数学分析A2(罗罗)

数学分析(A2)第一次小测答案

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一、计算(给出必要的计算步骤)(每小题10分)
(1)对方程 $e^z - xyz = 0$, 求$\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$.
(2)设 $f(x,y) = (e^x \cos y, e^x \sin y)$, 求 $\mathbf{J}f$ 和 $\mathbf{J}(f^{-1})$.

解 (1) 记 $F(x,y,z) = e^z - xyz$, 则有:

$$\begin{aligned} F'_x &= -yz, \\ F'_y &= -xz, \\ F'_z &= e^z - xy. \end{aligned}$$

由隐函数定理, 可得:

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F'_x}{F'_z} = \frac{yz}{e^z - xy} = \frac{z}{x(z-1)}, \\ \frac{\partial z}{\partial y} &= -\frac{F'_y}{F'_z} = \frac{xz}{e^z - xy} = \frac{z}{y(z-1)}. \end{aligned}$$

其中用到 $e^z = xyz$.

故有:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) \\ &= \frac{\frac{\partial z}{\partial x} x(z-1) - z(z + x\frac{\partial z}{\partial x} - 1)}{x^2(z-1)^2} \\ &= \frac{z(-z^2 + 2z - 2)}{x^2(z-1)^3}. \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) \\ &= \frac{\frac{\partial z}{\partial x} y(z-1) - zy\frac{\partial z}{\partial x}}{y^2(z-1)^2} \\ &= -\frac{z}{xy(z-1)^3}. \end{aligned}$$

(2) 直接计算, 得:

$$\mathbf{J}f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}.$$

其中, $f_1(x,y) = e^x \cos y$, $f_2(x,y) = e^x \sin y$.

注意到 $\det(\mathbf{J}f(x,y)) = e^{2x} \neq 0$, 由逆映射定理, 有:

$$\mathbf{J}(f^{-1}) = (\mathbf{J}f)^{-1} = \begin{pmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{pmatrix}.$$

二、(15分)
将函数 $z = \frac{1}{1-xy}$ 在 $(0,0)$ 处做Taylor展开, 并求出 $\frac{\partial^{m+n} z}{\partial x^n \partial x^m}(0,0)$, 其中 n 和 m 是任意非负整数.

解 由熟知的级数求和公式(也即等比数列求和公式), 在 $(0,0)$ 附近且当 $(x,y)\to(0,0)$ 时, 有:

$$\begin{aligned}\frac{1}{1-xy}&=1+xy+(xy)^2+(xy)^3+\cdots\\&=1+xy+(xy)^2+\cdots+(xy)^n+\frac{1}{1-xy}(xy)^{n+1}\\&=1+xy+(xy)^2+\cdots+(xy)^n+o(x^ny^n)\\&=1+xy+(xy)^2+\cdots+(xy)^n+o((\sqrt{x^2+y^2})^n).\end{aligned}$$

上式即为 z 在 $(0,0)$ 处的Taylor展开.

比较上式与一般的Taylor展开表达式的系数可知:

$$\frac{\partial^{m+n}z}{\partial x^n\partial x^m}(0,0)=\delta_{mn}m!n!.$$

其中 δ_{mn} 为Kronecker函数, 即 $\delta_{mn}=\begin{cases}1,\ m=n\\0,\ m\neq n\end{cases}.$

三、(15分)

设 $f(x,y)=|x-y|\phi(x,y)$, 其中 $\phi(x,y)$ 在点 $(0,0)$ 的某个邻域内连续, 问:

(1)当且仅当 $\phi(x,y)$ 满足什么条件时, 偏导数 $f'_x(0,0)$ 和 $f'_y(0,0)$ 存在?(需说明理由)

(2)当且仅当 $\phi(x,y)$ 满足什么条件时, $f(x,y)$ 在 $(0,0)$ 处可微?(需说明理由)

解 (1) 偏导数 $f'_x(0,0)$ 存在等价于极限 $\lim\limits_{x\to0}\frac{f(x,0)-f(0,0)}{x}$ 存在. 注意到:

$$\begin{aligned}\lim_{x\to0^+}\frac{f(x,0)-f(0,0)}{x}&=\lim_{x\to0^+}\frac{|x|\phi(x,0)}{x}=\phi(0,0),\\\lim_{x\to0^-}\frac{f(x,0)-f(0,0)}{x}&=\lim_{x\to0^-}\frac{|x|\phi(x,0)}{x}=-\phi(0,0).\end{aligned}$$

故上面的极限存在等价于 $\phi(0,0)=-\phi(0,0)$, 等价于 $\phi(0,0)=0$.

同理可得偏导数 $f'_y(0,0)$ 存在等价于 $\phi(0,0)=0$.

(2) 当 f 在 $(0,0)$ 处可微时, 偏导数 $f'_x(0,0)$ 和 $f'_y(0,0)$ 均存在, 由(1)有 $\phi(0,0)=0$.

另一方面, 当 $\phi(0,0)=0$ 时, 有:

$$\begin{aligned}0&\leqslant\lim_{(x,y)\to(0,0)}\frac{|f(x,y)|}{\sqrt{x^2+y^2}}\\&=\lim_{(x,y)\to(0,0)}\frac{|x-y|\cdot|\phi(x,y)|}{\sqrt{x^2+y^2}}\\&\leqslant\lim_{(x,y)\to(0,0)}\frac{(|x|+|y|)\cdot|\phi(x,y)|}{\sqrt{x^2+y^2}}\\&\leqslant\lim_{(x,y)\to(0,0)}\frac{\sqrt{x^2+y^2}\cdot|\phi(x,y)|}{\sqrt{x^2+y^2}}\\&=\lim_{(x,y)\to(0,0)}\phi(0,0)\\&=0.\end{aligned}$$

由两边夹定理, $\lim\limits_{(x,y)\to(0,0)}\frac{|f(x,y)|}{\sqrt{x^2+y^2}}=0$, 即 f 在 $(0,0)$ 处可微且微分为0.

综上, 题目所求条件为 $\phi(0,0)=0$.

四、(10分)

设 $F(x,y,z)$ 和 $G(x,y,z)$ 有连续偏导数, $\frac{\partial(F,G)}{\partial(x,z)} \neq 0$, 曲线 $\Gamma : \begin{cases} F(x,y,z) = 0 \\ G(x,y,z) = 0 \end{cases}$ 过点 (x_0, y_0, z_0). 记 Γ 在 Oxy 平面上的投影曲线为 L, 求 L 上过点 (x_0, y_0) 的切线方程.

解 由条件 $\frac{\partial(F,G)}{\partial(x,z)} \neq 0$ 和隐映射定理, 曲线 Γ 在点 (x_0, y_0, z_0) 附近可以表示为 $(x(y), y, z(y))$, 且有:

$$\begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}y} \\ \dfrac{\mathrm{d}z}{\mathrm{d}y} \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial z} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial y} \end{pmatrix} = \frac{1}{\frac{\partial F}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z}} \begin{pmatrix} \dfrac{\partial F}{\partial y}\dfrac{\partial G}{\partial z} - \dfrac{\partial F}{\partial z}\dfrac{\partial G}{\partial y} \\ \dfrac{\partial F}{\partial x}\dfrac{\partial G}{\partial y} - \dfrac{\partial F}{\partial y}\dfrac{\partial G}{\partial x} \end{pmatrix}.$$

Γ 在 Oxy 平面上的投影曲线在 (x_0, y_0) 附近可表示为 $(x(y), y)$, 其切向量为 $(\frac{\mathrm{d}x}{\mathrm{d}y}, 1)$, 或者也可以取 $(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial G}{\partial y}, \frac{\partial F}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z})$. 由此可求得切线方程为:

$$\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z}\right)(x - x_0) = \left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial G}{\partial y}\right)(y - y_0).$$

五、(10分)

设函数 $u = f(x,y)$ 具有二阶连续偏导数, 且满足 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. 证明函数 $v = f(x^2 - y^2, 2xy)$ 也满足 $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

证 记 $f = f(\xi, \eta)$, 则题中条件化为 $\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = 0$. 直接计算得:

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial f}{\partial \xi}\frac{\partial(x^2 - y^2)}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial(2xy)}{\partial x} \\ &= 2x\frac{\partial f}{\partial \xi} + 2y\frac{\partial f}{\partial \eta}, \\ \frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial v}{\partial x}) \\ &= 2\frac{\partial f}{\partial \xi} + 2x\frac{\partial}{\partial x}(\frac{\partial f}{\partial \xi}) + 2y\frac{\partial}{\partial x}(\frac{\partial f}{\partial \eta}) \\ &= 2\frac{\partial f}{\partial \xi} + 2x\left(\frac{\partial^2 f}{\partial \xi^2}\frac{\partial(x^2 - y^2)}{\partial x} + \frac{\partial^2 f}{\partial \eta \partial \xi}\frac{\partial(2xy)}{\partial x}\right) + 2y\left(\frac{\partial^2 f}{\partial \xi \partial \eta}\frac{\partial(x^2 - y^2)}{\partial x} + \frac{\partial^2 f}{\partial \eta^2}\frac{\partial(2xy)}{\partial x}\right) \\ &= 2\frac{\partial f}{\partial \xi} + 4x^2\frac{\partial^2 f}{\partial \xi^2} + 4xy\frac{\partial^2 f}{\partial \eta \partial \xi} + 4xy\frac{\partial^2 f}{\partial \xi \partial \eta} + 4y^2\frac{\partial^2 f}{\partial \eta^2}. \end{aligned}$$

同理可得:

$$\begin{aligned} \frac{\partial v}{\partial x} &= -2y\frac{\partial f}{\partial \xi} + 2x\frac{\partial f}{\partial \eta}, \\ \frac{\partial^2 v}{\partial x^2} &= -2\frac{\partial f}{\partial \xi} + 4y^2\frac{\partial^2 f}{\partial \xi^2} - 4xy\frac{\partial^2 f}{\partial \eta \partial \xi} - 4xy\frac{\partial^2 f}{\partial \xi \partial \eta} + 4x^2\frac{\partial^2 f}{\partial \eta^2}. \end{aligned}$$

故:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (4x^2 + 4y^2)\left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}\right) = 0.$$

□

六、(10分)

设平面点集 $D \subset \mathbb{R}^2$, 且 D 上每个连续函数都有界, 证明 D 是紧致集.

证 只需证明 D 是有界闭集.

先取 D 上的函数 $f(x)=||x||$. 由三角不等式可知 f 是连续的. 由题目条件得 f 有界. 故 D 有界.

下面反设 D 不是闭集, 则存在 $x_0\in\overline{D}\setminus D$. 考虑函数 $g(x)=\frac{1}{||x-x_0||}$. 由 $x_0\notin D$ 知 g 是良定的. 注意到:

$$\begin{aligned}|g(x_1)-g(x_2)|&=\left|\frac{1}{||x_1-x_0||}-\frac{1}{||x_2-x_0||}\right|\\&=\frac{\big|||x_2-x_0||-||x_1-x_0||\big|}{||x_1-x_0||\cdot||x_2-x_0||}\\&\leqslant\frac{||x_2-x_1||}{||x_1-x_0||\cdot||x_2-x_0||}.\end{aligned}$$

故 g 是连续的, 得到 g 是有界的.

但 $x_0\in\overline{D}\backslash D$ 推出 $x_0\in D'$, 即存在 D 中的点列 $\{x_n\}_{n=1}^{\infty}$, 使得 $\lim\limits_{n\to\infty}x_n=x_0$. 因此 $\lim\limits_{n\to\infty}g(x_n)=+\infty$, 矛盾!

综上, D 是有界闭集.

□

七、(10分)

设二元函数 $F(x,y)$ 在 $\mathbb{R}^2$ 上具有二阶连续偏导数, $F(x,y)=0$ 的解集形成一条不自交的曲线 L, 记 L 所围的区域为 D. 证明:

(1) $F(x,y)$ 必在区域 D 的内部取到最大值或最小值.

(2)若对 D 内任意点 (x,y), $F''_{xx}+F''_{yy}>0$, 则 $F(x,y)$ 在 D 内恒小于0.

证 (1) 即证: $F|_{\overline{D}}$ 必在 D 中取到最大或者最小值.

任取 $(x_0,y_0)\in D$, 由题可知 $F(x_0,y_0)\neq 0$. 若 $F(x_0,y_0)>0$, 因为 $\overline{D}$ 是紧集, 故 $F|_{\overline{D}}$ 必在 $\overline{D}$ 中某个点处取到最大值, 且这个最大值一定大于等于 $F(x_0,y_0)$, 故大于0. 而 $F|_L\equiv 0$, 因此 $F|_{\overline{D}}$ 在 D 中取到最大值. 同理, 若 $F(x_0,y_0)<0$, 则 $F|_{\overline{D}}$ 在 D 中取到最小值.

(2) 反证法, 假设存在 $(x_1,y_1)\in D$ 使得 $F(x_1,y_1)>0$. 则由(1)证明过程中的结论, 一定存在 $(x_2,y_2)\in D$, 使得 $F|_{\overline{D}}$ 在 (x_2,y_2) 处取到最大值. 故 (x_2,y_2) 是 F 的一个不严格的极大值点, 因此 F 在 (x_2,y_2) 处的Hesse阵是半负定的. 特别地, 有 $F''_{xx}(x_2,y_2)\geqslant 0$, $F''_{yy}(x_2,y_2)\geqslant 0$, 这与 $F''_{xx}(x_2,y_2)+F''_{yy}(x_2,y_2)>0$ 矛盾!

□

八、(10分)

设二元函数 $z=f(x,y)$ 在 $\mathbb{R}^2$ 上具有一阶连续偏导数, 且满足 $xf'_x(x,y)+yf'_y(x,y)=0$. 证明 $f(x,y)$ 是常数.

证 对任意的 $k\in\mathbb{R}$, 当 $x\neq 0$ 时, 有:

$$\frac{\mathrm{d}f(x,kx)}{\mathrm{d}x}=f'_x(x,kx)+kf'_y(x,kx)=\frac{1}{x}(xf'_x(x,kx)+kxf'_y(x,kx))=0.$$

故对任意的 $x_1>0,x_2>0$, 有 $f(x_1,kx_1)=f(x_2,kx_2)$. 得到对任意的 $x>0$, 有:

$$f(x,kx)=\lim_{x\to)0^+}f(x,kx)=f(0,0).$$

同理, 对任意的 $x<0$, 有 $f(x,kx)=f(0,0)$.

对于直线 $x=0$, 题中条件取 $x=0$, 有 $yf'_y(0,y)=0$. 故 $y\neq 0$ 时:

$$\frac{\mathrm{d}f(0,y)}{\mathrm{d}y}=f'_y(0,y)=0.$$

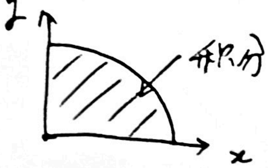
与前面的讨论同理, 对任意的 $y\neq 0$, 有 $f(0,y)=f(0,0)$.

我们已证对任意的 $(x,y)\in\mathbb{R}^2$, 有 $f(x,y)=f(0,0)$. 故 f 是常值函数.

$\square$

MA (A2) 小测 2.

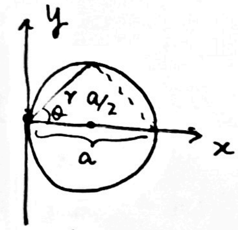
1 (a) $\int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy$



$= \int_0^{\pi/2} d\theta \int_0^1 r\,dr \cdot r$

$= \frac{\pi}{2} \cdot \frac{1}{3} = \pi/6$

(b) $\iint\limits_{x^2+y^2 \le ax} xy^2\, dx\, dy$



$= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{a\cos\theta} r^3 \cos\theta \sin^2\theta \cdot r\,dr$

$= \frac{2a^5}{5} \int_0^{\pi/2} \cos^6\theta \sin^2\theta\, d\theta$

$= \frac{2a^5}{5} \left(\int_0^{\pi/2} \cos^6\theta\, d\theta - \int_0^{\pi} \cos^8\theta\, d\theta \right)$

$= \frac{a^5\pi}{128}$

$\int_0^{\pi/2} \cos^n d\theta \overset{2\text{整除}n}{=\!=\!=} \frac{\pi}{2} \cdot \frac{(n-1)!!}{n!!}$

(c) $\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3}$

$= \frac{\ln 2}{2} - \frac{5}{16}$.

(d) 由对称性，直接视 $x = z$.

原式 $= \iiint \cos z\, dx\, dy\, dz$

$= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 \cos(r\cos\theta)\, r^2 \sin\theta\, dr$

$\cancel{= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin}$

$\overset{\text{先对}\theta\text{积分}}{=\!=\!=} 2\pi \cdot \int_0^1 \left(-r\sin(r\cos\theta)\right)\Big|_{\theta=0}^{\pi} dr$

$= 2\pi \int_0^1 2r\sin(r)\, dr \overset{\text{分部积分}}{=\!=\!=} 4\pi(\sin 1 - \cos 1)$.

(a) ~~连续~~

若 u 在内部取 max.

设 $u(x_0, y_0) = M$.

u 连续 $\Rightarrow \forall \varepsilon, \exists \delta, B_\delta(x_0, y_0) \subset D \rightarrow (x_0, y_0)$,

s.t. $\forall y \in B_\delta(x_0, y_0)$

$$|u(y) - M| < \varepsilon.$$

$$\Rightarrow \quad f_p(u) = \left(\frac{1}{\mu}\iint_D u^p\right)^{\frac{1}{p}}$$

$$\geq \left(\frac{1}{\mu}\int_{B_\delta(x_0,y_0)} u^p\right)^{\frac{1}{p}}$$

$$\geq \left(\frac{1}{\mu}\right)^{\frac{1}{p}}\left(\int_{B_\delta(x_0,y_0)} |u|^p\right)^{\frac{1}{p}}$$

$$\geq \left(\frac{1}{\mu}\right)^{\frac{1}{p}}\left(\int_{B_\delta(x_0,y_0)} |M-\varepsilon|^p\right)^{\frac{1}{p}}$$

$$= \left(\frac{|B_\delta(x_0,y_0)|}{\mu}\right)^{\frac{1}{p}}(M-\varepsilon)$$

$$\longrightarrow M-\varepsilon \quad \text{as} \quad p \to +\infty$$

由 ε 任意性，知 $\lim\limits_{p\to+\infty} f_p(u) \geq M$.

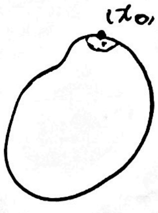
反向是显然的.

若 u 在边界处 (x_0, y_0) 取 max.

由于 D 是闭区域，所以 $\overline{D} = \overline{D^\circ}$

故 $B_\delta(x_0, y_0) \cap D^\circ \neq \emptyset, \forall \delta$

(x_0, y_0)



取 $(x_1, y_1) \in B_{\delta_1}(x_0, y_0) \cap D^\circ$.

δ_1 足够小，使得任一 $B_{\delta_1}(x_0, y_0)$ 中的点 p，$|u(p) - M| < \varepsilon$.

再取 $B_\delta(x_1, y_1) \subset \underbrace{B_{\delta_1}(x_0, y_0) \cap D^\circ}$

since 这是开集

于是 回到了 (1) 的情形.

$$f_p(u) \geq \left(\frac{1}{\mu}\right)^{\frac{1}{p}} \left(\iint_{B_\delta(x_1, y_1)} |M-\varepsilon|^p \right)^{\frac{1}{p}}$$

$$\to M - \varepsilon \quad \text{as} \quad p \to +\infty.$$

对 $p \to -\infty$ 的情形, 同理可证.

(b).

$$\lim_{p\to 0} f_p(u) = \exp\left(\lim_{p\to 0} \frac{\ln\left(\frac{1}{\mu}\iint_D u^p\right)}{p} \right)$$

$$\overset{L'H}{=\!=} \exp^{\lim\limits_{p\to 0}\left(\frac{\frac{\frac{1}{\mu}\iint_D u^p \ln u}{\frac{1}{\mu}\iint_D u^p}}{1} \right)}$$

$$= \exp\left(\frac{\frac{1}{\mu}\iint_D \ln u}{\frac{1}{\mu}\underbrace{|D|}_{=\mu}} \right)$$

$$= \exp\left(\frac{\iint_D \ln u}{\mu} \right).$$

数分 A2 小测 I 1,2,3,6, 评分标准

@rosefantasie

2021 年 6 月 6 日

1 （40 分）

2 （10 分）

证明：

$$1 \leq \iint_{[0,1]^2} (\sin x^2 + \cos y^2) \mathrm{d}x\mathrm{d}y \leq \sqrt{2}.$$

证明.

$$\begin{aligned}
\iint_{[0,1]^2} (\sin x^2 + \cos y^2) \mathrm{d}x\mathrm{d}y &= \iint_{[0,1]^2} \sin x^2 \mathrm{d}x\mathrm{d}y + \iint_{[0,1]^2} \cos y^2 \mathrm{d}x\mathrm{d}y \\
&= \iint_{[0,1]^2} (\sin x^2 + \cos x^2) \mathrm{d}x\mathrm{d}y = \int_0^1 \sin x^2 + \cos x^2 \mathrm{d}x \\
&= \sqrt{2} \int_0^1 \sin(x^2 + \frac{pi}{4}) \mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&x^2 \in [0,1] \\
&x^2 + \frac{\pi}{4} \in [\frac{\pi}{4}, 1 + \frac{\pi}{4}] \\
&\sin(x^2 + \frac{\pi}{4}) \in [1, \sqrt{2}] \\
&1 \leq \iint_{[0,1]^2} (\sin x^2 + \cos y^2) \mathrm{d}x\mathrm{d}y \leq \sqrt{2}
\end{aligned}$$

□

3 （15 分）

计算 $(x^2 + y^2 + z^2)^{2021} = z^{4041}$ 围成的区域的体积.

Solution. 用球坐标变换

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

积分区域:

$$r^{2n} \leq r^{2n-1}\cos^{2n-1}\theta$$
$$0 \leq r \leq \cos^{2n-1}\theta \text{ and } \theta \in [0, \frac{pi}{2}]$$

所以体积 =

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\cos^{2n-1}\theta} r^2\sin\theta \mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}r \\
&= 2\pi\int_0^{\frac{\pi}{2}}\frac{1}{3}\cos^{6n-3}\theta\sin\theta\mathrm{d}\theta \\
&= \frac{\pi}{3(3n-1)}
\end{aligned}$$

代入 $n = 2021$. □

4 （15 分）

计算下列 n

$$\int\cdots\int_{\Omega}\sqrt{x_1+x_2+\cdots+x_n}\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n,$$
$$\Omega = \{(x_1,x_2,\cdots,x_n)|x_1+x_2+\cdots+x_n \leq 1, x_i \geq 0, i = 1,2,\cdots,n\}.$$

Solution. 做变换

$$\begin{cases} y_1 = x_1+x_2+\cdots+x_n \\ y_2 = x_2+\cdots+x_n \\ \vdots \\ y_n = x_n \end{cases}$$

积分区域:

$$y_1 \leq 1, y_i - y_{i+1} \leq 0$$
$$\triangle : 0 \leq y_n \leq \cdots \leq y_2 \leq y_1 \leq 1$$

$$\begin{aligned}
\text{原式} &= \int\cdots\int_{\triangle}\sqrt{y_1}\mathrm{d}y_1\cdots\mathrm{d}y_n \\
&= \int_0^1\sqrt{y_1}\mathrm{d}y_1\int_0^{y_1}\mathrm{d}y_2\cdots\int_0^{y_{n-1}}\mathrm{d}y_n \\
&= \frac{2}{(n-1)!(2n+1)}
\end{aligned}$$

□

5 （10 分）

设函数 $f(x,y)$ 定义在有界矩形 $[a,b]\times[c.d]$ 上。对于任意 $x\in[a,b], f(x,y)$ 关于 y 在 $[c,d]$ 上为单调增函数，对于任意 $y\in[c,d], f(x,y)$ 关于 x 在 $[a,b]$ 上为单调增函数，证明 f 在 $[a,b]\times[c,d]$ 上的间断点全体是二维零测集.

证明.（老师给的做法）用定理 10.1.8(3) 找一个分割使得

$$\bar{S}(f,\pi) - \underline{S}(f,\pi) < \varepsilon.$$

用均匀分割 $||\pi_x|| = ||\pi_y|| = \frac{1}{n}$. 对任意的 $\varepsilon, \delta = \frac{1}{n}$

$$\begin{aligned}
&\bar{S}(f,\pi) = \sum_{i,j=1}^{n} f(x_i,y_j)\triangle x_i \triangle y_j. \\
&\underline{S}(f,\pi) = \sum_{i,j=1}^{n} f(x_{i-1},y_{j-1})\triangle x_i \triangle y_j. \\
&\bar{S}(f,\pi) - \underline{S}(f,\pi) = \sum_{i,j=1}^{n} (f(x_i,y_j) - f(x_{i-1},y_{j-1}))\triangle x_i \triangle y_j \\
&< \sum_{i,j=1}^{n} (f(x_i,y_j) - f(x_{i-1},y_{j-1}))\delta^2 \\
&< \delta^2 (\sum_{x_i=b \text{ or } y_j=d} f(x_i,y_j) - \sum_{x_i=a \text{ or } y_j=c} f(x_i,y_j)) \\
&< (f(b,d) - f(a,c))\frac{2n}{n^2} < \varepsilon
\end{aligned}$$

□

6 （10 分）

1. (a) $\Gamma: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $a>0, b>0$. $x<0, y>0$. $\int_\Gamma xy\,ds$.

$$\begin{cases} x=a\cos\theta \\ y=b\sin\theta \end{cases} \quad \theta\in(\tfrac{\pi}{2},\pi). \quad ds=\sqrt{dx^2+dy^2}=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

$$\int_\Gamma xy\,ds = ab\int_{\frac{\pi}{2}}^{\pi}\cos\theta\sin\theta\sqrt{a^2(1-\cos^2\theta)+b^2\cos^2\theta}\,d\theta$$

$$=-ab\int_{\frac{\pi}{2}}^{\pi}\cos\theta\sqrt{a^2(1-\cos^2\theta)+b^2\cos^2\theta}\,d\cos\theta$$

$$\overset{u=\cos\theta}{=}-ab\int_0^{-1}u\sqrt{a^2+(b^2-a^2)u^2}\,du$$

$$=\frac{ab}{3(b^2-a^2)}\left(a^2+(b^2-a^2)u^2\right)^{\frac{3}{2}}\Big|_{-1}^{0}=\frac{ab}{3(b^2-a^2)}(a^3-b^3)=\frac{-ab(a^2+ab+b^2)}{3(a+b)}$$

(b) $\Gamma: x^2+y^2+z^2=1$ 交 $x+y+z=0$. 第一卦限看逆. 求 $\int_\Gamma dx+ydy$.

$\vec{n}=\frac{1}{\sqrt{3}}(1,1,1)$.

$\vec{e_2}=\frac{1}{\sqrt{2}}(1,-1,0)$.

$\vec{e_1}=\vec{n}\times\vec{e_2}=\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)=\frac{1}{\sqrt{6}}(1,1,-2)$.

$\Gamma: \vec{r}=\vec{e_1}\cos\theta+\vec{e_2}\sin\theta$. $\quad ds=d\theta$.

$$\begin{cases} x=\frac{1}{\sqrt{6}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta \\ y=\frac{1}{\sqrt{6}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta \\ z=-\frac{2}{\sqrt{6}}\cos\theta \end{cases} \quad \begin{cases} dx=(-\frac{1}{\sqrt{6}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta)d\theta \\ dy=(-\frac{1}{\sqrt{6}}\sin\theta-\frac{1}{\sqrt{2}}\cos\theta)d\theta \\ dz=\frac{2}{\sqrt{6}}\sin\theta\,d\theta \end{cases}$$

$$\int_\Gamma dx+ydy=\int_0^{2\pi}-\frac{1}{6}\sin\theta+\frac{1}{2}\cos\theta+\left(\frac{1}{\sqrt{2}}\sin\theta-\frac{1}{\sqrt{6}}\cos\theta\right)\left(\frac{1}{\sqrt{6}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right)d\theta$$

$$=\int_0^{2\pi}\frac{1}{2\sqrt{3}}(\sin^2\theta-\cos^2\theta)+\left(\frac{1}{2}-\frac{1}{6}\right)\sin\theta\cos\theta\,d\theta$$

$$=-\frac{1}{2\sqrt{3}}\int_0^{2\pi}\cos2\theta\,d\theta+\frac{1}{6}\int_0^{2\pi}\sin2\theta\,d\theta=0.$$

(c) $\Sigma: x^2+y^2+z^2=a^2$ in I. $\int_\Sigma x^2d\sigma$.

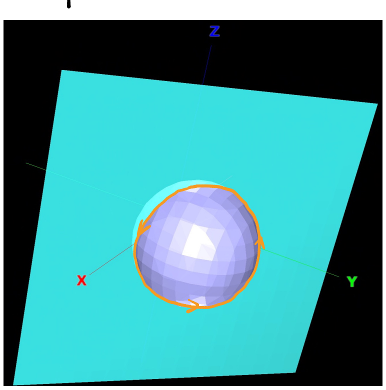
$$\int_\Sigma x^2d\sigma=\int_\Sigma y^2d\sigma=\int_\Sigma z^2d\sigma$$

$$\Rightarrow \int_\Sigma x^2d\sigma=\frac{1}{3}\int_\Sigma x^2+y^2+z^2d\sigma=\frac{a^2}{3}\int_\Sigma d\sigma=\frac{a^2}{3}\sigma(\Sigma)=\frac{a^2}{3}\cdot\frac{1}{8}\cdot4\pi a^2=\frac{\pi}{6}a^4.$$

(d) $\Sigma: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 外侧. $\iint_\Sigma dydz+ydzdx+y^2dxdy$.

$$\overset{\text{Gauss}}{=}\iiint_\Omega 1\,dxdydz=\mu(\Omega)=\frac{4}{3}\pi abc.$$

(e) $\int_\Gamma ydx+zdy+xdz$. $\Gamma: x+y=2$ 和 $x^2+y^2+z^2=2(x+y)$ 交成圆周. 从原点看顺时针.

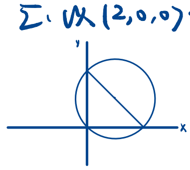


$\vec{n}=\frac{1}{\sqrt{2}}(1,1,0)$.

由Stokes公式. 原式 $=\iint_\Sigma\frac{1}{\sqrt{2}}\begin{vmatrix}1 & 1 & 0\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & z & x\end{vmatrix}d\sigma$

$$=\frac{1}{\sqrt{2}}\iint_\Sigma(-1-1)d\sigma=-\sqrt{2}\,\sigma(\Sigma)=-\sqrt{2}\cdot\pi(\sqrt{2})^2=-2\sqrt{2}\pi.$$

Σ: 以 $(2,0,0)$ 到 $(0,2,0)$ 的线段为直径的圆



2. $V=(P,Q,R)$ 为向量场 $\in C^2$. 证明 $\nabla(\nabla\cdot V)-\nabla\times(\nabla\times V)=\Delta V$.

$\nabla(\nabla\cdot V)=\nabla\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)$

$\frac{\partial}{\partial x}(\nabla\cdot V)=\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 Q}{\partial x\partial y}+\frac{\partial^2 R}{\partial x\partial z}$

$\nabla\times V=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$

$\nabla\times(\nabla\times V)=\left(Q_{xy}-P_{yy}-P_{zz}+R_{xz},\ R_{yz}-Q_{zz}-Q_{xx}+P_{xy},\ P_{xz}-R_{xx}-R_{yy}+Q_{yz}\right)$

第1个分量 of LHS $=P_{xx}+Q_{xy}+R_{xz}-Q_{xy}+P_{yy}+P_{zz}-R_{xz}=P_{xx}+P_{yy}+P_{zz}=\Delta P$.

同理, 另2个分量亦相等.

3. $V=\big(\underbrace{(2x+y+z)yz}_{P},\ \underbrace{(x+2y+z)zx}_{Q},\ \underbrace{(x+y+2z)xy}_{R}\big)$. 证明 V 有势场. 求所有势函数.

由多项式函数基本性质. $V\in C^2(\mathbb{R}^3)$.

$R_y-Q_z=xy+(x+y+2z)x-zx-(x+2y+z)x=0$.

$P_z-R_x=(2x+y+z)y+yz-(x+y+2z)y-xy=0$.

$Q_x-P_y=(x+2y+z)z+zx-(2x+y+z)z-yz=0$.

$\Rightarrow \nabla\times V=0 \Rightarrow V$ 是无旋场 $\Rightarrow V$ 是有势场. 保守场 (积分与路径无关).

势函数 F. $\begin{cases}\frac{\partial F}{\partial x}=(2x+y+z)yz\\ \frac{\partial F}{\partial y}=(x+2y+z)zx\\ \frac{\partial F}{\partial z}=(x+y+2z)xy\end{cases}$ $\Rightarrow$ $F=\int_{(a,b,c)}^{(x,y,z)}V\cdot d\vec{p}$.

$$=\underbrace{\int_{(a,b,c)}^{(x,b,c)}}_{\mathrm{I}}+\underbrace{\int_{(x,b,c)}^{(x,y,c)}}_{\mathrm{II}}+\underbrace{\int_{(x,y,c)}^{(x,y,z)}}_{\mathrm{III}}(Pdx+Qdy+Rdz).$$

$\mathrm{I}=\int_a^x Pdx=\int_a^x(2u+b+c)bc\,du=bc(x^2-a^2)+(b+c)bc(x-a)=bc\cdot x^2+(b^2c+bc^2)x-a^2bc-ab^2c-abc^2$.

$\mathrm{II}=\int_b^y(x+2u+c)cx\,du=cx(y^2-b^2)+(x+c)cx(y-b)=cxy^2-b^2cx+cx^2y-bcx^2+c^2xy-bc^2x$

$\mathrm{III}=\int_c^z(x+y+2u)xy\,du=(x+y)xy(z-c)+xy(z^2-c^2)=x^2yz+xy^2z-cx^2y-cxy^2+xyz^2-c^2xy$.

$F=\mathrm{I}+\mathrm{II}+\mathrm{III}=x^2yz+xy^2z+xyz^2-a^2bc-ab^2c-abc^2$.

$F(x,y,z)=x^2yz+xy^2z+xyz^2+C$ 为所有势函数.

4. 光滑曲面 $\Sigma\subset\mathbb{R}^3$. 围成闭区域 Ω. 外法向 $\vec{n}$ 指向 S 正侧. $u,v\in C^2(\Omega)$.

(a) $\int_\Omega v\Delta u\,dxdydz=-\int_\Omega Du\cdot Dv\,dxdydz+\int_\Sigma v\frac{\partial u}{\partial\vec{n}}d\sigma$.

$\int_\Omega(v\Delta u+Du\,Dv)dxdydz=\int_\Omega D\cdot(vDu)\,dxdydz\overset{\text{Gauss}}{=\!=}\int_\Sigma vDu\cdot\vec{n}\,d\sigma=\int_\Sigma v\frac{\partial u}{\partial\vec{n}}d\sigma$.

(b) $\int_\Omega(v\Delta u-u\Delta v)dxdydz=\int_\Sigma\left(v\frac{\partial u}{\partial\vec{n}}-u\frac{\partial v}{\partial\vec{n}}\right)d\sigma$.

由(a)即得.

(c) u harmonic on Ω. $\Leftrightarrow$ Ω 中 $\forall$ 简单闭曲面 S. $\int_S\frac{\partial u}{\partial\vec{n}}d\sigma=0$.

$\int_{\tilde{\Omega}}\Delta u\,dxdydz=\int_S\frac{\partial u}{\partial\vec{n}}d\sigma$. $\partial\tilde{\Omega}=S$.

"$\Rightarrow$" 由上式即得

"$\Leftarrow$": $\forall\vec{x}\in\Omega^\circ$. $u\in C^2\Rightarrow\Delta u\in C(\Omega)$.

$$\int_{B_\varepsilon(\vec{x})} \Delta u \, dx dy dz = 0. \quad \forall \varepsilon > 0. \quad B_\varepsilon(x) \subset \Omega.$$

若$\Delta u(\vec{x}) > 0$(或<0). 则 $\exists \varepsilon$. $\Delta u|_{B_\varepsilon(\vec{x})} > \frac{1}{2}\Delta u(\vec{x})$.

$\Rightarrow \int_{B_\varepsilon(\vec{x})} \Delta u \, dx dy dz > \frac{1}{2}\Delta u(\vec{x}) \pi \varepsilon^2 > 0$. 矛盾.

故$\Delta u(\vec{x}) = 0$. $\forall \vec{x} \in \Omega^\circ$.

再由Δu在Ω上连续知 $\Delta u = 0$ in Ω.

(d) u harmonic in Ω. Ω内部的值由Σ上的值唯一确定.

$$\begin{cases} \Delta u = 0. \ \Omega. \\ u|_\Sigma = g. \end{cases} \quad \text{只须} \quad \begin{cases} \Delta u = 0 \ \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \text{只有0解}.$$

$$0 = \int_\Omega u \cdot \Delta u \, dx = -\int_\Omega |Du|^2 dx + \int_\Sigma u Du \cdot \vec{n} \, d\sigma$$

$$= -\int_\Omega |Du|^2 dx.$$

$\Rightarrow$ $Du = 0$ on Ω (由连续性).

$\Rightarrow u = \text{Const}$ 而 $u|_{\partial\Omega} = 0$. $u \in C(\Omega) \Rightarrow u = 0$ on Ω.

(e) u harmonic on u. $(x_0, y_0, z_0) \in \Omega^\circ$. $B_r(x_0, y_0, z_0) \subset \Omega$.

$$u(x_0, y_0, z_0) = \frac{1}{4\pi r^2}\int_{\partial B_r(x_0, y_0, z_0)} u(x, y, z) \, d\sigma$$

$$\varphi(r) = \frac{1}{4\pi r^2}\int_{\partial B_r(\vec{x}_0)} u(\vec{x}) \, d\sigma(\vec{x}). \quad d\sigma(\vec{x}) = r^2 d\sigma(\vec{y}).$$

$$\vec{x} = \vec{x}_0 + r\vec{y}$$

$$= \frac{1}{4\pi}\int_{\partial B_1(0)} u(\vec{x}_0 + r\vec{y}) \, d\sigma(\vec{y}).$$

$$\varphi'(r) = \frac{1}{4\pi}\int_{\partial B_1(0)} \frac{\partial u}{\partial r} \, d\sigma$$

$$= \frac{1}{4\pi}\int_{\partial B_1(0)} Du(\vec{x}_0 + r\vec{y}) \cdot \vec{y} \, d\sigma(\vec{y})$$

$$= \frac{1}{4\pi r^2}\int_{\partial B_r(\vec{x}_0)} Du(\vec{x}) \cdot \frac{\vec{x} - \vec{x}_0}{r} \, d\sigma(\vec{x}).$$

$B_r(\vec{x}_0)$的单位外法向.

$$= \frac{1}{4\pi r^2}\int_{\partial B_r(\vec{x}_0)} Du \cdot \vec{n} \, d\sigma$$

$$= \frac{1}{4\pi r^2}\int_{B_r(\vec{x}_0)} \Delta u \, dx \overset{\text{harmonic}}{=} 0.$$

$$\Rightarrow \varphi(r) = \varphi(0) = \lim_{r \to 0} \frac{1}{4\pi r^2}\int_{\partial B_r(\vec{x}_0)} u(\vec{x}) \, d\sigma(\vec{x}) = u(\vec{x}_0).$$

一. 令 $\begin{cases} u = xy \\ v = \frac{y^2}{x} \end{cases}$. 则 $\begin{cases} x = (uv)^{\frac{1}{3}} \\ y = u^{\frac{2}{3}} v^{-\frac{1}{3}} \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{1}{3}} & \frac{1}{3}u^{\frac{1}{3}}v^{-\frac{2}{3}} \\ \frac{2}{3}(uv)^{-\frac{1}{3}} & -\frac{1}{3}u^{\frac{2}{3}}v^{-\frac{4}{3}} \end{pmatrix} = -\frac{1}{3}v^{-1}$$

$$原式 = \iint_{[1,3]^2} \frac{2(uv)^{\frac{1}{3}}}{u^{\frac{4}{3}}v^{-\frac{2}{3}} + u^{\frac{2}{3}}v^{-\frac{2}{3}}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_{[1,3]^2} \frac{2}{3} \frac{1}{v^2(1+u)} du\,dv$$

$$= \frac{2}{3}\int_1^3 \frac{1}{v^2} dv \int_1^3 \frac{1}{1+u} du$$

$$= \frac{4}{9}\ln 2$$

二. 令 $F(u,x,y,z) = \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} - 1$

由隐函数定理

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = -\frac{1}{F_u}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^{-1} \begin{pmatrix} \frac{2x}{u+a^2} \\ \frac{2y}{b^2+u} \\ \frac{2z}{c^2+u} \end{pmatrix}$$

$$2r\cdot \mathrm{grad}\, u = 4\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]^{-1}\left(\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right)$$

$$= \frac{4}{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}}$$

$$= |\mathrm{grad}\, u|^2$$

三. $\int_L yz\,dx + zx\,dy + xy\,dz$ $\qquad L: [0,1] \to \mathbb{R}^3$

$= \int_0^1 3abc\,t^2\,dt$ $\qquad t \mapsto (at, bt, ct)$

$= abc$

① $\sqrt[3]{a\frac{b}{\sqrt{3}}\frac{c}{\sqrt{6}}} \leq \sqrt{\frac{a^2+\frac{b^2}{3}+\frac{c^2}{6}}{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow abc \leq \frac{\sqrt{2}}{\sqrt{3}}$, "=" 成立 $\Leftrightarrow (a,b,c) = (\frac{1}{\sqrt{3}}, 1, \sqrt{2})$

② 令 $F(x,y,z,\lambda) = xyz - \lambda(x^2 + \frac{y^2}{3} + \frac{z^2}{6} - 1)$

$$\begin{cases} F_x = yz - 2x\lambda = 0 \\ F_y = xz - \frac{2y}{3}\lambda = 0 \\ F_z = xy - \frac{z}{3}\lambda = 0 \\ F_\lambda = -(x^2 + \frac{y^2}{3} + \frac{y^2}{6} - 1) = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{3}} \\ y = 1 \\ z = \sqrt{2} \\ \lambda = \sqrt{\frac{3}{2}} \end{cases}$$

又由于边界处 xyz 的值为0.

知 $(\frac{1}{\sqrt{3}}, 1, \sqrt{2})$ 处即为最大值点. 最大值 $\frac{\sqrt{2}}{\sqrt{3}}$

四. 解: 令 $F(x,y,z) = x^2+y^2+z^2-yz-1$

在 $P(x_0, y_0, z_0)$ 处法向量为 $(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) = (2x_0, 2y_0-z_0, 2z_0-y_0)$

题目中的要求等价于 $2z_0-y_0=0$

这与 $F(x_0, y_0, z_0)=0$ 联立, 即得

$$L=\{(\cos\theta, \tfrac{2}{\sqrt{3}}\sin\theta, \tfrac{1}{\sqrt{3}}\sin\theta): 0\le\theta<2\pi\}$$

S在L上方的部分可写为 $z=z(x,y)$ 的形式.

在L上有 $x^2+\frac{3y^2}{4}=1$. Σ在 Oxy 平面上的投影为 $D:=\{(x,y)\in\mathbb{R}^2: x^2+\frac{3y^2}{4}\le 1\}$.

因此原式 $= \iint_D \frac{(x+3)\sqrt{|y-2z|}}{\sqrt{4+y^2+z^2-4yz}} \sqrt{(\frac{\partial F}{\partial x})^2+(\frac{\partial F}{\partial y})^2+(\frac{\partial F}{\partial z})^2}\, \frac{1}{|\frac{\partial F}{\partial z}|}\, dx\,dy$

$$= \iint_D \frac{x+3}{\sqrt{|y-2z|}}\, dx\,dy$$

$$= \frac{1}{\sqrt{2}}\iint_D \frac{x+3}{\sqrt[4]{1-x^2-\frac{3}{4}y^2}}\, dx\,dy$$

$$\xlongequal{\begin{cases}x=r\cos\theta\\ y=\frac{2}{\sqrt{3}}r\sin\theta\end{cases}} \frac{1}{\sqrt{2}}\int_0^1\int_0^{2\pi} \frac{2}{\sqrt{3}}\frac{r\cos\theta+3}{\sqrt[4]{1-r^2}}\, r\, d\theta\, dr$$

$$= 2\sqrt{6}\pi \int_0^1 \frac{r}{\sqrt[4]{1-r^2}}\, dr$$

$$\cancel{= \frac{\sqrt{3}}{\sqrt{2}}\pi\times 3}$$

$$= 2\sqrt{6}\pi\times(-\tfrac{2}{3})(1-r^2)^{\frac{3}{4}}\Big|_0^1$$

$$= \frac{4\sqrt{2}}{\sqrt{3}}\pi$$

五. 令 $\Sigma' := \{(x,y,e)\in\mathbb{R}^3: x^2+y^2\le 1\}$ 向上为正向

$\vec{F} = (4xz, -2yz, x^2-z^2)$

$\nabla\cdot\vec{F}=0$

记 $D:=\{(x,y)\in\mathbb{R}^2: x^2+y^2\le 1\}$

由 Gauss formula

原式 $= -\iint_{\Sigma'} \vec{F}\cdot\vec{n}\, d\sigma$

$$= -\iint_D (x^2-e^2)\, dx\,dy$$

$$= -\int_0^1\int_0^{2\pi} (r^2\cos^2\theta - e^2)\, r\, d\theta\, dr$$

$$= \pi e^2 - \frac{\pi}{4}$$

六. 记交成的区域为 Ω

$D := \{(x,y,z) \in \overline{\Omega}: \ x \geq y \geq 0,\ z \geq 0\}$

则由对称性

$$V_\Omega = 16 \iiint_D dx\,dy\,dz$$

$$= 16 \iint_{\substack{x^2+y^2 \leq a^2 \\ x \geq y \geq 0}} dx\,dy \int_0^{\min(\sqrt{a^2-x^2},\ \sqrt{a^2-y^2})} dz$$

$$= 16 \iint_{\substack{x^2+y^2 \leq a^2 \\ x \geq y \geq 0}} \sqrt{a^2-x^2}\, dx\,dy$$

$$\overset{\text{极坐标换元}}{=} 16 \int_0^{\frac{\pi}{4}} \int_0^a r\sqrt{a^2-r^2\cos^2\theta}\, dr\, d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} -\frac{1}{3\cos^2\theta} (a^2 - r^2\cos^2\theta)^{\frac{3}{2}} \Big|_0^a d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} \frac{a^3}{3\cos^2\theta} (1 - \sin^3\theta)\, d\theta$$

$$= \frac{16a^3}{3} \left(\tan\theta - \cos\theta - \frac{1}{\cos\theta} \right) \Big|_0^{\frac{\pi}{4}}$$

$$= 8a^3 (2 - \sqrt{2})$$

七. $\int_L yf\,dx - xf\,dy \overset{\text{Green}}{=} -\int_{D'} xf_x + yf_y + 2f\ dx\,dy$

其中 D' 为 L 内部

只要证 $xf_x + yf_y + 2f \equiv 0$

对 $f(tx, ty) = t^{-2} f(x,y)$ 两边对 t 求导，得

$x f_x(tx, ty) + y f_y(tx, ty) = -2t^{-3} f(x,y)$.

令 $t = 1$ 得 $xf_x + yf_y + 2f \equiv 0$.

由 (x,y) 的任意性即知 $xf_x + yf_y + 2f \equiv 0$.

八. (1). $m\,g(x) \leq f(x) \leq M g(x)$. $\forall\ x \in [0,1]$.

故 $g(x) = \int_0^1 f(y)\, K(x,y)\, dy$

$\geq \int_0^1 m g(y)\, K(x,y)\, dx$

$= m f(x)$. $\forall\ x \in [0,1]$.

i.e. $\frac{f}{g} \leq \frac{1}{m}$.

由 M 的定义. $\frac{1}{m} \geq M$. $\Rightarrow$ $mM \geq 1$.

$\cancel{g(x) = \int_0^1 f(y)\, K(x,y)\, dy}$

~~$\leq \int_0^1 M g(y) K(x,y) dy$~~

~~$= M f(x). \quad \forall x \in [0,1]$~~

~~i.e. $\frac{f}{g} \leq \frac{1}{M}$~~

~~由 m 的定义. m~~

由对称性及 $\min\limits_{0\leq x\leq 1} \frac{g(x)}{f(x)} = \frac{1}{M}$. $\max\limits_{0\leq x\leq 1} \frac{g(x)}{f(x)} = \frac{1}{m}$

知 $\frac{1}{mM} \geq 1$. i.e. $mM \leq 1$

故 $mM = 1$.

(2). 由 m 的定义. $\exists x_0 \in [0,1]$. $f(x_0) = m g(x_0)$.

$m\int_0^1 g(y) K(x_0,y) dy = m f(x_0) = g(x_0) = \int_0^1 f(y) K(x_0,y) dy$

$\int_0^1 (f(y) - m g(y)) K(x_0,y) dy = 0$

由 $f(y) - m g(y) \geq 0$ 及 $K(x_0,y) > 0$.

得 $f(x) \equiv m g(x)$. $x \in [0,1]$.

$\Rightarrow m = M \xRightarrow{mM=1} m = M = 1$.