y_1 - y_0 \neq 0 \text{ 且 } y_1' + y_1 = f(x) \Rightarrow y_1 - y_0 = h, \quad h' + h = 0 \Rightarrow$$

$$h(x) = \int_{x_0}^x C e^{x_0-s} ds \quad \text{若 } x_0 = -\infty \Rightarrow h = 0 \quad \text{若 } x \neq -\infty, h \text{ 无界}$$

故不存在 $h \Rightarrow y_0$ 为唯一有界解.

$$\text{若 } f(x) \text{ 以 } w \text{ 为周期} \Rightarrow y_0(x+w) = \int_{-\infty}^{x+w} f(s) e^{s-x-w} ds \quad \text{令 } t = s-w$$

$$= \int_{-\infty}^x f(t) e^{t-x} dt = y_0(x)$$

故得证.

P45/1 (2) 解: 令 $u = x + 2y$ $\frac{du}{dx} = 1 + 2 \frac{dy}{dx} = 1 + 2 \frac{u+1}{2u-1} = \frac{4u+1}{2u-1}$

$\Rightarrow \int \frac{2u+1}{2u-1} du = \int dx$ 若 $4u+1=0 \Rightarrow 4x+8y+1=0$

若 $4u+1 \neq 0$ $\int \frac{2u-1}{4u+1} du = \int dx \Rightarrow 8y - 4x - 3 \ln|4x+8y+1| + C = 0$

P45/2 13) 解: $\frac{dy}{dx} = \frac{2x}{y} \cdot \frac{2x^2 - y^2}{x^2 + y^2 + 3}$

$\Rightarrow \frac{dy^2}{dx^2} = \frac{4(y^2+1) - 2(x^2+2)}{(y^2+1) + (x^2+2)}$ 取 $P = x^2+2$ $Q = y^2+1$

$\Rightarrow \frac{dQ}{dP} = \frac{4Q - 2P}{P + Q}$ 取 $Q = Pk$

$\Rightarrow k + P \frac{dk}{dP} = \frac{4k-2}{k+1}$

$\Rightarrow P \frac{dk}{dP} = \frac{(2-k)(k-1)}{k+1}$

若 $k=2 \Rightarrow y^2 - 2x^2 - 3 = 0$ 若 $k=1 \Rightarrow y^2 - x^2 - 1 = 0$

若 $k \neq 1, 2 \Rightarrow \int \frac{dP}{P} = \int \frac{(k+1)}{(2-k)(k-1)} dk$

$\ln|P| + C = 2 \ln|k-1| - 3 \ln|2-k|$

$\Rightarrow C \cdot |2P-Q|^3 = |Q-P|^2$

$\Rightarrow C |2x^2 - y^2 + 3|^3 = |y^2 - x^2 - 1|$ C 为常数.

P45/4 解: 取 $y = e^{\int u dx}$ 则 $y' = yu$ $y'' = yu^2 + yu'$

代入方程 $\Rightarrow u' + u^2 + p(x)u + q(x) = 0$

为关于 u 的里卡蒂方程.

3-1.

1. (2) 我们验证在 $x=0$ 附近, $y \ln|y|$ 满足 Osgood 条件.

由对称性, 只考虑 $y>0$ 的情形

对于 $f(y) = y \ln y$, 求导易知 $f(y)$ 在 $(0, \frac{1}{e}]$ 单调 $\downarrow$

则对 $0 < x < y \leq \frac{1}{e}$, 我们有.

$$|y \ln y - x \ln x| \leq |x - y| |\ln(x - y)|$$

事实上, 由于 $0 < x < y \leq \frac{1}{e} \Rightarrow$

$$0 \leq x \ln x - y \ln y = x \ln x - (y - x) \ln y - x \ln y.$$

$$= x \ln \frac{x}{y} - (y - x) \ln y.$$

$$\leq -(y - x) \ln y = (y - x) |\ln y|$$

$$\leq (y - x) |\ln(y - x)|$$

于是取 $F(r) = r |\ln r|$, 显然 $\int_0^1 \frac{dr}{F(r)} = +\infty$

$\Rightarrow f(y)$ 满足 Osgood 条件, 由此即可知在 $x=0$ 附近, 方程解存在唯一.

对于 $|y| > N$, $y \ln|y|$ 满足局部 Lip 条件.

进而可知上述方程解存在唯一.

Remark: Osgood 条件考试要具体验证, 很多同学作业并未验证.

2. $y_0(x) = 0$

$$y_1(x) = y_0 + \int_0^x t + y_0 + 1 dt = \int_0^x t + 1 dt = \frac{1}{2} x^2 + x$$

$$y_2(x) = y_0 + \int_0^x t + y_1(t) + 1 dt = \int_0^x t + \frac{1}{2} t^2 + t + 1 dt = x + x^2 + \frac{1}{6} x^3.$$

$$y_3(x) = y_0 + \int_0^x t + y_2(t) + 1 dt = \int_0^x t + t + t^2 + \frac{1}{6} t^3 + 1 dt = x + x^2 + \frac{1}{3} x^3 + \frac{1}{24} x^4$$

$$\text{可以归纳证明 } y_n(x) = 2 \left(x + \frac{x^2}{2} + \frac{x^{n+1}}{(n+1)!} \right) - x - \frac{x^{n+1}}{(n+1)!}$$

$$= 2 \left(1 + x + \frac{x^2}{2} + \frac{x^{n+1}}{(n+1)!} - 1 \right) - x - \frac{x^{n+1}}{(n+1)!}$$

$$\lim_{n \rightarrow \infty} y_n(x) = 2(e^x - 1) - x$$

3. proof:

method 1: 能量法

假设初值问题 (E) 在右侧有两个解 y_1, y_2 , 满足 $y_1(x_0) = y_2(x_0)$

$$\text{令 } r(x) = y_1(x) - y_2(x),$$

$$\text{则考察 } E(x) = \frac{1}{2} r^2(x),$$

$$\Rightarrow E'(x) = r(x) r'(x) = (y_1(x) - y_2(x)) (f(x, y_1) - f(x, y_2))$$

由于 $f(x, y)$ 关于 y 是递减的, 则有 $(y_1(x) - y_2(x)) (f(x, y_1) - f(x, y_2)) \leq 0$.

$$\Rightarrow E'(x) \leq 0, \text{ 而 } E(x_0) = 0, \quad E(x) \geq 0.$$

$$\text{于是必有 } E(x) \equiv 0 \quad \forall x \geq x_0 \Rightarrow y_1(x) \equiv y_2(x).$$

method 2: 反证法

假设解不唯一, 则必存在 $x_1 > x_0$, s.t. $y_1(x_1) \neq y_2(x_1)$

不妨设 $y_2(x_1) > y_1(x_1)$.

$$\text{则令 } \bar{x} = \sup \{x | y_1(x) = y_2(x)\}, \text{ 由 } x_1 \text{ 的定义, 我们知道 } \bar{x} < x_1$$

且, $\forall x \in (\bar{x}, x_1)$, 有 $y_2(x) > y_1(x)$.

$$\text{则由 } f \text{ 的单调性 } \Rightarrow F(x, y_2(x)) - F(x, y_1(x)) \leq 0.$$

$$\Rightarrow \frac{d(y_2 - y_1)}{dx} \leq 0, \text{ 而 } (y_2 - y_1)(\bar{x}) = 0$$

$$\Rightarrow y_2 - y_1 \leq 0 \quad \forall x \in (\bar{x}, x_1). \text{ 这是一个矛盾.}$$

于是假设不成立 $\Rightarrow$ 右侧解存在唯一.

左侧解不一定唯一.

$$\text{eg } \begin{cases} f(x, y) = \begin{cases} |y| & y < 0 \\ -\sqrt{y} & y \geq 0 \end{cases} \\ y(0) = 0 \end{cases} \quad \text{左侧解不唯一}$$

左侧 $y=0$
 $y = -\frac{1}{4}x^2$
均是方程解

Rmk: (Gronwall's inequality) 特殊情况.

$$\eta \geq 0, \quad \eta \in C^1[0, T], \quad \eta' \leq \phi \eta \text{ on } [0, T], \quad \eta(0) = 0$$

Then $\eta \equiv 0$ on $[0, T]$.

proof: 由于 $\eta' - \phi \eta \leq 0 \Rightarrow (e^{-\int_0^x \phi dt} \eta)' \leq 0$.

and $\eta(0) = 0$, $e^{-\int_0^x \phi dt} \eta \geq 0$.

$\Rightarrow \eta \equiv 0$ on $[0, T]$.

$\phi \in C([0, T])$

变形: $\xi(t) \leq C \int_0^t \xi(s) ds$, $\xi \geq 0$, 则 $\xi \equiv 0$.

令 $\eta = \int_0^t \xi(s) ds$, 再用上述不等式即可

应用, 在 Picard 迭代中, 证明唯一性时, 我们由 Lip 条件得到如下不等式

$$|u(x) - v(x)| \leq L \int_{x_0}^x |u(x) - v(x)| dx$$

于是由上述 Rmk, 可知 $|u(x) - v(x)| \equiv 0$.

3-3

2 (1) 首先任取初值 (x_0, y_0) , 满足 $x_0 \neq 0$, 不妨设 $x_0 > 0$,

则由于 $\frac{1}{x^2 + y^2} \leq \frac{1}{x^2} \quad \forall \quad x \in (0, +\infty)$.

于是由定理 3.5 可知, 该初值解以 $(0, +\infty)$ 为极大存在区间.

若 $x_0 < 0$, 则同理可知, 该初值解以 $(-\infty, 0)$ 为极大存在区间

若取定初值 $(x_0, y_0) = (1, y_0)$, $y_0 \neq 0$.

则首先由于 $\frac{1}{x^2 + y^2}$ 在 (x_0, y_0) 附近满足 Lip 条件, 则在 $\overline{x=0}$ 附近, 解存在唯一, 再由前述讨论, 我们知道 x 方向可延拓到无穷, 于是此时的极大存在区间为 $\mathbb{R}$.

综上该方程解的极大存在区间为 $(0, +\infty)$, $(-\infty, 0)$ 或 $\mathbb{R}$.

$$(3) |F(x, y)| = |y \sin \alpha y| \leq |y|$$

直接用定理 3.5 的结论可知, 解的极大存在区间为 $\mathbb{R}$.

4. 令 $y^2 - 2y - 3 = 0 \Rightarrow y = 3 \text{ or } y = -1$

若 $y_0 > 3$, 则首先易知解曲线不可能向下穿过 $y = 3$,

否则, 由解的存在唯一性可知, $y \equiv 3$, 这是一个矛盾.

于是首先必有 $y > 3 \quad \forall x \in (a, b)$

进而 $\frac{dy}{dx} = (y^2 - 2y - 3)e^{(x+y)^2} > 0$, $\Rightarrow$ 此时解曲线单调递增.

则考虑区域 $G: x \leq x_0, y \in \mathbb{R}$, 由延拓定理知, $y(x)$ 可以延拓到 G 的边界

而 $y(x)$ 关于 x 单调增, 且不可能穿过 $y = 3$, 于是 $\Rightarrow$ 必有 $a = -\infty$

若 $y_0 = 3$, 由解的存在唯一性定理, $y \equiv 3$

若 $1 < y_0 < 3$, 则同样易知解曲线不能穿过 $y = 1$ or $y = 3$, 否则由解的存在唯一性将得出矛盾

进而在此范围内, $\frac{dy}{dx} < 0$, 则类似于 $y_0 > 3$ 中的讨论, 我们同样可以得到

$a = -\infty, b = +\infty$

若 $y_0 \equiv 1$, 则由解的唯一性 $y \equiv 1$.

若 $y_0 < 1$, 则类似于前面的讨论, $y(x)$ 此时不能穿过 $y = 1$, 且 $\frac{dy}{dx} > 0$ 即曲线

单调递增, 考虑区域 $G: x \geq x_0, y \in \mathbb{R}$, 再利用延拓定理, 我们同样可以得到

$b = +\infty$

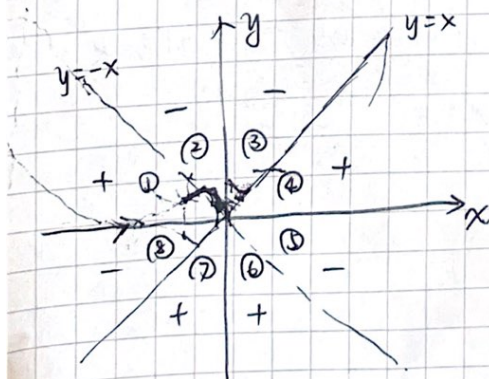
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5. proof: 首先由于 $y f(x, y) > 0$

若 $y > 0$, 则 $f(x, y) > 0$

由 $f(x, y)$ 的连续性 $f(x, 0) = 0$.

若 $y < 0$, 则 $f(x, y) < 0$



对于给定的 $x_0 < 0$, 我们证明当 $|y_0| < |x_0|$ 时, (E) 的解可延拓到 $(-\infty, +\infty)$

不妨设 $y_0 > 0$, $y_0 < 0$ 由对称性同理可以考虑

由解的延拓定理, 我们知道方程的解均可延拓到无穷远, 则由于在①中, $(x^2 - y^2)f(x, y) > 0$

则由此易知解曲线必向右穿过 $y = -x$, 则将进入

②, ③区域.

由于在 ②、③ 区域单调减，在 ① 单调增，所以曲线不可能再次进入 ①，则

曲线从 $y=x$ ($x>0$) 上的一点穿出，若 $(x, y)=(10, 0)$ ，则由于 $y=0$ 是方程的一个解

则此时可延拓到 $+\infty$ ，若 $x>0$ ，则由于在 $y=x$ 上，曲线斜率为 0，则必将从右侧

穿出 $y=x$ 进入 ④，再结合在 ③ 中 $\downarrow$ ，在 ④ 中 $\nearrow$ ，可知曲线将一直在区域 ④

中，又由于在右侧可延拓到无穷远 $\Rightarrow$ 右侧极大存在时刻为 $+\infty$

在考虑 x 左侧的情形，曲线在 ① 中单调增，在 ② 中单调减，可知曲线不会穿过

$y=0$ 进入 ③，于是此时曲线将一直在 ① 中，则左侧极大时刻为 $-\infty$

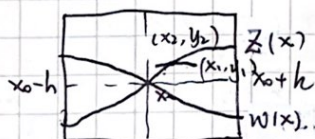
综上所述解可延拓到 $-\infty < x < +\infty$

3-4

1. proof: 不妨设 $x_0 \leq x_1 \leq x_0 + h$, 左侧同理可以证明

若 $y_1 \in [w(x_1), z(x_1)]$, 我们证明存在通过 (x_0, y_0) 的积分曲线, 同样也经过 (x_1, y_1) .

考察下述初值问题 (E):
$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$



由 Peano 存在定理, 上述初值问题在某 $|x - x_0| \leq \sigma$, $|y - y_0| \leq \delta$ 上有解

又由解的延拓定理, 可知该解向左延拓到 $\mathbb{R}$ 的边界

① 若恰好经过 (x_0, y_0) , 则由最大解, 最小解的定义可知, $y(x)$ 始终介于 $w(x)$ 和 $z(x)$ 之间, 故易知解的极大存在区间为 $[x_0 - h, x_0 + h]$.

② 假设 $(x, y(x))$ 不经过 (x_0, y_0) , 则由于向左可延伸到 $\mathbb{R}$ 的边界, 则在 x_0 右侧必和 $(x, w(x))$ or $(x, z(x))$ 相交, 不妨设和 $z(x)$ 相交于一点 (x_2, y_2)

由于在交点相切 (即切线斜率为 $f(x_2, y_2)$)

进而易知 $y'(x) = \begin{cases} z'(x) & x_0 \leq x \leq x_2 \\ y'(x) & x_0 \leq x \leq x_0 \end{cases}$ 也是方程的一个解, 且过 (x_0, y_0) , (x_1, y_1) .

由前面的讨论可知 $y'(x)$ 可延拓到 $[x_0 - h, x_0 + h]$.

于是 $y(x, y_1)$, $|x_1 - x_0| = h$, $w(x_1) \leq y(x_1) \leq z(x_1)$

在 $|x - x_0| \leq h$ 至少存在一个解, 满足 $y(x_0) = y_0$.

##

4-1

$$(2) \quad y = p \ln x + (xp)^2 \Rightarrow p = \frac{dy}{dx} = \frac{dp}{dx} x \ln x + p(\ln x + 1) + 2(xp) \cdot (p + x \frac{dp}{dx})$$

$$\text{则有 } p'x \ln x + p \ln x + 2xp^2 + 2x^2 p p' = 0$$

$$\Leftrightarrow (\ln x + 2xp)(p'x + p) = 0$$

$$\text{① 若 } \ln x + 2xp = 0 \Rightarrow p = -\frac{\ln x}{2x} \quad \text{则} \quad y = -\frac{\ln x}{2x} x \ln x + \left(\frac{\ln x}{2}\right)^2 = -\frac{1}{4}(\ln x)^2$$

$$\text{若 } p'x + p = 0 \Rightarrow p = \frac{C}{x}$$

$$\Rightarrow y = p x \ln x + (xp)^2 = \frac{C}{x} \cdot x \ln x + \left(\frac{C}{x} \cdot x\right)^2 = C \ln x + C^2$$

$$\text{综上 } y = -\frac{1}{4}(\ln x)^2 \text{ or } y = C \ln x + C^2 \quad (C \in \mathbb{R})$$

2

$$(2) \text{ 令 } x = \cosh t = \frac{e^t + e^{-t}}{2}$$

$$\text{则 } \sqrt{x^2 - 1} = \sinh t = \frac{e^t - e^{-t}}{2}$$

$$\Rightarrow dy = p dx = \frac{\sqrt{3}}{b}(e^t - e^{-t}) dx = \frac{\sqrt{3}}{b}(e^t - e^{-t}) \cdot \frac{e^t - e^{-t}}{2} dt = \frac{\sqrt{3}}{12}(e^{2t} + e^{-2t} - 2) dt$$

$$\Rightarrow y = \frac{\sqrt{3}}{12} \sinh t - \frac{\sqrt{3}}{6} t + C.$$

$$(3) \text{ 令 } x = u, \quad p = v.$$

$$\text{则有 } y = u^2 - v^2$$

$$\Rightarrow v = p = \frac{dy}{dx} = \frac{2u du - 2v dv}{du} = 2u - 2v \frac{dv}{du}$$

$$\Rightarrow 2 \frac{dv}{du} + 1 = 2 \frac{u}{v}.$$

$$\text{令 } v = uk, \text{ 则 } \frac{dv}{du} = k + u \frac{dk}{du}.$$

$$\Rightarrow 2(k + u \frac{dk}{du}) + 1 = \frac{2}{k} \Rightarrow \frac{dk}{du} = \frac{2k + 1 - \frac{2}{k}}{-2u}, \quad \text{若 } 2k + 1 - \frac{2}{k} \neq 0.$$

$$\Rightarrow -\frac{1}{2u} du = \frac{k dk}{2k^2 + k - 2} = \frac{\frac{1}{2} d(k + \frac{1}{4})^2 - \frac{1}{4} dk}{2(k + \frac{1}{4})^2 - \frac{17}{8}}$$

$$\Rightarrow -\frac{1}{2} \ln|u| = \frac{1}{4} \ln \left| (k + \frac{1}{4})^2 - \frac{17}{16} \right| - \frac{1}{4\sqrt{17}} \ln \left| \frac{k + \frac{1}{4} - \frac{\sqrt{17}}{4}}{k + \frac{1}{4} + \frac{\sqrt{17}}{4}} \right| + C.$$

$$\Rightarrow \ln|u| = -\frac{1}{2} \ln \left| (k + \frac{1}{4})^2 - \frac{17}{16} \right| + \frac{1}{2\sqrt{17}} \ln \left| \frac{k + \frac{1}{4} - \frac{\sqrt{17}}{4}}{k + \frac{1}{4} + \frac{\sqrt{17}}{4}} \right| + C.$$

$$\Rightarrow u = C \left| (k + \frac{1}{4})^2 - \frac{17}{16} \right|^{-\frac{1}{2}} \cdot \left(\frac{k + \frac{1}{4} - \frac{\sqrt{17}}{4}}{k + \frac{1}{4} + \frac{\sqrt{17}}{4}} \right)^{\frac{1}{2\sqrt{17}}}$$

$$\text{则 } y = u^2 - v^2 = u^2 - k^2 u^2 = (1 - k^2) u^2.$$

$$\text{若 } 2k + 1 - \frac{2}{k} = 0 \Rightarrow k = \frac{\sqrt{17}-1}{4}, \frac{\sqrt{17}+1}{4} \Rightarrow y = u^2(1 - k^2) = \frac{\sqrt{17}-1}{8} u^2 \text{ or } \frac{-\sqrt{17}-1}{8} u^2$$

5-3

1. 对于初值问题

$$(E) \quad \frac{dy}{dx} = f(x, y), \quad y(x_0) = \eta$$

我们作平移 $\tilde{x} = x_0 + x$, $\tilde{y} = y + \eta$

则初值问题转化为

$$\begin{aligned} \frac{dy}{dx} &= f(x_0 + x, y + \eta) & y(0) &= 0 \\ &= \tilde{f}(x, y, \eta) \end{aligned}$$

则由于 $f(x, y)$ 在区域 $|x - x_0| \leq a$, $|y - y_0| \leq b$ 上满足 Lip 条件, 即 $\exists L > 0$

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad \forall y_1, y_2 \in \{y \mid |y - y_0| \leq b\}$$

进而若 $|x| \leq \frac{h}{2}$, $|y_1| \leq \frac{b}{2}$ 时 (in fact $|y_1| \leq \frac{b}{2} \Rightarrow |y + \eta - y_0| \leq |\eta - y_0| + |y_1| \leq b$)

$$|\tilde{f}(x, y_1, \eta) - \tilde{f}(x, y_2, \eta)| \leq L|y_1 - y_2|$$

$\Rightarrow \tilde{f}$ 也满足 Lip 条件

则由 Thm 5.1, 我们便可知 $\varphi(x, \eta)$ 是 $|x - x_0| \leq \frac{h}{2}$, $|\eta - y_0| \leq \frac{b}{2}$ 的

连续函数.

Remark: 也可类似于 thm 5.1 的证明, 取 $\lambda = \eta$ 即可.

2. 解对初值依赖性的准确定义 (抱歉上课讲的有点问题)

(Definition).

设 n 维向量值函数 $f(x, y)$ 在 (x, y) 中的某个开区域 G 上连续, $y(x; x_0, y_0)$ 是

初值问题 $\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$ 的解, 解的极大存在区间为 J , 则若 J 的闭子区间 $[a, b]$ 满足 $\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } |x'_0 - x_0| < \delta, |y'_0 - y_0| < \delta$ 时, 以

(x'_0, y'_0) 为初值的解 $y(x; x'_0, y'_0)$ 在区间 $[a, b]$ 上存在, 且

$|y(x; x_0, y_0) - y(x; x'_0, y'_0)| < \varepsilon, \forall x \in [a, b]$, 则称解 $x = y(x; x'_0, y'_0)$ 在点 (x_0, y_0) 处连续依赖于初值 x'_0, y'_0

proof: $\forall (x_0, y_0) \in R$ (这里 R 我们假设是开区域).

由题设条件 $\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$ 的解是存在唯一的, 设最大存在区间为 J ,

我们任取 J 的闭子集 $[a, b], x_0 \in [a, b]$.

则 $\Gamma = \{(x, y) \mid y = y(x), a \leq x \leq b\}$ 是 R 中有界闭集

由有限覆盖定理, 存在 $\sigma > 0$, s.t.

$$\Sigma_\sigma = \{a \leq x \leq b, |y - y(x)| \leq \sigma\} \text{ 落在 } R \text{ 中.}$$

作开集 Ω , s.t. $\Sigma_\sigma \subset \Omega$, 且 $\Omega \subset R$ (即 Ω 是 R 的开子集, 且 $\bar{\Omega}$ 是紧集)

我们首先证明 $\Lambda = \{(t, x, y) \mid t \in I_{x, y}, \text{ 其中 } I_{x, y} \text{ 为初值解 } \begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x) = y \end{cases} \text{ 在 } \Sigma_\sigma \text{ 中的极大存在区间}\}$

是有界闭集, 事实上, 任取序列 $(t_n, x_n, y_n) \xrightarrow{\in \Lambda} (t_0, x_0, y_0)$

则由题设可知 $(t_{n_k}, y(t_{n_k}; x_n, y_n)) \in \Sigma_\sigma$

记 $y_n(t) = y(t; x_n, y_n)$ (由解的存在唯一性, 初值给定, $y(t)$ 唯一)

则由于 $t_n \rightarrow t_0$, 进而当 n 充分大时, 我们总可将 $y_n(t)$ 延拓到 t_0 附近固定的邻域 $[t_0 - h, t_0 + h]$ 对于充分小的 h . (这是由于 $y_n(t_n) \in \Sigma_\sigma \subset \Omega$, $\text{dist}(\Sigma_\sigma, \partial\Omega) > 0$.)

且斜率 $f(x, y)$ 在紧集 $\bar{\Omega}$ 上是一致有界的, 且由解的延拓定理, 解可延拓到 Ω 边界.

于是把 $y_n(t)$ 写成积分方程, 则有在 $[t_0-h, t_0+h]$ 上

$$y_n(t) = y_n + \int_{x_n}^t f(x, y_n(x)) dx$$

则首先显然 y_n 是一致有界的, 其次

$$|y_n(t) - y_n(s)| = \left| \int_s^t f(x, y_n(x)) dx \right| \leq |s-t| \sup_{\Omega} |f|$$

$\Rightarrow \{y_n(t)\}$ 等度连续. 于是由 Ascoli 引理, $\{y_n(t)\}$ 在 $[t_0-h, t_0+h]$ 上

存在一致收敛子列 $\{y_{n_k}(t)\}$

$$\text{记 } y_{n_k}(t) \xrightarrow{\text{uniformly}} \psi(t)$$

则在积分方程

$$y_{n_k}(t) = y_{n_k} + \int_{x_{n_k}}^t f(x, y_{n_k}(x)) dx$$

两侧, 令 $n_k \rightarrow +\infty$, 再结合 f 在紧子集上的一致连续性, 我们有

$$\psi(t) = y_0 + \int_{x_0}^t f(x, \psi(x)) dx$$

则 $\psi(x_0) = y_0$. 再由解的存在唯一性 $\Rightarrow \psi(t) = y(t; x_0, y_0)$

$$\begin{aligned} \text{进而由于 } |y_{n_k}(t_{n_k}) - \psi(t_0)| &= |y_{n_k}(t_{n_k}) - \psi(t_{n_k}) + \psi(t_{n_k}) - \psi(t_0)| \\ &\leq |y_{n_k}(t_{n_k}) - \psi(t_{n_k})| + |\psi(t_{n_k}) - \psi(t_0)| \\ &\rightarrow 0 \quad \text{as } n_k \rightarrow +\infty \end{aligned}$$

再结合 $(t_{n_k}, y_{n_k}(t_{n_k})) \in \Sigma_\sigma \Rightarrow (t_0, \psi(t_0)) \in \Sigma_\sigma$

这就证明了 Λ 是有界闭集

下面我们证明 $y(t; x, y)$ 是 Λ 上的连续函数

事实上, 假设不连续, 则 $\exists (t_n, x_n, y_n) \rightarrow (t_0, x_0, y_0)$ 以及 $\varepsilon_0 > 0$

$$|y(t_n, x_n, y_n) - y(t_0, x_0, y_0)| \geq \varepsilon_0$$

类似于上面的证明, 我们可以得到 $y(t; x_n, y_n)$ 的一个子列 $\{y(t; x_{n_k}, y_{n_k})\}$

s.t. $y(t; x_{n_k}, y_{n_k}) \xrightarrow{\text{uniformly}} y(t; x_0, y_0)$ 在 t 的某个小区间上

这将会与假设 $|y(t_n, x_{n_k}, y_{n_k}) - y(t_0, x_0, y_0)| \geq \varepsilon_0 \quad \forall n_k$ 矛盾

$\Rightarrow y(t; x, y)$ 是 Λ 上的一个连续函数

由于 Λ 是紧集, 进而 $\varphi(t; x, y)$ 在 Λ 上一致连续

则 $\forall \varepsilon > 0, \exists \delta > 0, s, t$, if $|x'_0 - x_0| < \delta, |y'_0 - y_0| < \delta, x'_0, y'_0 \in \Lambda$, 我们有

$$|\varphi(t; x'_0, y'_0) - \varphi(t; x_0, y_0)| < \varepsilon.$$

特别地, 我们取 $\varepsilon = \frac{\sigma}{2}$.

则 $\exists \delta > 0, |x'_0 - x_0| < \delta, |y'_0 - y_0| < \delta$ 时,

$$|\varphi(t; x'_0, y'_0) - \varphi(t; x_0, y_0)| < \frac{\sigma}{2}.$$

而 $\varphi(t; x'_0, y'_0)$ 可延拓至 Σ 边界, 上述条件限制了只可延拓至 $x=a, x=b$

$\Rightarrow \varphi(t; x'_0, y'_0)$ 在 $[a, b]$ 上也存在解.

进而由上面的证明, 解对初值是连续依赖的.

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(2020 Jan. 5)

考虑二阶 ODE

$$\begin{cases} x''(t) + x(t) + x^3(t) = 0 \\ (x(0), x'(0)) = (x_0, 0) \end{cases}$$

其中 $x(t)$ 为实值, 则 $x_0 \in \mathbb{R}$, 上述方程存在唯一周期解.

Proof: 二阶方程化成一阶方程组

$$\text{令 } y = x', \text{ 则 } y' = x'' = -x - x^3.$$

$$\text{则有 } \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x - x^3 \end{pmatrix} = \underset{\text{局部}}{\vec{f}(x, y)}. \quad \begin{pmatrix} x \\ y \end{pmatrix}_{(0)} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$$

由于 $\vec{f}(x, y)$ 在 $\mathbb{R}^2$ 上光滑, 故满足局部 Lip 条件.

$\Rightarrow$ 上述初值问题存在唯一解.

下面我们证明解的极大存在区间为 $(-\infty, +\infty)$

事实上, 对原方程, 两边同乘 $x'(t)$.

$$\text{则有 } x''(t)x'(t) + x(t)x'(t) + x^3(t)x'(t) = 0$$

$$\Rightarrow \left(\frac{1}{2}(x'(t))^2\right)' + \left(\frac{1}{2}x^2(t)\right)' + \frac{1}{4}(x^4(t))' = 0$$

$$\begin{aligned} \Rightarrow \frac{1}{2}(x'(t))^2 + \frac{1}{2}x^2(t) + \frac{1}{4}x^4(t) &= \frac{1}{2}(x'(0))^2 + \frac{1}{2}x^2(0) + \frac{1}{4}x^4(0) \\ &= \frac{1}{2}x_0^2 + \frac{1}{4}x_0^4 \end{aligned}$$

$$\Leftrightarrow \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = \frac{1}{2}x_0^2 + \frac{1}{4}x_0^4 \Rightarrow y^2 + x^2 + \frac{1}{2}x^4 = x_0^2 + \frac{1}{2}x_0^4$$

$\Rightarrow (x, y)$ 有界, 而由解的延拓定理易知解不能延拓到无穷, 而由 (x, y) 有界, 必有存在区间为 $(-\infty, +\infty)$.

下面证明 $x(t)$ 是周期的.

若 $x_0 = 0$, 则显然 $x \equiv y \equiv 0$ 是方程的唯一解.

$$\begin{aligned} \text{若 } x_0 \neq 0, \text{ 则易知 } \left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 &= x^2 + x^6 + 2x^4 + y^2 \geq x^2 + y^2 + \frac{1}{2}x^4 \\ &\geq x_0^2 + \frac{1}{2}x_0^4 \end{aligned}$$

$$\text{则 } L(\gamma) = \int_0^S \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt \geq S(x_0^2 + \frac{1}{2}x_0^4),$$

$$\gamma(t) = (x(t), y(t)) \text{ satisfy } y^2 + x^2 + \frac{1}{2}x^4 = x_0^2 + \frac{1}{2}x_0^4$$

则首先易知 $y^2 + x^2 + \frac{1}{2}x^4 = x_0^2 + \frac{1}{2}x_0^4$ 是有限长简单闭曲线,

则必存在 $t_1 \neq t_2$, $\gamma(t_1) = \gamma(t_2)$, 否则有 $L(\gamma) \rightarrow +\infty$ as $S \rightarrow +\infty$ 矛盾

不妨设 $t_1 < t_2$, 令 $T = t_2 - t_1$

则我们证明 $\gamma(t) = \gamma(t+T)$. 事实上, $\gamma(t_1) = \gamma(t_1+T)$

由解的存在唯一性, 由 $\gamma(t) = \gamma(t+T) \quad \forall t \in \mathbb{R}$

$$\Rightarrow x(t) = x(t+T) \quad \forall t \in \mathbb{R}$$

$\Rightarrow$ 方程存在唯一-周期解

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P156.3

proof: 方程两边同时对 η 求导, 我们有

$$\frac{d}{dx} \left(\frac{\partial y}{\partial \eta} \right) = x \cos(xy) \frac{\partial y}{\partial \eta} \quad \frac{\partial y}{\partial \eta} \Big|_{(0)} = \frac{\partial \eta}{\partial \eta} = 1.$$

$$\Rightarrow \frac{\partial y}{\partial \eta} = e^{\int_0^x \cos(ty) dt} > 0 \quad \forall x, \eta$$

P168. 2 (2).

解: 首先考察第一个方程 $\begin{cases} \frac{dx}{dt} = \frac{2t}{1+t^2} x \\ x(0) = 0 \end{cases}$

由方程解的存在唯一性, 易知 $x(t) \equiv 0$

代入方程 (2), 则有

$$\frac{dy}{dt} = -\frac{1}{t}y + t \Rightarrow \frac{dy}{dt} + \frac{1}{t}y = t \quad t > 0.$$

$$\Rightarrow \frac{d}{dt} \left(e^{\int_1^t \frac{1}{s} ds} y \right) = e^{\int_1^t \frac{1}{s} ds} t$$

$$\begin{aligned} \Rightarrow ty &= y(0) + \int_1^t s^2 ds \\ &= \frac{4}{3} + \frac{1}{3}(t^3 - 1) = \frac{1}{3}t^3 + 1 \end{aligned}$$

$$\Rightarrow y(t) = \frac{1}{3}t^2 + \frac{1}{t} \quad (t > 0)$$

$$\Rightarrow \text{方程的解为 } \begin{cases} x(t) = 0 \\ y(t) = \frac{1}{3}t^2 + \frac{1}{t} \end{cases} \quad (t > 0).$$

4.

proof: 由于方程组 $\frac{dy}{dx} = A(x)y$, $\frac{dy}{dx} = B(x)y$ 有相同的基解矩阵, 不妨

设为 $\Phi(x)$, 则 $\Phi(x)$ 可逆

$$\Rightarrow \frac{d\Phi(x)}{dx} = A(x)\Phi(x) \Rightarrow A(x) = \frac{d\Phi(x)}{dx} \Phi^{-1}(x).$$

同理, 我们有 $B(x) = \frac{d\Phi(x)}{dx} \Phi^{-1}(x).$

进而, 我们有 $A(x) \equiv B(x) \quad \forall x.$

6. proof: ① 首先证明方程(6.1)至多有 $n+1$ 个线性无关解.

任取方程(6.1)特解 $y^*(x)$, 线性方程基解矩阵为 $\Phi(x) = (\vec{y}_1(x), \vec{y}_2(x), \dots, \vec{y}_n(x))$

则通解 $y(x) = y^*(x) + \Phi(x)\vec{c}$.

$\Rightarrow y(x) \in \text{span}\{y^*(x), \vec{y}_1(x), \dots, \vec{y}_n(x)\}$

而 $\dim \text{span}\{y^*(x), y_1(x), \dots, y_n(x)\} \leq n+1$

进而方程(6.1)至多有 $n+1$ 个线性无关解

② 下证方程(6.1)恰好存在 $n+1$ 个线性无关解

事实上, 考察 $\{y^*(x), (y^*(x) + y_1(x)), \dots, (y^*(x) + y_n(x))\}$.

则若存在 $\lambda_1, \lambda_2, \dots, \lambda_{n+1} \in \mathbb{R}$, s.t.

$$\lambda_1 y^*(x) + \lambda_2 (y^*(x) + y_1(x)) + \dots + \lambda_{n+1} (y^*(x) + y_n(x)) = 0.$$

$$\Leftrightarrow (\lambda_1 + \lambda_2 + \dots + \lambda_{n+1}) y^*(x) + \lambda_2 y_1(x) + \dots + \lambda_{n+1} y_n(x) = 0 \quad \forall x$$

$$\text{记 } y(x) = (\lambda_1 + \lambda_2 + \dots + \lambda_{n+1}) y^*(x) + \lambda_2 y_1(x) + \dots + \lambda_{n+1} y_n(x).$$

$$\text{则 } y'_0 = \frac{dy}{dx} - A(x)y = (\lambda_1 + \lambda_2 + \dots + \lambda_{n+1}) f(x).$$

$$\text{由于 } f(x) \text{ 不恒为 } 0 \Rightarrow \lambda_1 + \lambda_2 + \dots + \lambda_{n+1} = 0.$$

$$\text{进而 } \lambda_2 \vec{y}_1(x) + \dots + \lambda_{n+1} \vec{y}_n(x) = 0 \quad \forall x.$$

$$\text{由于 } \vec{y}_1(x), \dots, \vec{y}_n(x) \text{ 线性无关} \Rightarrow \lambda_i = 0 \quad \forall 2 \leq i \leq n+1$$

$$\Rightarrow \lambda_1 = 0$$

$$\Rightarrow \lambda_i = 0 \quad \forall 1 \leq i \leq n+1$$

于是 $y^*(x), y^*(x) + y_1(x), \dots, y^*(x) + y_n(x)$ 线性无关, 且为原方程的解

进而方程(6.1)恰有 $n+1$ 个线性无关的解.

■

Rmk: 注意有且至多 $n+1$ 个线性无关解 $\Leftrightarrow$ ① 存在 $n+1$ 个线性无关的解

② 至多维数为 $n+1$

p189 1.4)

解: A 的特征多项式

$$f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda+5 & 10 & 20 \\ -5 & \lambda-5 & -10 \\ -2 & -4 & \lambda-9 \end{vmatrix} = (\lambda-5)(\lambda^2-4\lambda+5) \\ = (\lambda-5)(\lambda-2+i)(\lambda-2-i)$$

令 $f(\lambda)=0 \Rightarrow \lambda=5, 2-i, 2+i$

若 $\lambda=5$, 则 $\begin{pmatrix} 10 & 10 & 20 \\ -5 & 0 & -10 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$ 特解为 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

若 $\lambda=2+i$, 则 $\begin{pmatrix} 7+i & 10 & 20 \\ -5 & -3+i & -10 \\ -2 & -4 & -7-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$ 特解为 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5+5i \\ 5 \\ -4-2i \end{pmatrix}$

若 $\lambda=2-i$, 则 $\begin{pmatrix} 7-i & 10 & 20 \\ -5 & -3-i & -10 \\ -2 & -4 & -7-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow$ 特解为 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5-5i \\ 5 \\ -4+2i \end{pmatrix}$

则方程的一个复值解为 $e^{(2+i)x} \begin{pmatrix} 5+5i \\ 5 \\ -4-2i \end{pmatrix} = e^{2x} \begin{pmatrix} (5+5i)e^{ix} \\ 5e^{ix} \\ (-4-2i)e^{ix} \end{pmatrix} \\ = e^{2x} \begin{pmatrix} 5\cos x - 5\sin x \\ 5\cos x \\ -4\cos x + 2\sin x \end{pmatrix} + i \begin{pmatrix} 5\cos x + 5\sin x \\ 5\sin x \\ -2\cos x - 4\sin x \end{pmatrix}$

进而得到方程的两个实值解 $e^{2x} \begin{pmatrix} 5\cos x - 5\sin x \\ 5\cos x \\ -4\cos x + 2\sin x \end{pmatrix}$ $e^{2x} \begin{pmatrix} 5\cos x + 5\sin x \\ 5\sin x \\ -2\cos x - 4\sin x \end{pmatrix}$

$\Rightarrow$ 方程的通解为 $C_1 e^{5x} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 5\cos x - 5\sin x \\ 5\cos x \\ -4\cos x + 2\sin x \end{pmatrix} + C_3 e^{2x} \begin{pmatrix} 5\cos x + 5\sin x \\ 5\sin x \\ -2\cos x - 4\sin x \end{pmatrix}$

共轭

Rmk: 对复值解, 只用求其中一个特征值对应的特解, 再将实部, 虚部

拆开即可得实通解

1. (b)

解: A 的特征多项式:

$$f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 & -1 & -1 \\ -1 & \lambda-1 & 1 & 1 \\ -1 & 1 & \lambda-1 & 1 \\ -1 & 1 & 1 & \lambda-1 \end{vmatrix} = (\lambda+2)(\lambda-2)^3$$

令 $f(\lambda) = 0 \Rightarrow \lambda = -2$ or $\lambda = 2$

若 $\lambda = -2$, 则 $\begin{pmatrix} -3 & -1 & -1 & -1 \\ -1 & -3 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ -1 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Rightarrow$ 特解 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

若 $\lambda = 2$, 则 $\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Rightarrow$ 特解为 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ 方程通解为 $c_1 e^{-2x} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{2x} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{2x} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

3. (3) $f(\lambda) = \begin{vmatrix} \lambda-4 & 3 \\ -2 & \lambda+1 \end{vmatrix} = (\lambda-4)(\lambda+1)+6$
 $= \lambda^2 - 3\lambda - 4 + 6 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$

令 $f(\lambda) = 0 \Rightarrow \lambda = 1$ or $\lambda = 2$

若 $\lambda = 1$, 则考察 $\begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow$ 特解 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

若 $\lambda = 2$, 则 $\begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow$ 特解 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\Rightarrow$ 基解矩阵 $\Phi(x) = \begin{pmatrix} e^x & 3e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \Rightarrow \Phi^{-1}(x) = \begin{pmatrix} -2e^{-x} & 3e^{-x} \\ e^{-2x} & -e^{2x} \end{pmatrix}$

进而代入公式可知.

$$\begin{aligned}\vec{y} &= \Phi(x) \int_0^x \Phi^{-1}(s) f(s) ds \\&= \begin{pmatrix} e^x & 3e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \int_0^x \begin{pmatrix} -2e^{-s} & 3e^{-s} \\ e^{-2s} & -e^{-2s} \end{pmatrix} \begin{pmatrix} \sin s \\ -2\cos s \end{pmatrix} ds \\&= \begin{pmatrix} e^x & 3e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} \begin{pmatrix} -2e^{-x} \sin x + 4e^{-x} \cos x - 4 \\ 1 - e^{-2x} \cos x \end{pmatrix} \\&= \begin{pmatrix} -2\sin x + \cos x - 4e^x + 3e^{2x} \\ -2\sin x + 2\cos x - 4e^x + 2e^{2x} \end{pmatrix}.\end{aligned}$$

5. proof: 首先假设 A 的所有特征根都具有负的实部

记 $m = \max_{1 \leq i \leq 5} \operatorname{Re} \lambda_i$, 则 $m < 0$.

由于方程的基解矩阵

$$\Phi(x) = (e^{\lambda_1 x} p_1^{(1)}(x) \quad \cdots \quad e^{\lambda_5 x} p_5^{(5)}(x))$$

易知 $\|e^{\lambda_j x} p_j^{(j)}(x)\| \leq e^{mx} \|p_j^{(j)}(x)\|$.

由于 $p_j^{(j)}(x)$ 是关于 x 的向量值多项式.

$\Rightarrow x \rightarrow +\infty$, 我们有 $\|e^{\lambda_j x} p_j^{(j)}(x)\| \rightarrow 0 \quad \forall j$.

而方程任一解可写成 $\Phi(x)$ 列向量线性组合,

于是方程的任一解都 $\rightarrow 0$ as $x \rightarrow +\infty$

反之, 若方程任一解都 $\rightarrow 0$ as $x \rightarrow +\infty$

则 $\|e^{\lambda_j x} p_j^{(j)}(x)\| = e^{\operatorname{Re} \lambda_j x} \|p_j^{(j)}(x)\| \rightarrow 0 \quad \forall j$.

若 $\operatorname{Re} \lambda_i > 0$, 则由于 $p_j^{(j)}(x)$ 是向量值多项式, 且 $p_j^{(j)}(x) \neq 0 \quad \forall x$

进而 $\exists M, s.t. \quad x \rightarrow +\infty, \quad e^{\operatorname{Re} \lambda_i x} \|p_j^{(j)}(x)\| \geq M > 0$

矛盾 $\Rightarrow \operatorname{Re} \lambda_i < 0 \quad \forall i$, 进而 A 所有特征值实部为负.

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3.

(1) proof: 易知 $y(x)$ 与 $\psi(x)$ 的 Wronsky 行列式为

$$w(x) = \begin{vmatrix} y(x) & \psi(x) \\ y'(x) & \psi'(x) \end{vmatrix} = y(x)\psi'(x) - \psi(x)y'(x)$$

$$\begin{aligned} \Rightarrow w'(x) &= y'(x)\psi'(x) + y(x)\psi''(x) - \psi'(x)y'(x) - \psi(x)y''(x) \\ &= y(x)(-\psi(x)\psi'(x)) - \psi(x)(-y(x)y'(x)) = 0 \end{aligned}$$

$\Rightarrow w(x)$ 是常数

(2) 若方程有特解 $y = e^x$, 代入则有

$$e^x + g(x)e^x = 0 \Rightarrow g(x) = -1$$

$\Rightarrow$ 方程为 $y'' - y = 0 \Rightarrow$ 方程通解为 $y = c_1 e^x + c_2 e^{-x}$

4. proof: ① 首先, 若 $x_0 \in I$, s.t. $y(x_0) = 0, y'(x_0) = 0$.

则由线性方程解的存在唯一性知 $y(x) \equiv 0$.

这与 $y(x)$ 是 I 上的非零解矛盾

$\Rightarrow y(x)$ 在区间 I 上只有简单零点

进一步, 若 I 是闭区间, 假设 $y(x)$ 在 I 上有无穷多个不同零点, 不妨记为 $\{x_n\}$, 则由于 I 是紧集, 则存在收敛子列 $x_{n_k} \rightarrow x_0 \in I$.

进而由连续性, 首先有 $y(x_0) = 0$, 其次

$$y'(x_0) = \lim_{n_k \rightarrow +\infty} \frac{y(x_{n_k}) - y(x_0)}{x_{n_k} - x_0} = 0$$

$\Rightarrow x_0$ 为简单零点, 这是一个矛盾

于是在闭区间 I 上至多有有限个零点, 进一步可知每一个零点也是孤立的,

② 我们证明 $y(x)$ 在 (a, b) 上有零点, 且不恒为 0 时.

$$y(x) = y(x_0) \int_{x_0}^x \frac{1}{y^2(s)} e^{-\int_{x_0}^s p(t) dt} ds \quad \text{依旧为原方程的一个解.}$$

事实上, 我们假设在 (a, b) 上恰有 1 个零点的情形, 其余情形同理可以证明

设该零点为 $x_1 \in (a, b)$, 只要证明 $y(x)$ 在 $x = x_1$ 处良定即可

$$\begin{aligned} \text{由 L'Hospital 法则, } \lim_{x \rightarrow x_1} y(x) &= \lim_{x \rightarrow x_1} \frac{\int_{x_0}^x \frac{1}{y^2(s)} e^{-\int_{x_0}^s p(t) dt} ds}{\frac{1}{y(x)}} \\ &= \lim_{x \rightarrow x_1} \frac{\frac{1}{y^2(x)} e^{-\int_{x_0}^x p(t) dt}}{\frac{y'(x)}{y^2(x)}} \\ &= \frac{e^{-\int_{x_0}^{x_1} p(t) dt}}{y'(x_1)}. \end{aligned}$$

$\Rightarrow y(x)$ 在 $x = x_1$ 处良定

$$\Rightarrow y(x) = y(x_1) \int_{x_1}^x \frac{1}{y^2(s)} e^{-\int_{x_1}^s p(t) dt} ds \quad \text{依旧是原方程的一个解.}$$

Rmk: 这里由解的存在唯一性和延拓性, 再结合解本身的连续性, 可知这样得到的

$$7. \text{ 令 } x = e^t \Rightarrow \frac{dx}{dt} = x$$

解自然满足 C^1, C^2 条件.

$$\text{进而 } \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \cdot \frac{1}{x} \frac{d^2y}{dt^2} = \frac{1}{x^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt} \right)$$

用数学归纳法证明

$$\frac{d^k y}{dx^k} = \frac{1}{x^k} \left(\frac{d^k y}{dt^k} + \sum_{i=1}^{k-1} b_i \frac{d^i y}{dt^i} \right) \quad (b_i \text{ 是常数}).$$

$$\begin{aligned} \frac{d^{k+1} y}{dx^{k+1}} &= -\frac{1}{x^{k+1}} \left(\frac{d^k y}{dt^k} + \sum_{i=1}^{k-1} b_i \frac{d^i y}{dt^i} \right) + \frac{1}{x^k} \left(\frac{1}{x} \frac{d^{k+1} y}{dt^{k+1}} + \sum_{i=1}^{k-1} b_i \frac{d^{i+1} y}{dt^{i+1}} \cdot \frac{1}{x} \right) \\ &= \frac{1}{x^{k+1}} \left(\frac{d^{k+1} y}{dt^{k+1}} + \sum_{i=1}^k a_i \frac{d^i y}{dt^i} \right) \end{aligned}$$

进而代入原方程可得

$$\frac{d^n y}{dt^n} + c_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + c_n y = 0.$$

#

10 (1) 特征多项式

$$\begin{aligned} f(\lambda) &= \lambda^5 + 2\lambda^3 + \lambda = \lambda(\lambda^4 + 2\lambda^2 + 1) \\ &= \lambda(\lambda^2 + 1)^2 = \lambda(\lambda + i)^2(\lambda - i)^2 \end{aligned}$$

则 $\lambda=0$ 是 1 重特征值, $\pm i$ 分别为 2 重特征值

进而可以得到方程的基本解组为

$$e^0, e^{ix}, xe^{ix}, e^{-ix}, xe^{-ix}.$$

$\Rightarrow$ 原方程通解为 $y = C_1 + C_2 \cos x + C_3 \sin x + C_4 x \sin x + C_5 x \cos x$

(10) 首先易知 $y=e^x$ 是方程的特解.

$$y^{(4)} + y = 0 \text{ 的特征多项式为 } \lambda^4 + 1 = 0 \Rightarrow \lambda^4 = e^{i\pi}.$$

$$\Rightarrow \lambda_1 = e^{i\frac{\pi}{4}}, \lambda_2 = e^{i\frac{3\pi}{4}}, \lambda_3 = e^{i\frac{5\pi}{4}}, \lambda_4 = e^{i\frac{7\pi}{4}}.$$

则方程的通解为

$$y = e^x + C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x}$$

$$\text{代入 } y(0) = y'(0) = y''(0) = y'''(0) = 1.$$

可知 $y = e^x$.

13. 令 $x = e^t$, 则有 $\frac{dt}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$$

$$\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2}$$

$$\text{代入可得 } \frac{d^2 y}{dt^2} - \frac{dy}{dt} + 5 \frac{dy}{dt} + 13y = 0$$

$$\Leftrightarrow \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = 0.$$

$$\text{特征多项式 } f(\lambda) = \lambda^2 + 4\lambda + 13 \quad \text{令 } f(\lambda) = 0 \Rightarrow \lambda = -2 \pm 3i.$$

$$\text{则 } y = C_1 e^{(-2+3i)t} + C_2 e^{(-2-3i)t}$$

$$= \frac{1}{x^2} (C_1 e^{3it} + C_2 e^{-3it}) \Rightarrow y = \frac{1}{x^2} (C_1 \sin(3 \ln x) + C_2 \cos(3 \ln x))$$

8.2

渐近

1. proof: 首先全局稳定显然可知渐近稳定性

下面我们证明渐近稳定也可得全局稳定

考察方程 $\frac{d\vec{x}}{dt} = A(t)\vec{x}$

则若渐近稳定, 则 $\exists \delta > 0$, s.t. $|\vec{x}_0| < \delta$ 时, 我们有

$$\lim_{t \rightarrow +\infty} \vec{x}(t, 0, \vec{x}_0) = 0$$

下面证明零解也是全局稳定的, $\forall \vec{x}_0 \in \mathbb{R}^n$, 易知 $\exists M > 0$, s.t. $\frac{\|\vec{x}_0\|}{M} < \delta$.

则以 $(0, \frac{\vec{x}_0}{M})$ 为初值的解满足 $\lim_{t \rightarrow +\infty} \vec{x}(t, 0, \frac{\vec{x}_0}{M}) = 0$.

由线性性可知, $\vec{x}(t, 0, \vec{x}_0)$ 为初值的解为 $M \vec{x}(t, 0, \frac{\vec{x}_0}{M})$.

则此时显然也有 $\lim_{t \rightarrow +\infty} M \vec{x}(t, 0, \frac{\vec{x}_0}{M}) = 0$.

进而易知零解全局渐近稳定.

2. 首先在 $t=0$ 处给定初值 $\vec{x}_0$, 由积分因子法易知.

$$\vec{x}(t) = \vec{x}_0 e^{\int_0^t a(s) ds}$$

记 $f(t) = \int_0^t a(s) ds$, 记 $M = \sup_{t \in [0, +\infty)} \left\{ \int_0^t a(s) ds \right\}$.

下证明零解稳定 $\Leftrightarrow M < +\infty$

$\Rightarrow$ 若零解稳定, 则 $\forall \varepsilon > 0$, $\exists \delta$, $|\vec{x}_0| < \delta$ 时, 有

$$|\vec{x}(t)| \leq \varepsilon \quad \Rightarrow \quad \left| e^{\int_0^t a(s) ds} \right| \leq \frac{\varepsilon}{|\vec{x}_0|} \quad \forall t \in [0, +\infty).$$

$\Rightarrow M < +\infty$

$\Leftarrow$ 任给 $\varepsilon > 0$, 取 $\delta = \frac{\varepsilon}{2e^M}$, 则 $|\vec{x}_0| < \delta$ 时.

$$|\vec{x}(t)| \leq |\vec{x}_0| \left| e^{\int_0^t a(s) ds} \right| \leq e^M \cdot \frac{\varepsilon}{2e^M} < \varepsilon.$$

$\Rightarrow$ 零解是稳定的.

其次证明 渐近稳定 $\Leftrightarrow \int_0^{+\infty} a(s) ds = -\infty$

$\Rightarrow$ 由渐近稳定, 则 $|x_0|$ 充分小时, $\lim_{t \rightarrow +\infty} x_0 e^{\int_0^t a(s) ds} = 0$.

$$\Rightarrow \int_0^{+\infty} a(s) ds = -\infty$$

$\Leftarrow$ 若 $\int_0^{+\infty} a(s) ds = -\infty$, 首先显然零解稳定

其次 $\forall x_0$, 都有 $\lim_{t \rightarrow +\infty} x_0 e^{\int_0^t a(s) ds} = 0$.

$\Rightarrow$ 零解渐近稳定

In general, 对于方程

$$\frac{d\vec{x}}{dt} = A(t)\vec{x}. \quad (i)$$

则 ① 零解稳定 $\Leftrightarrow$ (i) 的所有解都有界.

② 零解渐近稳定 $\Leftrightarrow$ (i) 的所有解都趋于 0.

proof: 考察基本解组 $\Phi(x) = (\vec{y}_1(x), \vec{y}_2(x), \dots, \vec{y}_n(x))$.

不妨设 $\vec{y}_i(0) = \vec{e}_i$ $\vec{e}_i = (0, 0, \dots, \underset{\substack{\uparrow \\ i}}{1}, 0, \dots)$

否则作可逆线性变换即可, 依旧保持基本解组

" $\Leftarrow$ ", 由题目假设可知 $\|\vec{y}_i(x)\|$ 有界, 记 $M = \sup_{\substack{x \in [0, +\infty) \\ 1 \leq i \leq n}} \|\vec{y}_i(x)\|$.

则 $\forall \varepsilon > 0$, 取 $\delta = \frac{\varepsilon}{\sum_{i=1}^n M}$, 则 $\vec{x}_0 = (c_1, c_2, \dots, c_n)$. $\|\vec{x}_0\| < \delta$.

易知以 $(0, \vec{x}_0)$ 为初值的解为 $\vec{y}(x) = \sum_{i=1}^n c_i \vec{y}_i(x)$.

$$\text{则 } \|\vec{y}(x)\| \leq \sum_{i=1}^n |c_i| M < M \sum_{i=1}^n \frac{\varepsilon}{\sum_{i=1}^n M} < \varepsilon.$$

三. 零解稳定.

反之, 若零解稳定, 则初值 $\|\vec{x}_0\| < \delta$ 时, 必有 $\|\vec{y}(x, 0, \vec{x}_0)\| < M$.

1. 对上述基本解组, 考察 $\frac{\delta}{\varepsilon} \vec{y}(x)$, 则 $\|\frac{\delta}{\varepsilon} \vec{y}(x)\| < \delta$.

$\Rightarrow \|\frac{\delta}{\varepsilon} \vec{y}(x)\| \leq M \Rightarrow \|\vec{y}(x)\|$ 一致有界

$\Rightarrow$ 所有解均有界

②. 同理可以证明

(Corollary). 对于 $y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0$

零解稳定 $\Leftrightarrow$ 所有解有界

$\Rightarrow$ 渐近稳定 $\Leftrightarrow$ 所有解都趋于 0

4. ① 若 $p > 0$ 且 $q < 0$, 则 $\text{tr}(A) < 0$, $\det(A) > 0$.

则若 A 有共轭特征值 $\lambda, \bar{\lambda}$, 显然 $\text{tr}(A) = 2\text{Re}\lambda < 0$.

若 A 无复特征值, 则显然为实特征值 $\lambda_1, \lambda_2 < 0$.

则此时由书上的结论知零解渐近稳定.

② 若 $p > 0$ 且 $q = 0$, 则 $\text{tr}(A) < 0$, $\det(A) = 0$.

$\Rightarrow A$ 有两个实特征值 $0, \lambda$, 且 $\lambda < 0$.

此时显然稳定, 而不渐近稳定

若 $p = 0, q > 0$, 则此时 A 有两复特征值 $\lambda, \bar{\lambda}$

且 $\text{Re}\lambda = 0$.

则由书上结论知零解稳定, 而不渐近稳定

③ 由于 $A(t)$ 是常矩阵, 则渐近稳定 $\Leftrightarrow$ 特征值实部为负.

若两个实特征值, 则 $\Leftrightarrow \text{tr}(A) < 0, \det A > 0 \Leftrightarrow p > 0, q < 0$.

共轭复 $\Leftrightarrow \text{tr}(A) < 0, \det A > 0 \Leftrightarrow p > 0, q < 0$.

则渐近稳定 $\Leftrightarrow p > 0, q < 0$.

稳定 $\Leftrightarrow$ 特征值实部非正, 且若为 0 Jordan 块为 1 阶

若有一个为负, 另一个非正, 则 $\text{tr}(A) < 0, \det A \geq 0 \Leftrightarrow p > 0, q \geq 0$.

若两个实部都为0，① 有两个共轭虚特征值 $\Leftrightarrow q > 0, p = 0$.

② 两个特征值都为0，则 $p = q = 0$ ，这与假设矛盾

$\Rightarrow$ 稳定 $\Leftrightarrow p > 0, q \geq 0$ or $q > 0, p = 0$

于是由上述讨论知，除了①、②假设中的条件，其他情形均不稳定。

part 0

回顾多变量：形式回顾，省略所有细节

史济怀：多变量微分几何空间： $\mathbb{R}^3 \rightarrow \mathbb{R}^n$ 也讲

第一型曲线积分 $\int_{\Gamma} f d\ell = \int_{t_1}^{t_2} f(\vec{r}(t)) |\vec{r}'(t)| dt$
 $\Gamma: \vec{r} = \vec{r}(t), t_1 \leq t \leq t_2$

第二型曲面积分 $\int_{\Sigma} f d\sigma = \int_U f(\vec{r}) |\vec{r}_u \times \vec{r}_v| du dv$
 $\Sigma: \vec{r} = \vec{r}(u, v)$

对于第二型曲面积分和第二型曲线积分 $\Rightarrow$ 转化为第一型曲线与第一型曲面

Q: 对于 n 维空间中的 k 维超曲面, f 在该超面上的积分如何?

$$\int_{\Sigma} f d\sigma = ?$$

$$\Sigma: \vec{r} = \vec{r}(u_1, \dots, u_k) = (r_1(u_1, \dots, u_k), \dots, r_n(u_1, \dots, u_k)), (u_1, \dots, u_k) \in U$$

$$J\vec{r} = \begin{pmatrix} \frac{\partial r_1}{\partial u_1} & \dots & \frac{\partial r_1}{\partial u_k} \\ \vdots & & \vdots \\ \frac{\partial r_n}{\partial u_1} & \dots & \frac{\partial r_n}{\partial u_k} \end{pmatrix} \in \mathbb{R}^{n \times k}$$

$$\Rightarrow \int_{\Sigma} f d\sigma = \int_U f(\vec{r}) \sqrt{(J\vec{r})^T J\vec{r}} \underbrace{du_1 \dots du_k}_{\text{体积元}}$$

$\uparrow$ 证明参考 Evans
 $\uparrow$ 学完实变函数后有兴趣再去了解
 $\uparrow$ 不认账

证明参考 Evans <<Measure Theory and fine properties of functions>>

学完实变函数后有兴趣再去了解

在PDE中, 我们重点关注 $\Sigma = S^n$ 的情形

Part I: Stokes formula --- Newton Leibnitz formula 的推广

回顾微分: $(fg)' = f'g + fg'$

注意

$$\int_{x_1}^{x_2} f' = f|_{x_1}^{x_2}$$

$$\int_{x_1}^{x_2} fg' dx = fg|_{x_1}^{x_2} - \int_{x_1}^{x_2} f'g dx \Leftrightarrow \int_{x_1}^{x_2} fg' + f'g dx = fg|_{x_1}^{x_2}$$

$$\int_{x_1}^{x_2} fg'' dx = fg'|_{x_1}^{x_2} - \int_{x_1}^{x_2} f'g' dx \Leftrightarrow \int_{x_1}^{x_2} fg'' + f'g' dx = fg'|_{x_1}^{x_2}$$

$$= fg'|_{x_1}^{x_2} - fg'|_{x_1}^{x_2} + \int_{x_1}^{x_2} f''g dx$$

$$\Leftrightarrow \int_{x_1}^{x_2} f''g dx$$

$$= fg'|_{x_1}^{x_2} - fg'|_{x_1}^{x_2}$$

在多变量微分上, 也有如上类似的公式, 但是对于多变量, 什么是

principle

$$f \rightarrow \vec{f}$$

$$\int_{x_1}^{x_2} \rightarrow \int_U$$

$$f' \rightarrow \nabla \vec{f}$$

$$f|_{x_1}^{x_2} \rightarrow \int_{\partial U} \vec{f} \cdot \vec{\nu} d\sigma$$

曲面积分

$\Rightarrow$

$$\nabla(fg) = (\nabla f)g + f\nabla g$$

$$\nabla(\vec{f} \cdot \vec{g}) = (\nabla \vec{f}) \cdot \vec{g} + \vec{f} \cdot \nabla \vec{g}$$

$$\star \int_U \nabla \vec{f} \cdot \vec{g} dx = \int_{\partial U} \vec{f} \cdot \vec{\nu} d\sigma \dots \text{史济怀上的 Gauss formula}$$

证明可参考

Rudin "Principles of mathematical analysis" chapter 9

标量版本

$$\vec{f} = (f_1, f_2, \dots, f_n), \nabla \vec{f} = \partial_i f_j, \vec{f} \cdot \vec{\nu} = \sum f_i \nu_i$$

$$\Rightarrow \star \int_U \partial_i f dx = \int_{\partial U} f \nu_i d\sigma$$

推广:

$$\int_U \Delta f dx = \int_U \nabla(\nabla f) \cdot \vec{\nu} d\sigma = \int_{\partial U} \frac{\partial f}{\partial n} d\sigma$$

$$\int (\nabla f) \cdot (\nabla g) dx = \int f \Delta g dx + \int f \nabla g \cdot \vec{\nu} d\sigma = - \int f \Delta g dx + \int f \nabla g \cdot \vec{\nu} d\sigma$$

$$\int_0^{x_2} f'g' dx = - \int_{x_1}^{x_2} f g'' dx + f g' \Big|_{x_1}^{x_2}$$

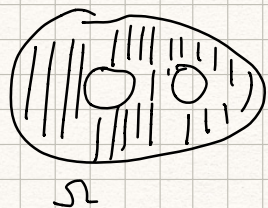
★(第二格林公式) (对应格林公式) $\int_{x_1}^{x_2} f g'' dx - \int_{x_1}^{x_2} f'' g dx = f g' \Big|_{x_1}^{x_2} - f' g \Big|_{x_1}^{x_2}$

$$\begin{aligned} \int_U f \circ g dx - \int_U f'' g dx &= \int_{\partial U} f \nabla g \cdot \vec{\nu} d\sigma - \int_{\partial U} (f' g) \cdot \vec{\nu} d\sigma \\ &= \int_{\partial U} f \frac{\partial g}{\partial n} d\sigma - \int_{\partial U} \frac{\partial f}{\partial n} g d\sigma \end{aligned}$$

Stokes 公式为什么重要: 正如 Newton-Leibniz 公式重要

↑ 可以起到转移变量的作用, 建立导数与原函数之间的联系

Application: 电势学:



Ω 内的电荷分布给定, $\partial\Omega$ 上的电势给定, 则 Ω 上的电势 ϕ 唯一确定

$$\begin{aligned} \Delta\phi &= 0 \\ \phi|_{\partial\Omega} &= \phi_0 \end{aligned} \Rightarrow \begin{cases} \Delta\phi_1 = 0 \\ \phi_1|_{\partial\Omega} = \phi_0 \end{cases} \quad \begin{cases} \Delta\phi_2 = 0 \\ \phi_2|_{\partial\Omega} = \phi_0 \end{cases} \Rightarrow \begin{cases} \Delta(\phi_1 - \phi_2) = 0 \\ (\phi_1 - \phi_2)|_{\partial\Omega} = 0 \end{cases}$$

$$\begin{aligned} \Rightarrow \int_{\Omega} \phi \Delta\phi &= \int_{\Omega} \nabla(\phi \nabla\phi) - |\nabla\phi|^2 dx = - \int_{\Omega} |\nabla\phi|^2 dx + \int_{\partial\Omega} \phi \frac{\partial\phi}{\partial n} d\sigma \\ &\stackrel{0}{=} \Rightarrow \nabla\phi = 0 \Rightarrow \phi = 0 \end{aligned}$$

Part 2: 换元公式

$f = f(x_1, \dots, x_n)$, if $x_i = x_i(y_1, \dots, y_n)$
 $x_n = x_n(y_1, \dots, y_n)$

$$\frac{\partial f}{\partial y_i} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial y_i} + \dots + \frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial y_i} = \left[\frac{\partial f}{\partial x_1} \quad \dots \quad \frac{\partial f}{\partial x_n} \right] \begin{bmatrix} \frac{\partial x_1}{\partial y_i} \\ \vdots \\ \frac{\partial x_n}{\partial y_i} \end{bmatrix}$$

$$\Rightarrow \left[\frac{\partial f}{\partial x_1} \dots \frac{\partial f}{\partial x_n} \right] = \left[\frac{\partial f}{\partial x_1} \dots \frac{\partial f}{\partial x_n} \right] \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}$$

应用: 计算 $\mathbb{R}^2$ 中的极坐标变换 Laplace

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \begin{aligned} x_1 &= r \cos \theta \\ x_2 &= r \sin \theta \end{aligned}$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x_1} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x_1} \quad \frac{\partial^2 f}{\partial x_1^2} = \frac{\partial^2 f}{\partial r^2} \left(\frac{\partial r}{\partial x_1} \right)^2 + \frac{\partial^2 f}{\partial \theta \partial r} \frac{\partial r}{\partial x_1} \frac{\partial \theta}{\partial x_1} + \frac{\partial f}{\partial r} \frac{\partial^2 r}{\partial x_1^2} \\ + \frac{\partial^2 f}{\partial \theta^2} \left(\frac{\partial \theta}{\partial x_1} \right)^2 + \frac{\partial^2 f}{\partial \theta \partial r} \frac{\partial \theta}{\partial x_1} \frac{\partial r}{\partial x_1} + \frac{\partial f}{\partial \theta} \frac{\partial^2 \theta}{\partial x_1^2}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial^2 f}{\partial r^2} \left(\frac{\partial r}{\partial x_2} \right)^2 + \frac{\partial^2 f}{\partial \theta \partial r} \frac{\partial r}{\partial x_2} \frac{\partial \theta}{\partial x_2} + \frac{\partial f}{\partial r} \frac{\partial^2 r}{\partial x_2^2} \\ + \frac{\partial^2 f}{\partial \theta^2} \left(\frac{\partial \theta}{\partial x_2} \right)^2 + \frac{\partial^2 f}{\partial \theta \partial r} \frac{\partial \theta}{\partial x_2} \frac{\partial r}{\partial x_2} + \frac{\partial f}{\partial \theta} \frac{\partial^2 \theta}{\partial x_2^2}$$

$$\begin{bmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{bmatrix} = Id \Rightarrow \begin{bmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{bmatrix}^{-1} \\ = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}^{-1} \\ = \frac{1}{r} \begin{bmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ = \begin{bmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{bmatrix}$$

$$\frac{\partial^2 r}{\partial x_2^2} = \frac{\partial}{\partial x_2} \left(\frac{\partial r}{\partial x_2} \right) = \frac{\partial}{\partial x_2} (\sin \theta) = \frac{\partial \sin \theta}{\partial \theta} \frac{\partial \theta}{\partial x_2} = \cos \theta \left(\frac{\cos \theta}{r} \right) = \frac{\cos^2 \theta}{r}$$

$$\frac{\partial^2 \theta}{\partial x_2^2} = \frac{\partial}{\partial x_2} \left(\frac{\partial \theta}{\partial x_2} \right) = \frac{\partial}{\partial x_2} \left(-\frac{\cos \theta}{r} \right) = \frac{\partial \left(-\frac{\cos \theta}{r} \right)}{\partial r} \frac{\partial r}{\partial x_2} + \frac{\partial \left(-\frac{\cos \theta}{r} \right)}{\partial \theta} \frac{\partial \theta}{\partial x_2} \\ = -\frac{\cos \theta}{r^2} \sin \theta + \frac{\sin \theta}{r} \frac{\cos \theta}{r} = -\frac{2 \sin \theta \cos \theta}{r^2}$$

$$\frac{\partial^2 r}{\partial x_1^2} = \frac{\partial}{\partial x_1} \left(\frac{\partial r}{\partial x_1} \right) = \frac{\partial}{\partial x_1} (\cos \theta) = \frac{\partial \cos \theta}{\partial \theta} \frac{\partial \theta}{\partial x_1} = (-\sin \theta) \left(-\frac{\sin \theta}{r} \right) = \frac{\sin^2 \theta}{r}$$

$$\frac{\partial^2 \theta}{\partial x_1^2} = \frac{\partial}{\partial x_1} \left(\frac{\partial \theta}{\partial x_1} \right) = \frac{\partial}{\partial x_1} \left(\frac{\sin \theta}{r} \right) = \frac{\sin \theta}{r^2} \frac{\partial r}{\partial x_1} + \frac{\cos \theta}{r} \frac{\partial \theta}{\partial x_1} \\ = \frac{2 \sin \theta \cos \theta}{r^2}$$

$$\begin{aligned}
 \Rightarrow \Delta f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} \left[\left(\frac{\partial r}{\partial x_1} \right)^2 + \left(\frac{\partial r}{\partial x_2} \right)^2 \right] + 2 \frac{\partial^2 f}{\partial r \partial \theta} \left[\frac{\partial r}{\partial x_1} \frac{\partial \theta}{\partial x_1} + \frac{\partial r}{\partial x_2} \frac{\partial \theta}{\partial x_2} \right] + \frac{\partial^2 f}{\partial \theta^2} \left[\left(\frac{\partial \theta}{\partial x_1} \right)^2 + \left(\frac{\partial \theta}{\partial x_2} \right)^2 \right] \\
 &\quad + \frac{\partial f}{\partial r} \left(\frac{\partial^2 r}{\partial x_1^2} + \frac{\partial^2 r}{\partial x_2^2} \right) + \frac{\partial f}{\partial \theta} \left(\frac{\partial^2 \theta}{\partial x_1^2} + \frac{\partial^2 \theta}{\partial x_2^2} \right) \\
 &= \frac{\partial^2 f}{\partial r^2} + 2 \frac{\partial^2 f}{\partial r \partial \theta} \left[-\frac{\sin \theta \cos \theta}{r} + \frac{\sin \theta \cos \theta}{r} \right] + \frac{\partial^2 f}{\partial \theta^2} \left[\frac{\sin^2 \theta}{r^2} + \frac{\cos^2 \theta}{r^2} \right] \\
 &\quad + \frac{\partial f}{\partial r} \left(\frac{\sin^2 \theta}{r} + \frac{\cos^2 \theta}{r} \right) + \frac{\partial f}{\partial \theta} \left(\frac{2 \sin \theta \cos \theta}{r^2} - \frac{2 \sin \theta \cos \theta}{r^2} \right) \\
 &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r} \frac{\partial f}{\partial r}
 \end{aligned}$$

对于三维情形, exercise

对于积分的换元公式

$$\begin{aligned}
 &\int_U f(x_1, \dots, x_n) dx_1 \dots dx_n \\
 &\quad \begin{matrix} x_1 = x_1(y_1, \dots, y_n) \\ \vdots \\ x_n = x_n(y_1, \dots, y_n) \end{matrix} \quad Jx = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \\
 \Rightarrow &\int_U f(x_1, \dots, x_n) dx_1 \dots dx_n = \int_{\vec{x}^{-1}(U)} f \circ \vec{x}(\vec{y}) |Jx| dy_1 \dots dy_n
 \end{aligned}$$

Part 3: 曲面面积公式

曲面面积已在 part 0 中介绍, 我们再来计算

$\mathbb{R}^3$ 中的 S 面积

$$\begin{aligned}
 x_1 &= \sin \theta \\
 x_2 &= \cos \theta \cos \varphi \\
 x_3 &= \cos \theta \sin \varphi \\
 \theta &\in [0, 2\pi], \varphi \in [0, \pi]
 \end{aligned}
 \quad \int_S f d\sigma = \int_{(\cos \theta) \times (\cos \theta)} f(\sin \theta, \cos \theta \cos \varphi, \cos \theta \sin \varphi) \sqrt{Jx^T Jx} d\theta d\varphi$$

$$Jx = \begin{bmatrix} \cos \theta & 0 \\ -\sin \theta \cos \varphi & -\cos \theta \sin \varphi \\ -\sin \theta \sin \varphi & \cos \theta \cos \varphi \end{bmatrix}$$

$$(Jx)^T (Jx) = \begin{bmatrix} \cos^2 \theta & -\sin \theta \cos \varphi & -\sin \theta \sin \varphi \end{bmatrix} \begin{bmatrix} \cos \theta \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \cos \theta \end{bmatrix}$$

$\mathbb{R}^{n+1}$ 中的 S^n

$$x_1 = r \sin \theta_1$$

$$x_2 = r \cos \theta_1 \sin \theta_2$$

$$x_3 = r \cos \theta_1 \cos \theta_2 \sin \theta_3$$

$\vdots$

$$x_n = r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-1} \sin \theta_n$$

$$x_{n+1} = r \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-1} \cos \theta_n$$

$$J = \begin{bmatrix} r \cos \theta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}^T$$

$$\vec{x}(r, \theta_1, \dots, \theta_n) = (x_1, \dots, x_{n+1})$$

$$\int_{B_r(0)} f(x_1, \dots, x_{n+1}) dx_1 \cdots dx_{n+1} = \int_0^r \int_0^{2\pi} \cdots \int_0^\pi f(\vec{x}(r, \theta_1, \dots, \theta_n)) \begin{vmatrix} \sin \theta_1 & r \cos \theta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} dr d\theta_1 \cdots d\theta_n$$

$$\int_{B_r(0)} f(x_1, \dots, x_{n+1}) dx_1 \cdots dx_{n+1} = \int_0^r \int_0^{2\pi} \cdots \int_0^\pi f(\vec{x}) \begin{vmatrix} \sin \theta_1 & r \cos \theta_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} J \bigg| d\theta_1 \cdots d\theta_n$$

$$\begin{bmatrix} \frac{\partial \vec{x}}{\partial r} \\ \vdots \\ \frac{\partial \vec{x}}{\partial \theta_n} \end{bmatrix}^T \begin{bmatrix} \frac{\partial \vec{x}}{\partial r} & \frac{\partial \vec{x}}{\partial \theta_1} & \cdots & \frac{\partial \vec{x}}{\partial \theta_n} \end{bmatrix} = \begin{bmatrix} \langle \frac{\partial \vec{x}}{\partial r}, \frac{\partial \vec{x}}{\partial r} \rangle & \cdots & \langle \frac{\partial \vec{x}}{\partial r}, \frac{\partial \vec{x}}{\partial \theta_n} \rangle \\ \vdots & \ddots & \vdots \\ \langle \frac{\partial \vec{x}}{\partial \theta_n}, \frac{\partial \vec{x}}{\partial r} \rangle & \cdots & \langle \frac{\partial \vec{x}}{\partial \theta_n}, \frac{\partial \vec{x}}{\partial \theta_n} \rangle \end{bmatrix}^{1/2}$$

$$\begin{aligned} \left| \begin{bmatrix} \frac{\partial \vec{x}}{\partial r} & \frac{\partial \vec{x}}{\partial \theta_1} & \cdots & \frac{\partial \vec{x}}{\partial \theta_n} \end{bmatrix} \right|^2 &= \begin{bmatrix} \left(\frac{\partial \vec{x}}{\partial r} \right)^T \\ \vdots \\ \left(\frac{\partial \vec{x}}{\partial \theta_n} \right)^T \end{bmatrix} \begin{bmatrix} \frac{\partial \vec{x}}{\partial r} & \frac{\partial \vec{x}}{\partial \theta_1} & \cdots & \frac{\partial \vec{x}}{\partial \theta_n} \end{bmatrix} \\ &= \begin{bmatrix} \langle \frac{\partial \vec{x}}{\partial r}, \frac{\partial \vec{x}}{\partial r} \rangle & \cdots & \langle \frac{\partial \vec{x}}{\partial r}, \frac{\partial \vec{x}}{\partial \theta_n} \rangle \\ \vdots & \ddots & \vdots \\ \langle \frac{\partial \vec{x}}{\partial \theta_n}, \frac{\partial \vec{x}}{\partial r} \rangle & \cdots & \langle \frac{\partial \vec{x}}{\partial \theta_n}, \frac{\partial \vec{x}}{\partial \theta_n} \rangle \end{bmatrix} \end{aligned}$$

$$\frac{\partial}{\partial r} \left(\frac{\partial \vec{x}}{\partial r} \frac{\partial \vec{x}}{\partial \theta_i} \right) = \frac{\partial^2 \vec{x}}{\partial r^2} \frac{\partial \vec{x}}{\partial \theta_i} + \frac{\partial^2 \vec{x}}{\partial r \partial \theta_i}$$

$$= 0 + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \tilde{x}}{\partial t} \frac{\partial \tilde{x}}{\partial t} \right) = 0$$

$$\Rightarrow \begin{bmatrix} 1 \\ J \end{bmatrix}$$

$$\Rightarrow \int_{B_R(0)} f dx_1 \dots dx_n = \int_0^R dr \int_{\partial B_r(0)} f d\sigma = \int_0^R$$

$$\Rightarrow \frac{d}{dR} \int_{B_R(0)} f dx_1 \dots dx_n = \int_{\partial B_R(0)} f d\sigma$$

更一般的公式 $\int_{\mathbb{R}^n} f |Du| dx = \int_0^\infty \int_{\{u>s\}} f ds dx$

part I: Lyapunov 稳定性 三个稳定性定义弄清楚

① 线性化方法 (证明用到 bootstrap 和 Gronwall Inequality)

$$\text{Thm 1: } \frac{d\vec{x}}{dt} = A \vec{x}$$

① 渐稳 $\Leftrightarrow A(t)$ 所有特征值有负实部

② 稳 $\Leftrightarrow \dots \dots \dots$ 非正

③ 不稳 $\Leftrightarrow \exists \dots \dots \dots$ 正实部

$$\text{Thm 2: } \frac{d\vec{x}}{dt} = A(t, x) \vec{x} = A(t) \vec{x} + N(t, x), \quad \lim_{|x| \rightarrow 0} \frac{|N(t, x)|}{|x|} = 0, \quad A(t) = A \text{ const}$$

$\Rightarrow$ ① $A(t, x)$ 所有特征值为负 $\Rightarrow$ 渐稳

② 至少有一个正的实部 $\Rightarrow$ 不稳

③ 特征值 ≤ 0 , 无法判断

$$\hookrightarrow \begin{cases} \frac{dx}{dt} = -x^3 \\ x(0) = x_0 \end{cases}$$

重点: 如何求线性部分

$$\frac{dx}{dt} = f(x, y, t)$$

$$\frac{dy}{dt} = g(x, y, t)$$

把 f, g 看成是 x, y 的多变量函数 $\Rightarrow$ Taylor 展开

$$f(x, y, t) = f(0, 0, t) + \overset{=0}{\frac{\partial f}{\partial x}(0, 0, t)} x + \frac{\partial f}{\partial y}(0, 0, t) y + N_1(x, y, t)$$

$$g(x, y, t) = g(0, 0, t) + \overset{=0}{\frac{\partial g}{\partial x}(0, 0, t)} x + \frac{\partial g}{\partial y}(0, 0, t) y + N_2(x, y, t)$$

$$\Rightarrow \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(0, 0, t) & \frac{\partial f}{\partial y}(0, 0, t) \\ \frac{\partial g}{\partial x}(0, 0, t) & \frac{\partial g}{\partial y}(0, 0, t) \end{bmatrix}}_{\text{is Linear part}} \begin{bmatrix} x \\ y \end{bmatrix} + N(t, x)$$

is Linear part

② Lyapunov fct

应用: 自治系统 $\frac{dx}{dt} = f(x)$ $\checkmark$ 与 t 无关 (不显含 t)

构造一个 Lyapunov function V 满足 $V(0)=0, V(x)>0$ if $x \neq 0$

$$\text{记 } V^* = (\partial_x V)f_1 + (\partial_y V)f_2 + \dots + (\partial_n V)f_n$$

$$\left(V^* = \frac{d}{dt} V(x(t)) = (\partial_x V) \frac{\partial x_1}{\partial t} + \dots + (\partial_n V) \frac{\partial x_n}{\partial t} \right)$$

Thm1: $V^*(y)$ 在 Ω 上 $\leq 0 \Rightarrow$ 稳定.

Thm2: $V^*(y)$ 在 Ω 上 $< 0 \Rightarrow$ 渐近稳定

Thm3: 在原点的任何邻域内 $\exists \alpha$, s.t. $V(\alpha) > 0 \Rightarrow$ 不稳定 $\checkmark$ V^* 正定

proof of thm1:

稳定定义 若 $\begin{cases} \frac{dy}{dt} = f(y) \\ y(0) = y_0 \end{cases}$ $\forall \epsilon > 0, \exists \delta$ 若 $|y_0 - 0| < \delta$, 则 $|y(t) - 0| < \epsilon, \forall t$

现在给 $\epsilon > 0$, 我们需要找一个 δ , s.t. $\forall |y_0 - 0| < \delta$, 有 $\begin{cases} y(t) \text{ 整体存在} \\ |y(t)| < \epsilon \end{cases}$ $\checkmark$
又需证 $|y(t)| < \epsilon$, 我们如何选取 δ ? 取 $B_\delta(0) \subset \Omega$

① 首先, 对任意解 $y(t)$ $\frac{d}{dt} V(y(t)) = V^* \leq 0, V(y(t)) \leq V(y_0)$

若取 δ 足够小, s.t. $|V(y_0)| < \min \{ V(y) \mid \epsilon \leq |y| \leq 1 \}, \forall |y_0| < \delta$

$$\Rightarrow V(y(t)) \leq V(y_0)$$

if $\exists t_0, |y(t_0)| \geq \epsilon, \Rightarrow |V(y(t_0))| \geq \min \{ V(y) \mid \epsilon \leq |y| \leq 1 \}$, 矛盾 -- ok

例1: 证明: 如果存在函数 $V(y)$ 使得在包含原点的区域 Ω 上, $V^* = \lambda V + W$, 其中常数 $\lambda > 0, W$ 恒非负且使得在原点的任何一个小邻域上存在点 $q, V(q)W(q) > 0$, 则原解是不稳定的

proof: $V^*(\phi(t)) = \lambda V(\phi(t)) + W(\phi(t)) \geq \lambda V(\phi(t))$

取 $\phi(t)$ 是以 $\phi(0)$ 为初值的解, 其中 $V(\phi(0))W(\phi(0)) > 0$

$$\Rightarrow \frac{d}{dt} V(\phi_t) \geq \lambda V(\phi_t)$$

线性化 $\frac{1}{V(\phi_t)} \frac{d}{dt} V(\phi_t) \geq \lambda$

$$\Rightarrow \ln V(\phi_t) \geq \lambda t + \ln V(\phi_0)$$

$$\Rightarrow V(\phi_t) \geq e^{\lambda t} C, \text{ where } C > 0$$

$\Rightarrow$ 不成立

事实 $\frac{d}{dt} V(\phi_t) \geq \lambda V(\phi_t) \Leftrightarrow \frac{1}{V(\phi_t)} \frac{d}{dt} V(\phi_t) \geq \lambda$

需要 $V(\phi_t) > 0, \forall t \in [0, \infty)$ 条件

下证 $V(\phi_t) > 0$

$$\frac{d}{dt} V(\phi_t) = \lambda V(\phi_t) + W(\phi_t), \quad V(\phi_0) W(\phi_0) > 0$$

法一: if $V(\phi_{t_0}) < 0, \exists t_0 = \inf \{t: V(\phi_t) = 0\}$, 则 $\frac{d}{dt} V(\phi_t) \Big|_{t=t_0} \leq 0$

$\Rightarrow V(\phi_t) \leq 0 \Rightarrow \exists t \in [0, t_0], \frac{d}{dt} V(\phi_t) < 0$, 矛盾

法二: 求解方程: 非线性方程, 常数变易

例2: $y'' + y + g(y) = 0$, $g(u)$ 是 $\{u | |u| \leq k\}$ 上的连续可微函数. 导数也连续, $g(u)$ 对 $\forall y$ 和 $yg(u) > 0$

$$\begin{aligned} \text{proof: } y_1 = y, y_2 = y', \quad \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} y_2 \\ -y_2 - g(y_1) \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_2 - g'(y_1)y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ h(y_1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -g'(y_1) & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ h(y_1) \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ -g'(y_1) & -1 \end{bmatrix} \quad |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ g'(y_1) & \lambda + 1 \end{vmatrix} = \lambda(\lambda + 1) + g'(y_1) = 0$$

$g' = 0$, 无法判断, 不好.

用老师课做过

例3: $y'' + h(y)y' + q(y) = 0$

h 在 $\{u | |u| \leq k_1 < k_2\}$ 上大于0

$$\frac{dx}{dt} = \frac{1}{2}$$

$$\frac{dy}{dt} = -h(x, y)y_2 - g(x)$$

① 稳定 $V = \frac{1}{2}y_2^2 + \int_0^x g(y)dy$

$$V \text{ 正定 } \dots \Rightarrow k$$

$$V^* = g(x)y_2 + y_2(-h(x, y)y_2 - g(x)) = -h(x, y)y_2^2 \leq 0 \text{ 在 } (y, 0) \text{ 上恒为 } 0$$

$\Rightarrow$ 稳定

② 渐近: $V_1 = \frac{1}{2}y_2^2 + \int_0^x g(y)dy + \beta g(x)y_2$

$$\begin{aligned} V_1^* &= \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y_2} \frac{dy}{dt} = \frac{\partial V}{\partial x} \cdot \frac{1}{2} + \frac{\partial V}{\partial y_2} (-h(x, y)y_2 - g(x)) \\ &= (g(x) + \beta g'(x)y_2) \frac{1}{2} - (y_2 + \beta g(x)) (h(x, y)y_2 + g(x)) \\ &= \cancel{g(x)\frac{1}{2}} + \beta g'(x)\frac{1}{2}y_2^2 - \underbrace{h(x, y)y_2^2}_{\text{负}} - \underbrace{\beta g(x)h(x, y)y_2}_{\text{负}} - \underbrace{g(x)\frac{1}{2}}_{\text{负}} - \beta g^2(x) \\ &> \beta g'(x)\frac{1}{2}y_2^2 - h(x, y)y_2^2 + \frac{\beta}{2}(g(x)^2 + h^2y_2^2) - \beta g^2(x) \\ &= \left[\beta g'(x) - h(x, y) + \frac{\beta}{2}h^2 \right] \frac{1}{2}y_2^2 - \frac{\beta}{2}g^2(x) \end{aligned}$$

改进 $V_2 = \frac{1}{2}y_2^2 + \int_0^x g(y)dy + \beta g(x)y_2 h(x, y)$

$$\begin{aligned} V_2^* &= \frac{\partial V_2}{\partial x} \frac{dx}{dt} + \frac{\partial V_2}{\partial y_2} \frac{dy}{dt} = [g(x) + \beta g'(x)y_2 h(x, y) + \beta g(x)y_2 \frac{\partial h}{\partial x}] \frac{1}{2} \\ &\quad - [y_2 + \beta g(x)h(x, y) + \beta g(x)y_2 \frac{\partial h}{\partial y}] [h(x, y)y_2 + g(x)] \\ &= \cancel{g(x)\frac{1}{2}} + \beta g'(x)h(x, y)\frac{1}{2}y_2^2 + \beta g(x)\frac{\partial h}{\partial x}\frac{1}{2}y_2^2 - \underbrace{h(x, y)y_2^2}_{\text{负}} - \beta g(x)h^2(x, y)\frac{1}{2}y_2^2 - \beta g(x)y_2\frac{\partial h}{\partial y}h(x, y)y_2 \\ &\quad - \cancel{g(x)\frac{1}{2}} - \beta g^2(x)h(x, y) - \beta g^2(x)\frac{\partial h}{\partial y}\frac{1}{2}y_2^2 \end{aligned}$$

① + ② $< -\frac{1}{2}h(x, y)y_2^2$, if β 足够小

$$\rho g(x) \frac{\partial h}{\partial x} \cdot \frac{1}{2} h^2 - h^2 < -\frac{1}{2} h^2 ?$$

$$\text{但是 } \rho g(x) \frac{\partial h}{\partial x} h^2 \frac{1}{2} = h^2 \left[\rho g(x) \frac{\partial h}{\partial x} - 1 \right] \leq h^2 [\rho g(x) (M-1)], \quad \rho g(x) < -\frac{1}{2} h^2$$

$$\text{3) 构造 } V_3 = \frac{1}{2} y^2 + \int_0^y g(y) dy + \rho h^2(x, y) g(x) \frac{1}{2}$$

$$V_3' = \underbrace{\rho g'(x) h^2 \frac{1}{2} + 2 \rho g(x) h \frac{\partial h}{\partial x} \frac{1}{2} - h^2 - \rho g(x) h^2(x, y) \frac{1}{2} - 2 \rho g(x) h \frac{\partial h}{\partial x} \frac{1}{2}}_{- \rho g^2(x) h^2(x, y) - 2 \rho g(x) h \frac{\partial h}{\partial x} \frac{1}{2}}$$

$$< -\frac{1}{2} h^2 - \underbrace{\rho g(x) h^2(x, y) \frac{1}{2}}_{\text{key}} - \underbrace{\rho g^2(x) h^2(x, y) \frac{1}{2}}_{\geq \rho g^2(x) (h^2 \frac{1}{2})^2} < -\frac{1}{2} h^2$$

Cauchy

$$V_4 = \frac{1}{2} y^2 + \int_0^y g(y) dy + \rho h^2(x, y) g(x) \frac{1}{2} \quad \dots \text{ok}$$

$$\text{例4 } \begin{cases} \frac{dx}{dt} = -y^3 - x^5 \\ \frac{dy}{dt} = x^3 - y^5 \end{cases}$$

$$\frac{dy}{dt} = x^3 - y^5$$

$$V(x, y) = \frac{1}{4} (x^4 + y^4)$$

$$V' = x^3(-y^3 - x^5) + y^3(x^3 - y^5) = -x^8 y^3 < 0 \quad \text{if } (x, y) \neq (0, 0)$$

$\Rightarrow$ 渐近稳定

$$\text{例5 } \begin{cases} \frac{dx}{dt} = 2x^2 + y^3 \\ \frac{dy}{dt} = -x^3 + 2x^5 \end{cases}$$

$$V(x, y) = xy$$

$$y(2x^2 + y^3) + x(-x^3 + 2x^5)$$

$$= \pi y + y^2 + 2x^2 > 0$$

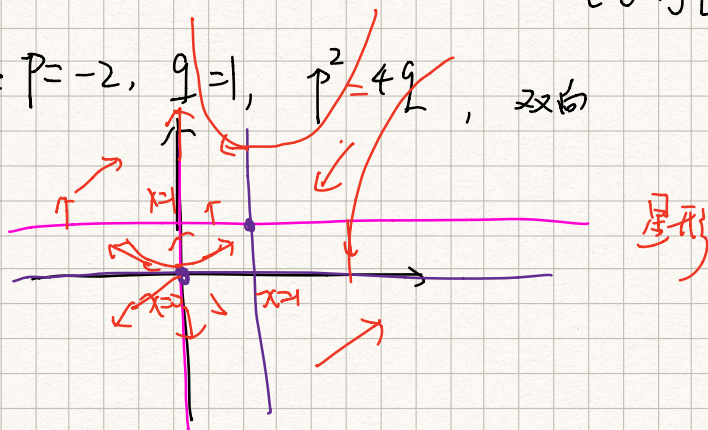
例6: $\begin{cases} y_1' = x \\ y_2' = -\sin y \end{cases}$

问稳定性

相图

$$\begin{cases} \frac{dx}{dt} = x(1-y) = x - xy \\ \frac{dy}{dt} = y(1-x) = y - xy \end{cases} \Rightarrow \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} xy \\ xy \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} xy \\ xy \end{bmatrix}$$

$\# p = -2, q = 1, p^2 = 4q$, 双曲



① 平衡点附近相图：关键在于判断结点类型和箭头方向，附近轨线形状不重要。不需要借助 nullcline

例1 $\begin{cases} \dot{x} = 2x + y + xy^2 \\ \dot{y} = x + 2y + x^2 + y^2 \end{cases}$ 在 (0,0) 附近相图

step 1: 把方程化为线性部分 + 非线性部分的形式

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y + xy^2 \\ x + 2y + x^2 + y^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} xy^2 \\ x^2 + y^2 \end{bmatrix}$$

step 2: 根据线性部分判断结点类型

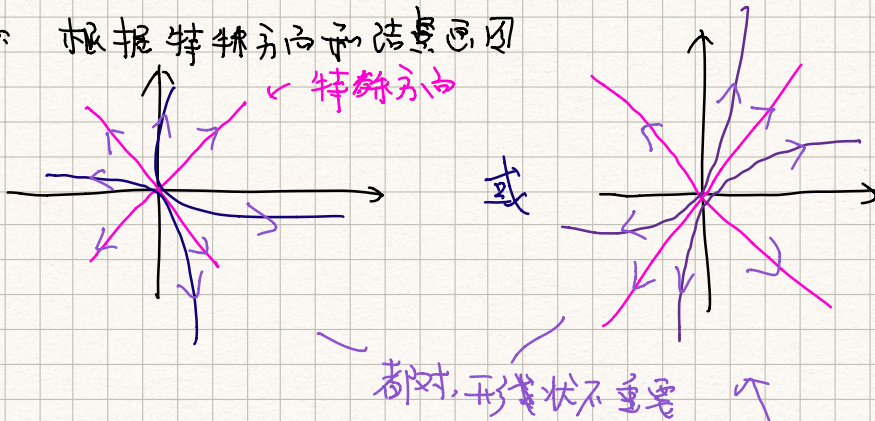
$$p = -tr = -4, \quad q = \det = 2, \quad p^2 > 4q, \Rightarrow \text{双向结点}$$

step 3: 确定结点的特殊方向：双向结点必有 2 个特征向量，方向不同，过原点的线

设 $y = kx$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ kx \end{bmatrix} = \lambda \begin{bmatrix} x \\ kx \end{bmatrix}, \text{ 解得 } k = \pm 1$$

step 4: 根据特殊方向和结点画图



step 5: 确定图中的箭头方向：代入特殊值

$$\begin{cases} \frac{dx}{dt} = 2x + y + xy^2 \\ \frac{dy}{dt} = x + 2y + x^2 + y^2 \end{cases}$$

$\Rightarrow (2x+y+xy^2, x+2y+x^2+y^2)$ 在 (1,1) 处指向第一象限

$\Rightarrow$ 不是双向结点！

例2: $\dot{x} = x + 2y$

$$\frac{dy}{dt} = y - x + x^3$$

step 1: $\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ x^3 \end{bmatrix}$

step 2: $p = -6, q = 9$

$p^2 = 4q, \Rightarrow$ 单向 / 星形 $\Rightarrow$ 计算特殊方向 $\begin{cases} 1^\circ, \text{单向} \\ 2^\circ, \text{星形} \end{cases}$

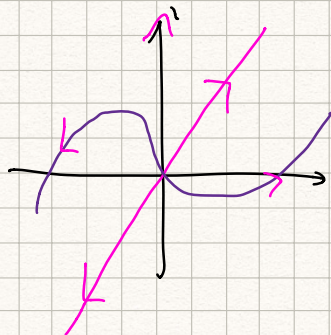
$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ kx \end{bmatrix} = \lambda \begin{bmatrix} x \\ kx \end{bmatrix}$$

$$\Rightarrow \frac{x+2kx}{-2x+1kx} = \frac{1}{k} \Rightarrow (2k+1)k = 5k-2$$

$$\Rightarrow k=1 \Rightarrow \text{单向}$$

step 3: 在 step 2 中已知特殊方向 $y=x$

step 4



形状不重要

确定箭头

step 5: $(x+2y, -2y+5x+x^3) \Big|_{x=1, y=1} = (3, 4)$

② 整体相图: 需要借助 nullcline .

首先不能平线化, $\Rightarrow$ 不确定平线化结论类型 $\Rightarrow$ 利用①画平线化点附近相图

$\Rightarrow$ 延长特殊方向, 根据 nullcline 画整体相图

例: $\frac{dx}{dt} = x^2$

$\alpha < 0$ 整体相图

$$\frac{dy}{dt} = -x(y - a(x-1)),$$

step 1: 确定平衡点

$$\begin{cases} x^2 - 1 = 0 \\ -x(y - a(x-1)) = 0 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ x=0 \text{ 或 } y=a(x-1) \end{cases} \Rightarrow \text{平衡点为 } (1,0) \text{ 或 } (-1,0)$$

step 2: 对原方程平移, 将平衡点移到原点, 再根据判别法计算结点类型

以 $(1,0)$ 为例, 令 $\tilde{x} = x-1, \tilde{y} = y$

$$\frac{d\tilde{x}}{dt} = \frac{dx}{dt} = x^2 - 1 = (\tilde{x}+1)^2 - 1 = \tilde{x}^2 + 2\tilde{x}$$

$$\begin{aligned} \frac{d\tilde{y}}{dt} &= -x(y - a(x-1)) = -(\tilde{x}+1)(\tilde{y} - a(\tilde{x}+1-1)) \\ &= -(\tilde{x}+1)\tilde{y} + a(\tilde{x}+1)^2 - a \\ &= -\tilde{x}\tilde{y} - \tilde{y} + a(\tilde{x}^2 + 2\tilde{x}) \\ &= -\tilde{y} + 2a\tilde{x} - \tilde{x}\tilde{y} + a\tilde{x}^2 \end{aligned}$$

$$\Rightarrow \frac{d}{dt} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2a & -1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} \tilde{x}^2 \\ -\tilde{x}\tilde{y} + a\tilde{x}^2 \end{bmatrix}$$

根据 $\begin{bmatrix} 2 & 0 \\ 2a & -1 \end{bmatrix}$ 判断结点 $\Rightarrow \Delta = -2 < 0 \Rightarrow$ 鞍点

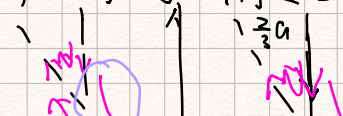
同理 $(-1,0)$ 也是鞍点

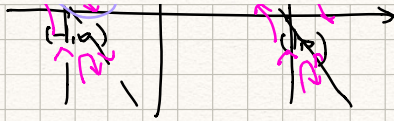
$$\Rightarrow \text{计算特征方向 } \begin{bmatrix} 2 & 0 \\ 2a & -1 \end{bmatrix} \begin{bmatrix} x \\ kx \end{bmatrix} = \lambda \begin{bmatrix} x \\ kx \end{bmatrix}$$

$$\frac{2}{2a-k} = \frac{1}{k} \Rightarrow 2k = 2a - k \quad k = \frac{2}{3}a \quad \tilde{y} = \frac{2}{3}a\tilde{x}$$

注意, 还有一个方向为 $x=0, \begin{bmatrix} 2 & 0 \\ 2a & -1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = - \begin{bmatrix} 0 \\ y \end{bmatrix}$
 $\tilde{x}=0$

所以平衡点附近图为

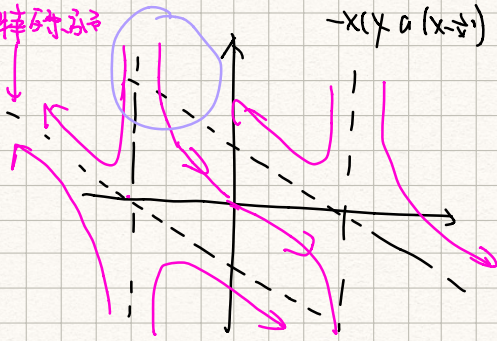




step3: 延长特殊方向, 根据 nullcline 画出整体

计算 nullcline $x^2-1=0 \Rightarrow x=\pm 1$

特殊方向



$$-x(x-a(x-\frac{1}{x}))=0 \Rightarrow x=0 \text{ 或 } y=a(x-\frac{1}{x})$$

说明: 在这个图里, nullcline 的作用
在于确定了 $\odot$ 里的流向渐
近方向