第一次习题课.

2021年3月17日 星期三 下午11:30

Part I. 作业.

8.1 2, 4 不讲.

5. 反设 $\exists x \in B_r(a)\cap B_r(b)$,

则有 $2r = \|a-b\| \underset{\text{范数的三角不等式}}{\le} \|a-x\| + \|x-b\| < r + r = 2r$. 矛盾. $\boxtimes$.

6. 左边. $\left(\sum_{i=1}^n 1^2\right)\left(\sum_{j=1}^n x_i^2\right) \ge \left(\sum_{k=1}^n |x_k|\right)^2$

$\Longleftrightarrow \frac{1}{\sqrt{n}}\sum_{i=1}^n |x_i| \le \|x\|$.

右边. $\|x\|^2 = \sum_{i=1}^n x_i^2 \le \sum_{i=1}^n x_i^2 + 2\sum_{i<j}|x_i x_j| = \left(\sum_{i=1}^n |x_i|\right)^2$

7. 左边. $(\max |x_i|)^2 \le \sum_{i=1}^n x_i^2 = \|x\|^2$.

右边. by 6, $\|x\| \le \sum_{i=1}^n |x_i| \le n \cdot \max |x_i|$.

8.2 1. by thm 8.2.2, 只需证

$\frac{1}{n} \to 0$, $\sqrt[n]{n} \to 1$ as $n \to +\infty$.

$\frac{1}{n}\to 0$: 显然. $\sqrt[n]{n} = e^{\frac{1}{n}\ln n} \longrightarrow e^0 = 1$.

3. 设 $\{x_n\}\subset \mathbb{R}^n$, $\lim_{n\to+\infty} x_n = x$ (i.e. $\lim_{n\to+\infty}\|x_n - x\| = 0$).

故 $\exists N \in \mathbb{N}^*$ s.t. $\|x_n - x\| < 1$ for $\forall n > N$.

故有 $\|x_n\| < 1 + \|x\|$ for $\forall n > N$.

欲证 $\{x_n\}$ 有界,

现取 $M = \sum_{k=1}^N \|x_k\| + \|x\| + 1$, 即有

$\|x_n\| < M$, $\forall n \in \mathbb{N}^*$.

4. 设 $\{x_n\}\subset\mathbb{R}^n$, $\{x_n\}$ 为 Cauchy 列.

故 $\exists N \in \mathbb{N}^*$ s.t. $\|x_m - x_n\| < 1$ for $\forall m, n > N$.

固定 $m = N+1$, 有 $\|x_n\| < 1 + \|x_{N+1}\|$, $\forall n > N$.

欲证 $\{x_n\}$ 有界,

现取 $M = \sum_{k=1}^{N+1}\|x_k\| + \|x\| + 1$, 即有

$\|x_n\| < M$, $\forall n \in \mathbb{N}^*$.

(使用 Cauchy 列 $\Leftrightarrow$ 收敛列 没有问题, 但该题本意还是希望让你们熟悉 Cauchy 列的定义).

5. 只需证 $\{x_n\}$ 为 Cauchy 列.

$\sum_{k=1}^{\infty}\|x_{k+1} - x_k\| < +\infty$

$\Rightarrow \sum_{k=m}^{\infty}\|x_{k+1}-x_k\| \to 0$ as $m \to +\infty$

$\Rightarrow \sum_{k=m}^{m+p}\|x_{k+1}-x_k\| \to 0$ as $m\to+\infty$, $\forall p$.

$\Rightarrow \forall \varepsilon > 0$, $\exists N$, s.t. $\forall m > N$, $\forall p$, 有

$\sum_{k=m}^{m+p}\|x_{k+1}-x_k\| < \varepsilon$.

而 LHS $\ge \left\|\sum_{k=m}^{m+p}(x_{k+1}-x_k)\right\| = \|x_{m+p+1} - x_m\|$. $\boxtimes$.

8.3. 1.

① $A^\circ = \emptyset$. (至多可数集无内点, 因为 $B_\delta(x)$ 不可数).

$\overline{A} = \{0\}\cup A$

$\partial A = \{0\}\cup A$. ($\clubsuit$: 全空间 $= \mathring{A} \underset{\text{不交并}}{\sqcup} \partial A \sqcup (A^c)^\circ$.)

② $A^\circ = A$

$\overline{A} = \{(x,y) : 0 \le y \le x+1\}$

$\partial A = \{(x,0) : x \ge -1\} \cup \{(x, x+1) : x \ge 1\}$.

③ $A^\circ = \emptyset$, $\overline{A} = A$, $\partial A = A$.

练习: 证明 $\partial A = \overline{A}\setminus \mathring{A}$.

2. $A^\circ = (A^c)^\circ = \emptyset$, $\partial A = \mathbb{R}^2$.

由有理数的稠密性, $B_\delta(x)\cap A$ 总不为空. 故 $A' = \mathbb{R}^2$.

由有理点至多可数, $A^\circ = \emptyset$.

$\partial A = \overline{A}\setminus A^\circ = (A' \cup A)\setminus A^\circ = \mathbb{R}^2$.

于是 $(A^c)^\circ = \mathbb{R}^2 \setminus (\partial A \cup A^\circ) = \emptyset$.

remark. $\overline{A} = A' \cup A = A' \sqcup \underbrace{(A\setminus A')}_{A\text{中的孤立点. 自证之.}}$

Part 2. 例题

8.3 5, 6, 8, 9, 10, 11, 12, 13 及问题.

6. $\mathbb{R}^n = A^\circ \sqcup \partial A \sqcup (A^c)^\circ$

A° 开, 于是 $(A^c)^\circ \cup \partial A$ 闭.

A° 为 A 包含最大开集, 于是 $(A^c)^\circ \cup \partial A = (A^\circ)^c$ 为

包含 A^c 的最小闭集 (why?). 于是 $(A^\circ)^c = \overline{A^c}$. 即 $A^\circ = (\overline{A^c})^c$

11. G_1 开. 于是 G_1^c 闭. $G_2 \subset G_1^c$, 于是 $\overline{G_2} \subset \overline{G_1^c} = G_1^c$ (why?).

于是 $\overline{G_2} \cap G_1 = \emptyset$. So does $G_2 \cap \overline{G_1}$.

问题. 1. $\mathbb{R}^n = E^\circ \sqcup \partial E \sqcup (E^c)^\circ$.

故只需证 $\partial E \cap E^\circ = \emptyset$ 且 $\partial E \cap (E^c)^\circ = \emptyset$.

$\forall x \in \partial E$,

① $\forall r$, $B_r(x) \cap (\overline{E})^c \neq \emptyset$, $(\overline{E})^c \subset E^c$,

于是 $B_r(x) \cap E^c \neq \emptyset$, 故 $x \notin E^\circ$.

② $\forall r$, $B_r(x) \cap \overline{E} \neq \emptyset$.

若 $x \in E$, 自然有 $x \notin (E^c)^\circ$.

若 $x \notin E$, 取 $y_n \in B_{\frac{1}{n}}(x) \cap \overline{E}$.

由于 $y_n \in \overline{E}$, 故可取 $x_n \in \underbrace{B_{\frac{1}{n}}(y_n) \cap E}_{\neq \emptyset}$ since $y_n \in \overline{E}$. 见习题 8.3.3

有 $\|x_n - x\| \leq \|x_n - y_n\| + \|y_n - x\|$

$< \frac{2}{n}$.

即 $x_n \in B_{\frac{2}{n}}(x) \cap E$

知 $x_n \to x$ as $n \to +\infty$. 从而 $x \in E'$.

于是 $x \in \overline{E} = E \cup E' = E^\circ \cup \partial E$ (why?)

故 $x \notin (E^c)^\circ$.

2. 对 $\forall x \in G$, 我们定义

$$O_x = \{ y : y = x, \text{ or } [x,y] \subset G, \text{ or } [y,x] \subset G \}.$$

G 有界, 不妨设 $G \subset [0,1]$.

取 $\xi_1 = 1/2$, $\xi_2 = \frac{1}{4}$, $\xi_3 = \frac{3}{4}$, $\xi_4 = \frac{1}{8}$, $\xi_5 = \frac{3}{8}$, $\xi_6 = \frac{5}{8}$, ...

定义 $B_i = \begin{cases} \emptyset, & 若\ \xi_i \notin G \\ O_{\xi_i}, & 若\ \xi_i \in G. \end{cases}$

Claim: $\bigcup_{i=1}^{\infty} B_i = G$.

proof. 首先, O_x 为开集. (why?) (从而 $\bigcup_{i=1}^{\infty} B_i$ 的为可数个开区间之并)

于是 $\bigcup_{i=1}^{\infty} B_i \subset G$ (since $B_i \subset G$, $\forall i$)

其次, 若 $\exists x \in G \setminus \bigcup_{i=1}^{\infty} B_i$

取 $B_\delta(x) \subset G$ since G 开.

① 若 $B_\delta(x) \cap \bigcup_{i=1}^{\infty} B_i \neq \emptyset$.

则 $\exists k$, $B_\delta(x) \cap B_k \neq \emptyset$.

则 $x \in B_k$ (why? 回顾 O_x 定义)

矛盾.

② 若 $B_\delta(x) \cap \bigcup_{i=1}^{\infty} B_i = \emptyset$.

取 $\xi_k \in B_\delta(x)$. (since ξ_k 在 $[0,1]$ 中稠密 (why?))

则应有 $B_\delta(x) \subset B_k$. (why?)

矛盾.

故不存在这样的 x. 于是 $G \subset \bigcup_{i=1}^{\infty} B_i$. ☑.

8-3-7. (1) $(A\cap B)^\circ = A^\circ \cap B^\circ$. (2) $\overline{A\cup B} = \overline{A}\cup\overline{B}$.

(1) $x\in (A\cap B)^\circ \Leftrightarrow \exists r>0.\ B_r(x)\subset A\cap B \Leftrightarrow \begin{cases} B_r(x)\subset A \Leftrightarrow x\in A^\circ \\ B_r(x)\subset B \Leftrightarrow x\in B^\circ \end{cases} \Leftrightarrow x\in A^\circ\cap B^\circ$.

(2) ① $\overline{A\cup B}\subset \overline{A}\cup\overline{B}$.

$x\in \overline{A\cup B} \Rightarrow x\in A\cup B\subset \overline{A}\cup\overline{B}$.

这里注意不是$\{x_n\}\subset A$或$\{x_n\}\subset B$.

或 $x\in (A\cup B)' \Rightarrow \exists \{x_n\}_{n=1}^{\infty}\subset A\cup B.\ x_n\to x \Rightarrow \begin{cases} \{x_n\}\text{中只有有限项不同} \Rightarrow x\in A\cup B\subset\overline{A\cup B}. \\ \{x_n\}\text{中有无限项在}A\text{中或}B\text{中} \Rightarrow x\in A'\subset\overline{A}\text{ 或 }x\in B'\subset\overline{B}. \end{cases}$

$\Updownarrow$

$(\forall r>0,\ B_r(\check{x})\cap(A\cup B)' \nRightarrow \forall r>0.\ B_r(\check{x})\cap A\neq\phi \text{ or } \forall r>0\ B_r(\check{x})\cap B\neq\phi)$

$\therefore x\in\overline{A}\cup\overline{B}$

$(0,+\infty) = \{r \mid B_r(\check{x})\cap A\neq\phi\}\cup\{r\mid B_r(\check{x})\cap B\neq\phi\}$

可能$B_r(\check{x})\cap B=\phi$ 可能$B_r(\check{x})\cap A=\phi$.

② $x\in\overline{A}\cup\overline{B} \Rightarrow x\in\overline{A}\subset\overline{A\cup B}$ 或 $x\in\overline{B}\subset\overline{A\cup B}$.

这次作业存在的几个主要问题

1.紧致集的定义。

(a) 要求证明紧致集A到x轴上的投影$P(A)$还是紧致集。
常见错误证明是：任取A的一个开覆盖，其有有限子覆盖，将该有限子覆盖投影到x轴上即成为$P(A)$的有限覆盖。
按照紧致集的定义，我们应当证明对于$P(A)$的任意开覆盖都有有限子覆盖，而不是从A的任意开覆盖构造出$P(A)$的有限覆盖。
两种处理方式。一个是绕开紧致集的定义，证明$P(A)$是有界闭集/列紧集。一个是任取$P(A)$的开覆盖$\{\mathcal{O}_\alpha\}$，则$\{\mathcal{O}_\alpha \times \mathbb{R}\}$为$A$的一个开覆盖。于是存在有限子覆盖$\{\mathcal{O}_k \times \mathbb{R}\}_{1\leq k\leq l<\infty}$. 于是$\{\mathcal{O}_k\}$为$P(A)$的一个有限子覆盖.

(b) 要求证明A与B都是紧致集等价于$A\times B$是紧致集。
右推左的证明有跟上面提到的同样的问题。必须先取A, B的开覆盖，构造出$A\times B$的开覆盖，再利用$A\times B$的紧致性构造出A, B的有限子覆盖。
左推右的证明问题也还是类似，但比较突出。除了不是任取$A\times B$的开覆盖外，常见错误证明是设$A\times B$的开覆盖形如$\{\mathcal{P}_k\times\mathcal{Q}_l\}$。这个假设当然是有问题的，因为不是每一个二维集合都能表示为两个一维集合的笛卡尔积。比较容易的处理方式还是转而证明$A\times B$是有界闭集/列紧集。强行直接用紧致集的定义写的话比较难说得清楚。

2.Frechet紧与列紧的等价性。
这个问题注意到的人很少。一个是，不是任何一个无穷子集都可以看作是数列：因为这个无穷子集可以不可数。必须要先从这个无穷子集里取出一个可数子集。
第二个是，对于一列点列，其可以仅仅包含有限个元素，比如$\{a,b,a,b,...\}$.
这种情况当然有子列收敛，但大部分人都忽略了。

3.$\overline{A\cup B}=\overline{A}\cup\overline{B}$.
这里的问题是要说明
$\forall r>0, B_r(x)\cap A\neq\emptyset$ 或 $B_r(x)\cap B\neq\emptyset$
等价于

$\forall r>0, B_r(x)\cap A\neq\emptyset$ 或 $\forall r>0, B_r(x)\cap B\neq\emptyset$.
其实写起来很容易，反证一下就行了。但大部分人直接等价过去了。其实这里没有那么显然。
另一种处理方法是取补，然后就回到了本题上一小问的结论。

第三次习题课

2021年4月3日 星期六 上午11:18

Week 3 & Week 4 HW.

8.7 2–7 ; 8.8 1, 2 ;

9.1 2–4, 5(2)(3), 6 (even)

9.2 1, 2, 4(2)(4), 5(2)(4), 6

9.3 2, 3(3)(4), 4, 6, 7

9.4 2, 3, 5, 7, 8, 9(2)(3), 10.

PS: Chap 9 中习题有几题
计算漏了. 思路就好.
如果发现证明过有问题(或感觉太模糊)
请告诉我更正.

§8.7

2. 反设 f 一致连续. $\exists\ \delta>0$, s.t.

$$\|(x,y)-(\tilde{x},\tilde{y})\|<\delta \Rightarrow |f(x,y)-f(\tilde{x},\tilde{y})|<1.$$

取 $x=\tilde{x}=1,\quad y=1-\frac{\delta}{1+\delta},\quad \tilde{y}=1-\frac{\delta}{1+2\delta}$

则 $|(x,y)-(\tilde{x},\tilde{y})|=\frac{\delta^2}{(1+\delta)(1+2\delta)}<\delta$

但 $|f(x,y)-f(\tilde{x},\tilde{y})|=\left|\frac{1+\delta}{\delta}-\frac{1+2\delta}{\delta}\right|=1.$ 矛盾. ☒.

3. (1) $p\in\bar{A}$

$\Leftrightarrow\ \exists\ \{p_n\}\subset A,\ \|p-p_n\|\to 0$ as $n\to+\infty$

$\Leftrightarrow\ \inf\limits_{a\in A}\|a-p\|=0$. i.e. $\rho(p,A)=0$.

(2) $\|p-a\|\leq\|q-a\|+\|q-p\|$ for $\forall\ a\in A$.

对 a 取 inf

$\Rightarrow\ \rho(p,A)\leq\rho(q,A)+\|q-p\|$

同理 $\rho(q,A)\leq\rho(p,A)+\|q-p\|$.

故 $|\rho(p,A)-\rho(q,A)|\leq\|q-p\|$. ☒.

4. (1) Claim: $\inf\limits_{a\in A}\rho(a,B)=\rho(A,B)$.

proof: $\|a-p\|\geq\inf\limits_{b\in B}\|a-b\|$ for $p\in B$.

对 $a\in A,\ p\in B$ 取下确界

$$\boxed{\Rightarrow\ \rho(A,B)\geq\inf_{a\in A}\rho(a,B).}$$

另一方面, 对 $a\in A,\ b\in B$ 总有

$$\|a-b\|\geq\inf_{p\in A,\,q\in B}\|p-q\|$$

对 b 取下确界

$\Rightarrow\ \rho(a,B)\geq\rho(A,B)$

再对 a 取下确界

$$\boxed{\Rightarrow\ \inf_{a\in A}\rho(a,B)\geq\rho(A,B)}\ .\ ☒.$$

现由于 A 有界闭 对 $\{a_n\}\subset A$ satisfying

$$|\rho(a_n,B)-\rho(A,B)|<1/n$$

可取子列 $\{a_{k_n}\}\subset\{a_n\}$ s.t.

$$a_{k_n} \longrightarrow a \overset{A\text{有界闭}}{\in} A$$

由 3(2) 知 $\rho(x, B)$ 连续

$\Rightarrow \rho(A,B) = \lim\limits_{n\to\infty} \rho(a_{k_n}, B) = \rho(a, B)$. ▨

(2). $\exists\ a \in A$ s.t. $\rho(a, B) = \rho(A, B)$ since A 紧的.

$$\rho(a,B) = \inf_{b\in B} \rho(a, b)$$

则 $\exists \{b_n\} \subset B$, $|\rho(a, b_n) - \rho(a, B)| < 1/n$

B 紧 $\Rightarrow \exists \{b_{k_n}\} \subset \{b_n\}$, $b_{k_n} \to b \in B$

$\Rightarrow \rho(A, B) = \rho(a, B) = \lim\limits_{n\to\infty} \rho(a, b_{k_n})$

$= \rho(a, b)$. ▨

(3). "$\Rightarrow$" $\rho(A, B) = 0$

$\Rightarrow \exists\ a$, $\rho(a, B) = 0$ since A 紧

$\Rightarrow \exists \{b_n\}$ s.t. $|\rho(a, b_n)| < 1/n$

$\Rightarrow \{b_n\} \subset \overline{B_1}(a) \cap B$ 有界闭

$\Rightarrow \exists \{b_{k_n}\} \subset \{b_n\}$, $b_{k_n} \to b \in B$

$\Rightarrow 0 = \lim\limits_{n\to\infty} \rho(a, b_n) = \rho(a, b)$

$\Rightarrow A \ni a = b \in B$. $\Rightarrow A \cap B \neq \emptyset$.

"$\Leftarrow$" 显然.

5. $A = \{(x, \log x) : x > 0\}$ $B = \{(0, y) : y \in \mathbb{R}\}$

6. (Tip: $\bigcup_{a\in A} \overline{B_c(a)} = \{p \in \mathbb{R}^n : \rho(p, A) \le c\}$ 不难, 但左式为紧的集并不显然.)

proof. 记 B 为 $\{p \in \mathbb{R}^n \mid \rho(p, A) \le c\}$.

一方面, B 有界. since $\underset{\in B}{\|b\|} \le \|b - a\| + \|a\| \le c+1 + M$

$\exists a \in A$, $\|b-a\| \le c+1$; M: A 有界.

另一方面, B 闭. 取 $x \in B^c$,

$\Rightarrow \rho(x, A) > c$

取 $r = \dfrac{\rho(x, A) - c}{2}$

则: $B_r(x) \subset B^c$.

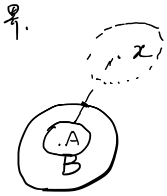
(因为 $\forall\ y \in B_r(x)$, 有

$|\rho(y, A) - \rho(x, A)| \le \|y - x\| < r$

$\Rightarrow \rho(y, A) \ge \rho(x, A) - r$

$= \dfrac{\rho(x, A) + c}{2}$

$> c$).



故 B^c 开. 故 B 闭. ▨

7. 反设 E 连通.

由于 $\exists\, p \in E,\ q \in E,\ f(p) < 0 < f(q)$,

由介值定理知 $\exists\, r \in E$, s.t. $f(r) = 0$. 与E的定义矛盾.

§ 8.8

1. $\forall\, y \in \overline{f(E)}$, $\exists\, \{x_n\} \subset E$

$x_n \to x$, $y = f(x)$.

f连续, 故 $y = f(\lim_{n\to\infty} x_n)$

$= \lim_{n\to\infty} f(x_n) \in \overline{f(E)}$. ▨.

E 有界即可. 此时, 我们证明 $\overline{f(E)} \subset f(\overline{E})$.

proof: $\overline{f(E)}$ 为包含 $f(E)$ 的最小闭集.

$\overline{E}$ 紧, 故 $f(\overline{E})$ 紧. 而 $f(E) \subset f(\overline{E})$,

故 $f(\overline{E})$ 为包含 $f(E)$ 的闭集.

故 $\overline{f(E)} \subset f(\overline{E})$.

2. (1) if $(x_n, f(x_n)) \to (x, y) \in (G(f))'$

then $\lim_{n\to+\infty} x_n = x \in E$ since E 闭

$\lim_{n\to\infty} f(x_n) = y$

$\parallel$ 应

$f(\lim_{n\to+\infty} x_n) = f(x)$

$\Rightarrow (x, y) = (x, f(x)) \in G(f)$ since $x \in E$. ▨

(2) E 紧 $\Rightarrow$ $f(E)$ 紧

$\Rightarrow G(f)$ 有界 since $\|(x, y)\| = (x^2 + \|y\|^2)^{1/2}$

$\leq (C_1 + C_2)^{1/2} < +\infty$

另一方面, $G(f)$ 闭 by (1).

故 $G(f)$ 紧.

(3) 我们任取 $x_n \to x \in E$.

只需证明, $f(x_n) \to f(x)$.

否则, $\exists\, \{x_{k_n}\} \subset \{x_n\}$,

$|f(x_{k_n}) - f(x)| > \varepsilon_0$

$G(f)$ 紧 $\Rightarrow \exists\, \{x_{l_{k_n}}\} \subset \{x_{k_n}\}$. 直接记为 $\{\tilde{x}_n\}$,

有 $(\tilde{x}_n, f(\tilde{x}_n)) \to (x, y)$

这里 $|y - f(x)| = |\lim_{n\to\infty} f(\tilde{x}_n) - f(x)| \geq \varepsilon_0$.

故 $y \neq f(x)$. 但 $(x,y) \in (G(f))'$ 而 $(x,y) \notin G(f)$. 矛盾. ▨

§ 9.1

只讲 2, 3

9.1.2

$$\lim_{t\to 0}\frac{f(t(\cos\theta,\sin\theta))-f(0)}{t}$$

$$=\lim_{t\to 0}\sqrt{|\cos^2\theta-\sin^2\theta|}\cdot\frac{|t|}{t}$$

$\exists \iff \cos 2\theta = 0 \iff \theta = \pm\frac{\pi}{4}, \pm\frac{3}{4}\pi.$

9.1.3

$$\lim_{t\to 0}\frac{f(t(\cos\theta,\sin\theta))-f(0)}{t}$$

$$=\lim_{t\to 0}\frac{t}{|t|}\cos\theta\sin\theta$$

$\exists \iff \cos\theta\sin\theta = 0 \iff \theta = 0, \pm\frac{\pi}{2}, \pi$

9.1.4(x)

$$\lim_{t\to 0}\frac{f(t\vec{u})-f(0)}{t}$$

$$=\lim_{t\to 0}\frac{|t|}{t}\,|u_x+u_y+u_z|$$

where $|\vec{u}|=1,\ \vec{u}=u_x\vec{e_x}+u_y\vec{e_y}+u_z\vec{e_z}$

$\exists \iff u_x+u_y+u_z=0$

$\iff \vec{u}\in\{(x,y,z): x+y+z=0\}$

$\iff \vec{u}\perp(1,1,1).$

9.1.5

(2) $f_x(1,2) = \frac{y}{1+xy}\Big|_{(1,2)} = 2/3$

$f_y(1,2) = \frac{x}{1+xy}\Big|_{(1,2)} = 1/3$

(3) $f_x(1,1) = e^{x+y^2} + 2x\cos(x^2y)\Big|_{(1,1)} = e^2+2\cos 1.$

$f_y(1,1) = 2ye^{x+y^2} + \cos x^2y\Big|_{(1,1)} = 2e^2+\cos 1.$

9.1.6

	f_x	f_y	f_z
(2)	$\frac{2xy^2}{x^4+y^2}$	$-\frac{1}{x^4+y^2}$	—
(4)	$\frac{1}{x+y^2}$	$\frac{2y}{x+y^2}$	—
(6)	$y\cos xy$	$x\cos xy$	—
(8)	$yz\,e^{xyz}$	$xz\,e^{xyz}$	$xy\,e^{xyz}$
(10)	$\frac{1}{x+y^2+z^3}$	$\frac{2y}{x+y^2+z^3}$	$\frac{3z^2}{x+y^2+z^3}$

(12) $z_{x_i} = \dfrac{2x_i}{\sqrt{1-\left(\sum_i x_i^2\right)^2}}$

9.2 1, 2, 4(2)(4), 5(2)(4), 6

9.3 2, 3(3)(4), 4, 6, 7

9.4 2, 3, 5, 7, 8, 9(2)(3), 1

9.2.1

$$\lim_{t\to 0}\frac{f(t(\cos\theta,\sin\theta))-f(0)}{t}$$

$\cos^2\theta\sin\theta$

$$= \lim_{t\to 0} \frac{}{t^2\cos^4\theta+\sin^2\theta}$$

$$= \begin{cases} 0, & \sin\theta = 0 \\ \dfrac{\cos^2\theta}{\sin\theta}, & \sin\theta \neq 0. \end{cases}$$

但 $p_n \triangleq (\frac{1}{n}, \frac{1}{n^2})$,

then $p_n \to 0$

but $f(p_n) = \frac{1}{2} \not\to f(0)$.

故 f 不连续. 于是不可微.

9.2.2

反设 $\sqrt{|xy|}$ 在原点处可微,

$f(\vec{h}) - f(0) = \lambda_1 h_1 + \lambda_2 h_2 + o(\|h\|)$, $\|h\| = (h_1^2+h_2^2)^{\frac{1}{2}} \to 0$

于是有

$f(c, \pm c) = (\lambda_1 \pm \lambda_2)c + o(|c|)$ as $c \to 0$

而 $f(c, c) = f(c, -c)$

$\Rightarrow$ $2\lambda_2 c + o(|c|) = 0$ as $c \to 0$

$\Rightarrow$ $\lambda_2 + o(1) = 0$ as $c \to 0$

$\Rightarrow$ $\lambda_2 = 0$.

同理, $\lambda_1 = 0$. 故 $f(c, c) = o(|c|)$ as $c \to 0$

但 $f(c, c) = |c| \neq o(|c|)$. 矛盾.

9.2.4

(2) $\left(\frac{1}{x+y-z} + e^{x+y}\sin z, \frac{1}{x+y-z} + e^{x+y}\sin z, -\frac{1}{x+y-z} + e^{x+y}\cos z\right)\Big|_{(1,2,1)}$

$= \left(\frac{1}{2} + e^3\sin 1, \frac{1}{2} + e^3\sin 1, -\frac{1}{2} + e^3\cos 1\right)$

(4) $\left(\cos(\sum_i x_i^i), 2x_2\cos(\sum_i x_i^i), \cdots, n x_n^{n-1}\cos(\sum_i x_i^i)\right)$.

9.2.5

(2) $Jf = \begin{pmatrix} 2xy\sin yz \\ x^2\sin yz + x^2yz\cos yz \\ x^2y^2\cos yz \end{pmatrix}$

(4) $Jf = \begin{pmatrix} \frac{x_1}{(\sum_i x_i^2)^{1/2}} \\ \vdots \\ \frac{x_n}{(\sum_i x_i^2)^{1/2}} \end{pmatrix}$

9.2.6

可微, 因为 $f(x,y) = o(\|(x,y)\|)$ as $(x,y) \to 0$

$\left(|f(x,y)| \leq \underbrace{\sqrt{x^2+y^2}}_{\to 0} \cdot \|(x,y)\| \right)$

但 $f_x = \cdots$

$f_y = \cdots$ 不连续.

9.3.2 (1) $\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}^T$ 写反了, 加个转置

(2) $\begin{pmatrix} \cos\theta & \sin\theta & \\ -r\sin\theta & r\cos\theta & \\ & & 1 \end{pmatrix}^T$

(3) $\begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix}^T$

9.3.3 (3) since $\frac{\partial}{\partial x_i}(fg) = g\frac{\partial f}{\partial x_i} + f\frac{\partial g}{\partial x_i}$, $\forall 1\le i\le n$.

(4) since

$$(LHS)_i = \frac{\partial}{\partial x_i}\left(\sum_{k=1}^m f_k g_k\right)$$

第i个分量

$$= \sum_{k=1}^m \left(f_k \frac{\partial g_k}{\partial x_i} + g_k\frac{\partial f_k}{\partial x_i}\right)$$

$$= (g(Jf))_i + (fJg)_i = (RHS)_i \quad \forall 1\le i\le n$$

9.3.4

$$J(\underbrace{\|f\|^2}_{=\text{Const}}) = 0$$

$$\Rightarrow J\langle f, f\rangle = 0$$

$$\overset{9.3.3(4)}{\Rightarrow} 2\langle f, Jf\rangle = 0$$

几何解释：圆的切线与位矢垂直

9.3.6 (1) $f(0) = f(0+0) = f(0) + f(0)$ □.

(2) $0 = f(x+(-x)) = f(x) + f(-x)$. □

(3) $f(x) = f(\sum_i \lambda_i e_i) = \sum_i \lambda_i f(e_i)$ □.

9.3.7 设 $f(e_i) = \sum_{j=1}^m \lambda_{ij}\tilde{e}_j$. then

$$\frac{\partial f_j(x)}{\partial x_i} = \frac{\partial f_j(\sum_i \mu_i e_i)}{\partial x_i}$$

$$= \lim_{h\to 0}\frac{(\mu_1\lambda_{1j} + \cdots + (\mu_i + h)\lambda_{ij} + \cdots + \mu_n\lambda_{nj}) - (\mu_1\lambda_{1j} + \cdots + \mu_n\lambda_{nj})}{h}$$

$$= \lim_{h\to 0}\lambda_{ij} = \lambda_{ij}.$$

则

$$Jf = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{n1} \\ \vdots & & \vdots \\ \lambda_{1m} & \cdots & \lambda_{nm} \end{pmatrix}$$

$$\left(= (\overrightarrow{f(e_1)}, \cdots, \overrightarrow{f(e_n)}) \right)$$

9.4.2

$$x u_x - y u_y$$

$$= x\cdot y f'(xy) - y\cdot x f'(xy)$$

$$= 0. \quad □$$

9.4.3

$$x u_x + y^2 u_y$$

$$= x\cdot\frac{1}{x} f'(\ln x + \tfrac{1}{y}) + y^2\cdot(-\tfrac{1}{y^2}) f'(\ln x + \tfrac{1}{y})$$

$$= 0. \quad □$$

9.4.5

① $\xi = x+y$, $\eta = xy$, $u_x = f_\xi + y f_\eta$, $u_y = f_\xi + x f_\eta$

② $\xi = x$, $\eta = xy$, $\varphi = xyz$, $u_x = f_\xi + y f_\eta + yz f_\varphi$,

$u_y = x f_\eta + xz f_\varphi$, $u_z = xy f_\varphi$

(3) $\xi = x/y,\ \eta = y/z,\quad u_x = f_\xi / y,\quad u_y = -\frac{x}{y^2} f_\xi + \frac{1}{z} f_\eta,$

$$u_z = -\frac{y}{z^2} f_\eta$$

9.4.7

$$u_r = \cos\theta\, u_x + \sin\theta\, u_y = \cos\theta\,(2xy - y^2) + \sin\theta\,(x^2 - 2xy) = \cdots$$

$$u_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y = \cdots$$

9.4.8

$$\begin{aligned} LHS &= x(F_u u_x + F_v v_x + F_w w_x) \\ &\quad + y(F_u u_y + F_v v_y + F_w w_y) \\ &\quad + z(F_u u_z + F_v v_z + F_w w_z) \\ &= (x u_x + y u_y + z u_z) F_u \\ &\quad + (x v_x + y v_y + z v_z) F_v \\ &\quad + (x w_x + y w_y + z w_z) F_w \end{aligned}$$

而 $u^2 = y^2 z^2 / x^2$

$$\Rightarrow 2u u_x = -\frac{2y^2z^2}{x^3}$$

$$2u u_y = \frac{2yz^2}{x^2}$$

$$2u u_z = \frac{2y^2 z}{x^2}$$

$$\Rightarrow x u_x + y u_y + z u_z$$

$$= \frac{1}{u}\left(-\frac{y^2z^2}{x^2} + \frac{y^2z^2}{x^2} + \frac{y^2z^2}{x^2}\right)$$

$$= \frac{1}{u}\cdot u^2 = u.$$

其余同理.

9.4.9

(2)
$$J(f\circ g) = \begin{pmatrix} \varphi'(e^t + e^{-t}) & \varphi'(e^t + e^{-t}) \\ \varphi'(e^t - e^{-t}) & -\varphi'(e^t - e^{-t}) \end{pmatrix} \begin{pmatrix} 0 & e^t \\ 0 & -e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^t\varphi'(e^t + e^{-t}) - e^{-t}\varphi'(e^t + e^{-t}) \\ 0 & e^t\varphi'(e^t - e^{-t}) + e^{-t}\varphi'(e^t - e^{-t}) \end{pmatrix}$$

(3)
$$\left.\begin{pmatrix} 2x & 1 & 1 \\ 2 & 1 & 2z \\ 0 & 0 & 0 \end{pmatrix}\right|_{g(u,v,w)} \begin{pmatrix} v^2w^2 & 2uvw^2 & 2uv^2w \\ 0 & w^2\cos v & 2w\sin v \\ 2ue^v & u^2e^v & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2uv^4w^4 + 2ue^v & 4u^2v^3w^4 + w^2\cos v + u^2e^v & 4u^2v^4w^3 + 2w\sin v \\ 2v^2w^2 + 4u^3e^{2v} & 4uvw^2 + w^2\cos v + 2u^4e^{2v} & 4uv^2w + 2w\sin v \\ 0 & 0 & 0 \end{pmatrix}$$

9.4.10 设 $\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = A = (a_{ij})$. 则 $AA^T = I$.

$$\frac{\partial f}{\partial e_1} = \lim_{t \to 0} \frac{f(\vec{x} + te_1) - f(\vec{x})}{t} = f_x a_{11} + f_y a_{12} + f_z a_{13}$$

$$\Rightarrow \quad LHS = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij} f_j \right)^2$$

（这里记 $f_x = f_1$, $f_y = f_2$, $f_z = f_3$）

$$= \sum_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij}^2 f_j^2 + \sum_{k<l} 2 a_{ik} a_{il} f_k f_l \right)$$

$$= \sum_j \left(\sum_i a_{ij}^2 \right) f_j^2 + \sum_{k<l} 2 f_k f_l \left(\sum_{i=1}^{3} a_{ik} a_{il} \right)$$

$$AA^T = I \Rightarrow \sum_{i=1}^{3} a_{ij}^2 = 1, \quad 1 \le j \le 3$$

$$\begin{pmatrix} a_{11} & \cdots & a_{13} \\ \vdots & & \vdots \\ a_{31} & \cdots & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{31} \\ \vdots & & \vdots \\ a_{13} & \cdots & a_{33} \end{pmatrix} \qquad 及 \quad \sum_{i=1}^{3} a_{ik} a_{il} = 0, \quad k \ne l.$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

9.5-2. $\vec{r}(t)=\left(\frac{2t}{1+t^2},\ \frac{1-t^2}{1+t^2},\ 1\right)$. 求证 $\vec{r}(t)$和$\vec{r}'(t)$正交.

$\vec{r}'(t)=\left(\frac{2-2t^2}{(1+t^2)^2},\ \frac{-4t}{(1+t^2)^2},\ 0\right)$

$\langle \vec{r}(t),\vec{r}'(t)\rangle=\frac{1}{(1+t^2)^3}(4t-4t^3-4t+4t^3)=0$. $x^2+y^2=1\Rightarrow$ 圆.

9.5-3. $\vec{r}(t)=(e^t\cos t,\ e^t\sin t)$ $t\in\mathbb{R}$. $\vec{r}'(t)$, $\vec{r}(t)$ 成夹角.

$\vec{r}'(t)=(e^t\cos t-e^t\sin t,\ e^t\sin t+e^t\cos t)$.

$\langle \vec{r}(t),\vec{r}'(t)\rangle= e^{2t}\cos^2 t-e^{2t}\cos t\sin t+e^{2t}\sin^2 t+e^{2t}\sin t\cos t=e^{2t}$.

$\|\vec{r}(t)\|^2=e^{2t}$, $\|\vec{r}'(t)\|^2=2e^{2t}$.

$\cos\theta=\frac{\langle\vec{r}(t),\vec{r}'(t)\rangle}{\|\vec{r}(t)\|\cdot\|\vec{r}'(t)\|}=\frac{\sqrt{2}}{2}\Rightarrow\theta=\frac{\pi}{4}$. $\theta\in[0,\pi)$.

9.5-5. (1) $(\lambda(t)\cdot\vec{a}(t))'=\lambda'(t)\vec{a}(t)+\lambda(t)\cdot\vec{a}'(t)$.

(2) $(\vec{a}(t)\cdot\vec{b}(t))'=\vec{a}'(t)\cdot\vec{b}(t)+\vec{a}(t)\cdot\vec{b}'(t)$.

(3) $(\vec{a}(t)\times\vec{b}(t))'=\vec{a}'(t)\times\vec{b}(t)+\vec{a}(t)\times\vec{b}'(t)$.

(1) $(\lambda(t)\cdot\vec{a}(t))'=\left((\lambda(t)\cdot a_1(t))',\cdots,(\lambda(t)\,a_m(t))'\right)$

$=\left(\lambda'(t)\cdot a_1(t)+\lambda(t)\,a_1'(t),\cdots,\lambda(t)'\,a_m(t)+\lambda(t)a_m'(t)\right)$

$=\lambda'(t)\,(a_1(t),\cdots,a_m(t))+\lambda(t)\,(a_1'(t),\cdots,a_m'(t))$.

$=\lambda'(t)\,\vec{a}(t)+\lambda(t)\vec{a}'(t)$.

(2) $(\vec{a}(t)\cdot\vec{b}(t))'=\left(\sum_{i=1}^m a_i(t)b_i(t)\right)'=\sum_{i=1}^m(a_i(t)\,b_i(t))'=\sum_{i=1}^m(a_i(t)b_i'(t)+a_i'(t)b_i(t))=\vec{a}(t)\cdot\vec{b}'(t)+\vec{a}'(t)\cdot\vec{b}(t)$

(3) $(\vec{a}(t)\times\vec{b}(t))'=(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)'=(a_2'b_3-a_3'b_2+a_2b_3'-a_3b_2',\ a_3'b_1-a_1'b_3+a_3b_1'-a_1b_3',\ a_1'b_2-a_2'b_1+a_1b_2'-a_2b_1')$

$=\vec{a}'(t)\times\vec{b}(t)+\vec{a}(t)\times\vec{b}'(t)$.

9.5-7 $\begin{cases}x=a\cos t\\ y=b\sin t\end{cases}$ $0\le t\le 2\pi$.

(1) 切向量

(2) 从一焦点出发的光线,经椭圆镜面反射后必经过另一个焦点.

(1) $\vec{r}'(t)=(-a\sin t,\ b\cos t)$.

(2) 即证明 $\forall t$, $\vec{r}(t)-\vec{F}_1$ 与 $\vec{r}'(t)$ 的夹角 θ_1 和 $\vec{r}(t)-\vec{F}_2$ 与 $\vec{r}'(t)$ 的夹角 θ_2 互补. $\vec{F}_1,\vec{F}_2$ 为焦点坐标.

不妨设 $a>b$. 则 $\vec{F}_1=(\sqrt{a^2-b^2},0)$. $\vec{F}_2=(-\sqrt{a^2-b^2},0)$. 设 $c=\sqrt{a^2-b^2}$.

$\langle\vec{r}(t)-\vec{F}_1,\vec{r}'(t)\rangle=-(a\cos t-c)\cdot a\sin t+b\sin t\cdot b\cos t=(b^2-a^2)\sin t\cos t+ac\sin t.=-c^2\sin t\cos t+ac\sin t=c\sin t\,(a-c\cos t)$

$\langle\vec{r}(t)-\vec{F}_2,\vec{r}'(t)\rangle=-(a\cos t+c)\,a\sin t+b\sin t\, b\cos t=(b^2-a^2)\sin t\cos t-ac\sin t=-c^2\sin t\cos t-ac\sin t=-c\sin t\,(a+c\cos t)$

$a>c>c\cos t$.

$\|\vec{r}(t)-\vec{F}_1\|^2=(a\cos t-c)^2+(b\sin t)^2=a^2\cos^2t+b^2\sin^2t+c^2-2ac\cos t=\underbrace{a^2\cos^2t+(b^2+c^2)\sin^2t}_{a^2}+c^2\cos^2t-2ac\cos t=(a-c\cos t)^2$

$\|\vec{r}(t)-\vec{F}_2\|^2=(a\cos t+c)^2+(b\sin t)^2=a^2\cos^2t+b^2\sin^2t+c^2+2ac\cos t=\underline{a^2\cos^2t+(b^2+c^2)\sin^2t}+c^2\cos^2t+2ac\cos t=(a+c\cos t)^2$.

$$\left.\begin{aligned}&\Rightarrow\frac{\langle\vec{r}(t)-\vec{F}_1,\vec{r}'(t)\rangle}{\|\vec{r}(t)-\vec{F}_1\|}=\frac{c\cdot\sin t\,(a-c\cdot\cos t)}{a-c\cdot\cos t}=c\cdot\sin t\\&\frac{\langle\vec{r}(t)-\vec{F}_2,\vec{r}'(t)\rangle}{\|\vec{r}(t)-\vec{F}_2\|}=\frac{-c\cdot\sin t\,(a+c\cdot\cos t)}{a+c\cdot\cos t}=-c\cdot\sin t\end{aligned}\right\}\Rightarrow\underbrace{\frac{\langle\vec{r}(t)-\vec{F}_1,\vec{r}'(t)\rangle}{\|\vec{r}(t)-\vec{F}_1\|\cdot\|\vec{r}'(t)\|}}_{\cos\theta_1}=-\underbrace{\frac{\langle\vec{r}(t)-\vec{F}_2,\vec{r}'(t)\rangle}{\|\vec{r}(t)-\vec{F}_2\|\cdot\|\vec{r}'(t)\|}}_{\cos\theta_2}.$$

$\Rightarrow\theta_1+\theta_2=\pi$ #.

9.5-8.(2) 求曲率: $\vec{r}(t)=(a(3t-t^3),\ 3at^2,\ a(3t+t^3))$. $a>0$.

$\vec{r}'(t)=(3a-3at^2,\ 6at,\ 3a+3at^2)$. $\|\vec{r}'(t)\|^2=9a^2[(1-t^2)^2+4t^2+(1+t^2)^2]=9a^2\cdot 2(t^2+1)^2 \Rightarrow \|\vec{r}'(t)\|=3\sqrt{2}a(1+t^2)$.

$\vec{r}''(t)=(-6at,\ 6a,\ 6at)$. $\|\vec{r}'(t)\times\vec{r}''(t)\|=\|(18a^2t^2-18a^2,\ -36a^2t,\ 18a^2+18a^2t^2)\|$

$=18a^2\cdot\|(t^2-1,\ -2t,\ 1+t^2)\|=18a^2\sqrt{2}(1+t^2)$

$$k(t)=\frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}=\frac{18\sqrt{2}a^2(1+t^2)}{54\sqrt{2}a^3(1+t^2)^3}=\frac{1}{3a(1+t^2)^2}$$

9.5-10. 求法向量和切平面方程.

(3) $3x^2+2y^2-2z-1=0$. $\vec{p}_0=(1,1,2)$.

$F(x,y,z)=3x^2+2y^2-2z-1$.

$\frac{\partial F}{\partial x}=6x$. $\frac{\partial F}{\partial y}=4y$. $\frac{\partial F}{\partial z}=-2$

$JF(\vec{p}_0)=(\frac{\partial F}{\partial x}(\vec{p}_0),\ \frac{\partial F}{\partial y}(\vec{p}_0),\ \frac{\partial F}{\partial z}(\vec{p}_0))=(6,4,-2)$

$\vec{n}=\frac{1}{\sqrt{15}}(3,2,-1)$

切平面. $3(x-1)+2(y-1)-(z-2)=0$.

化简: $3x+2y-z-3=0$.

(4) $z=y+\log\frac{x}{z}$. $\vec{p}_0=(1,1,1)$.

$F(x,y,z)=\log\frac{x}{z}+y-z$.

$\frac{\partial F}{\partial x}=\frac{1}{x}$. $\frac{\partial F}{\partial y}=1$. $\frac{\partial F}{\partial z}=-\frac{1}{z}-1$.

$JF(\vec{p}_0)=(1,1,-2)$.

$\vec{n}=\frac{1}{\sqrt{6}}(1,1,-2)$.

切平面: $(x-1)+(y-1)-2(z-1)=0$

化简: $x+y-2z=0$.

9.5-11. 求 $x^2+2y^2+3z^2=21$ 上所有平行于 $x+4y+6z=0$ 的切平面.

$JF(\vec{x})=(2x,4y,6z)$

在 (x_0,y_0,z_0) 点切平面 $x_0(x-x_0)+2y_0(y-y_0)+3z_0(z-z_0)=0$.

$$\begin{cases} x_0:2y_0:3z_0=1:4:6 \Rightarrow x_0:y_0:z_0=1:2:2 \\ x_0^2+2y_0^2+3z_0^2=21 \Rightarrow x_0^2+8x_0^2+12x_0^2=21 \Rightarrow x_0=\pm1\ \ y_0=z_0=2x_0.\quad \pm(1,2,2). \end{cases}$$

$\therefore$ 2个切平面: $(x-1)+4(y-2)+6(z-2)=0 \Rightarrow x+4y+6z-21=0$

$-(x+1)-4(y+2)-6(z+2)=0 \Rightarrow x+4y+6z+21=0$.

9.5-12. 曲面 $z=xe^{x/y}$ 上所有切平面都通过原点.

$F=xe^{x/y}-z$. $JF=(e^{x/y}+\frac{x}{y}e^{x/y},\ -\frac{x^2}{y^2}e^{x/y},\ -1)$

切平面: $(e^{x_0/y_0}+\frac{x_0}{y_0}e^{x_0/y_0})(x-x_0)-\frac{x_0^2}{y_0^2}e^{x_0/y_0}(y-y_0)-(z-z_0)=0$.

$x=y=z=0$ 代入. LHS $=-x_0e^{x_0/y_0}-\frac{x_0^2}{y_0}e^{x_0/y_0}+\frac{x_0^2}{y_0}e^{x_0/y_0}+z_0$

$=-x_0e^{x_0/y_0}+z_0=0$.

$\Rightarrow (0,0,0)$ 在切平面上. $\forall (x_0,y_0,z_0)$.

9.5-13. 试定出 $\lambda>0$, 使曲面 $xyz=\lambda$ 与 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 在某一点相切(有共同的切平面).

$F_1=xyz-\lambda$. $F_2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$JF_1=(yz,xz,xy)=(\frac{\lambda}{x},\frac{\lambda}{y},\frac{\lambda}{z})$. $JF_2=(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2})$.

$JF_1 \parallel JF_2 \Rightarrow \frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3} \Rightarrow \lambda=xyz=\sqrt{x^2y^2z^2}=\sqrt{\frac{1}{27}a^2b^2c^2}=\frac{abc}{3\sqrt{3}}$.

9.5-16. 求曲面 $x^2+y^2+z^2=x$ 的切平面，使其垂直于平面 $x-y-z=2$ 和 $x-y-\frac{z}{2}=2$.

$F=x^2+y^2+z^2-x$.

$\nabla F=(2x-1,\ 2y,\ 2z)\perp(1,-1,-1)$ 和 $(1,-1,-\frac{1}{2})$.

$$\begin{cases}2x-1-2y-2z=0\\ 2x-1-2y-z=0\end{cases}\Rightarrow\begin{cases}y=\frac{2x-1}{2}\\ z=0\end{cases}\Rightarrow \vec{n}=\frac{1}{\sqrt{2}}(1,1,0)$$

$$\begin{cases}x^2+y^2-x=0\\ 2y=2x-1\end{cases}\Rightarrow x=\frac{2\pm\sqrt{2}}{4},\ y=\pm\frac{\sqrt{2}}{4}$$

切平面为 $(x-\frac{2+\sqrt{2}}{4})+y-\frac{\sqrt{2}}{4}=0$ 和 $(x-\frac{2-\sqrt{2}}{2})+y+\frac{\sqrt{2}}{4}=0$,

化简: $x+y-\frac{1+\sqrt{2}}{2}$ $\qquad x+y-\frac{1-\sqrt{2}}{2}=0$.

9.5-19. 曲面 $F(\frac{x-a}{z-c},\frac{y-b}{z-c})=0$ 的切平面通过一个固定点.

$\frac{\partial F}{\partial x}=F_1\cdot\frac{1}{z-c}$, $\frac{\partial F}{\partial y}=F_2\cdot\frac{1}{z-c}$, $\frac{\partial F}{\partial z}=\frac{x-a}{-(z-c)^2}F_1+\frac{y-b}{-(z-c)^2}F_2$

在 (x_0,y_0,z_0) 处切平面: $\frac{1}{z_0-c}F_1(x-x_0)+\frac{1}{z_0-c}F_2(y-y_0)-\left(\frac{x_0-a}{(z_0-c)^2}F_1+\frac{y_0-b}{(z_0-c)^2}F_2\right)(z-z_0)=0$. F_i 亦是在 (x_0,y_0,z_0) 取值

$F_1(x-x_0)(z_0-c)+F_2(y-y_0)(z_0-c)-F_1(x_0-a)(z-z_0)-F_2(y_0-b)(z-z_0)=0$.

将 (a,b,c) 代入方程, $LHS=F_1(a-x_0)(z_0-c)+F_2(b-y_0)(z_0-c)-F_1(x_0-a)(c-z_0)-F_2(y_0-b)(c-z_0)=0$.

过固定点 (a,b,c).

9.5-22. E, F, G.

(1) 椭球面: $\vec{r}(u,v)=(a\sin u\cos v,\ b\sin u\sin v,\ c\cos u)$

(2) 单叶双曲面: $\vec{r}(u,v)=(a\cosh u\cos v,\ b\cosh u\sin v,\ c\sinh u)$

(3) 椭圆抛物面: $\vec{r}(u,v)=(u,v,\frac{1}{2}(\frac{u^2}{a^2}+\frac{v^2}{b^2}))$.

(1) $\vec{r}_u=(a\cos u\cos v,\ b\cos u\sin v,\ -c\sin u)$ $\qquad \vec{r}_v=(-a\sin u\sin v,\ b\sin u\cos v,\ 0)$.

$E=\|\vec{r}_u\|^2=a^2\cos^2u\cos^2v+b^2\cos^2u\sin^2v+c^2\sin^2u$.

$F=\langle\vec{r}_u,\vec{r}_v\rangle=-a^2\cos u\sin u\cos v\sin v+b^2\cos u\sin u\cos v\sin v$.

$G=\|\vec{r}_v\|^2=a^2\sin^2u\sin^2v+b^2\sin^2u\cos^2v$.

(2) $\vec{r}_u=(a\sinh u\cdot\cos v,\ b\sinh u\cdot\sin v,\ c\cdot\cosh u)$. $\qquad \vec{r}_v=(-a\cdot\cosh u\cdot\sin v,\ b\cosh u\cdot\cos v,\ 0)$.

$E=\|\vec{r}_u\|^2=a^2\sinh^2u\cos^2v+b^2\sinh^2u\cdot\sin^2v+c^2\cosh^2u$

$F=\langle\vec{r}_u,\vec{r}_v\rangle=-a^2\sinh u\cosh u\sin v\cos v+b^2\sinh u\cdot\cosh u\cdot\sin v\cos v$

$G=\|\vec{r}_v\|^2=a^2\cosh^2u\sin^2v+b^2\cosh^2u\cos^2v$.

(3) $\vec{r}_u=(1,0,\frac{u}{a^2})$. $\qquad \vec{r}_v=(0,1,\frac{v}{b^2})$.

$E=\|\vec{r}_u\|^2=1+\frac{u^2}{a^4}$.

$F=\langle\vec{r}_u,\vec{r}_v\rangle=\frac{uv}{a^2b^2}$

$G=\|\vec{r}_v\|^2=1+\frac{v^2}{b^4}$.

9.5-23. $I := E du^2 + 2F du dv + G dv^2$. $I = d\vec{r}^2$. $\vec{r} = \vec{r}(u,v)$.

$d\vec{r} = \vec{r}_u du + \vec{r}_v dv$

$d\vec{r}^2 = \langle \vec{r}_u du + \vec{r}_v dv, \vec{r}_u du + \vec{r}_v dv \rangle$

$= \|\vec{r}_u\|^2 du^2 + \vec{r}_u \cdot \vec{r}_v du dv + \vec{r}_v \cdot \vec{r}_u dv du + \|\vec{r}_v\|^2 dv^2$

$= E du^2 + 2F du dv + G dv^2$.

9.5-24. 已知曲面 $I = du^2 + (u^2+a^2) dv^2$. 求曲面上曲线 $u=v$ 从 v_1 到 v_2 的弧长. $v_2 > v_1$.

$\|\vec{r}_u\|^2 = 1$. $\|\vec{r}_v\|^2 = u^2+a^2$. $\langle \vec{r}_u, \vec{r}_v \rangle = 0$.

$u=v$ 曲线 $\vec{r}(v)$. $\vec{r}'(v) = \vec{r}_u \cdot \frac{\partial u}{\partial v} + \vec{r}_v \cdot \frac{\partial v}{\partial v} = \vec{r}_u + \vec{r}_v$.

$\Rightarrow \|\vec{r}'(v)\|^2 = \|\vec{r}_u\|^2 + 2\langle \vec{r}_u, \vec{r}_v \rangle + \|\vec{r}_v\|^2 = 1+u^2+a^2 = 1+v^2+a^2$.

$S = \int_{v_1}^{v_2} \|\vec{r}'(v)\| dv = \int_{v_1}^{v_2} \sqrt{1+a^2+v^2}\, dv = \int_{v_1}^{v_2} \sqrt{1+a^2} \cdot \sqrt{1+\frac{v^2}{1+a^2}}\, dv = \frac{1}{2} v\sqrt{1+a^2+v^2} + \frac{1}{2}(1+a^2) \log\left(v + \sqrt{1+a^2+v^2}\right)\Big|_{v_1}^{v_2}$.

补. 一、(15分) 去年小测1.

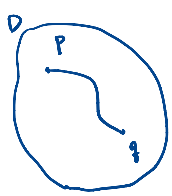
得分	

1. 叙述 $\mathbb{R}$ 上的开集与连通集合的定义; 2+2

2. 证明 $\mathbb{R}$ 的非空连通开集一定是开区间. 2+2

3. 已知 $D \subset \mathbb{R}^n$ 为区域, p, q 为 D 中不同的两点, 证明在 D 中存在 $\mathbb{R}^n$ 的闭区域 E 满足 $p, q \in E^\circ$. 4+3

Proof: 3.



1° 构造 E.

D 是区域 $\Rightarrow$ 道路连通. $\Rightarrow \exists$ 连续曲线 $l \in D$ 连接 p, q.

$\forall a \in l \subset D$ 开集. $\exists \overline{B_{r(a)}(a)} \subset D$.

$\{B_{r(a)}(a)\}_{a \in l}$ 构成紧集 l 的开覆盖.

$\exists$ 有限覆盖 $\{B_{r_i}(a_i)\}_{i=1}^{N} \supset l$.

取 $E = \bigcup_{i=1}^{N} \overline{B_{r_i}(a_i)}$.

2° E 是闭区域.

(a) E 闭: 闭集的有限并是闭集.

(b) E° 是区域, 即连通开集.

$E^\circ = \bigcup_{i=1}^{N} B_{r_i}(a_i)$

$\forall c \neq d, c, d \in E^\circ$.

$\exists m, n$. $c \in B_{r_m}(a_m)$. $d \in B_{r_n}(a_n)$.

若 $m=n$. 则由 $B_r(a)$ 连通得其道路.

若 $m \neq n$. 则 c, d 间有道路 $c - a_m - a_n - d$

$B_{r_m}(a_m)$ 连通 $\quad l \quad B_{r_n}(a_n)$ 连通.

$\Rightarrow E^\circ$ 道路连通 $\Rightarrow$ 连通.

8.3-5. $\partial A=\overline{A}\cap(A^\circ)^c$

$x\in\partial A\Leftrightarrow \forall r>0\ \left\{\begin{array}{l} B_r(x)\cap A\neq\phi \overset{8.3\text{-}3}{\Longleftrightarrow} x\in\overline{A} \\ B_r(x)\cap A^c\neq\phi \Leftrightarrow B_r(x)\not\subset A\Leftrightarrow x\notin A^\circ \end{array}\right\}\Leftrightarrow x\in\overline{A}\cap(A^\circ)^c.$

8.3-7. (1) $(A\cap B)^\circ=A^\circ\cap B^\circ$. (2) $\overline{A\cup B}=\overline{A}\cup\overline{B}$.

(1) $x\in(A\cap B)^\circ\Leftrightarrow\exists r>0.\ B_r(x)\subset A\cap B\Leftrightarrow\left\{\begin{array}{l} B_r(x)\subset A\Leftrightarrow x\in A^\circ \\ B_r(x)\subset B\Leftrightarrow x\in B^\circ \end{array}\right\}\Leftrightarrow x\in A^\circ\cap B^\circ.$

(2) ① $\overline{A\cup B}\subset\overline{A}\cup\overline{B}$.

$x\in\overline{A\cup B}\Rightarrow\ x\in A\cup B\subset\overline{A}\cup\overline{B}$.

这里注意不是$\{x_n\}\subset A$或$\{x_n\}\subset B$.

或 $x\in(A\cup B)'\Rightarrow\exists\{x_n\}_{n=1}^{\infty}\subset A\cup B.\ x_n\to x\Rightarrow\left\{\begin{array}{l}\{x_n\}\text{中只有有限项不同}\Rightarrow x\in A\cup B\subset\overline{A}\cup\overline{B}. \\ \{x_n\}\text{中有无限项在}A\text{中或}B\text{中}\Rightarrow x\in A'\subset\overline{A}\text{或}x\in B'\subset\overline{B}.\end{array}\right.$

$\therefore x\in\overline{A}\cup\overline{B}$

$(\forall r>0.\ B_r(\mathring{x})\cap(A\cup B)'\not\Rightarrow \forall r>0\ B_r(\mathring{x})\cap A\neq\phi$ or $\forall r>0\ B_r(\mathring{x})\cap B\neq\phi)$

$(0,+\infty)=\{r\mid B_r(\mathring{x})\cap A\neq\phi\}\cup\{r\mid B_r(\mathring{x})\cap B\neq\phi\}$

可能$B_r(\mathring{x})\cap B=\phi$ 可能$B_r(\mathring{x})\cap A=\phi$.

② $x\in\overline{A}\cup\overline{B}\Rightarrow\ x\in\overline{A}\subset\overline{A\cup B}$ 或 $x\in\overline{B}\subset\overline{A\cup B}$.

8.3-8 (1) 闭集列$\{F_i\}$. 使得$\bigcup_{i=1}^{\infty}F_i=B_1(0)$. $F_i=\{x\in\mathbb{R}^n:\|x\|\leq 1-\frac{1}{i}\}=\overline{B_{1-\frac{1}{i}}(0)}$.

(2) 开集列$\{G_i\}$. 使得$\bigcap_{i=1}^{\infty}G_i=\overline{B_1(0)}$ $G_i=\{x\in\mathbb{R}^n:\|x\|<1+\frac{1}{i}\}=B_{1+\frac{1}{i}}(0)$.

8.3-9 I为指标集. (1) $\overline{\bigcap_{\alpha\in I}A_\alpha}\subset\bigcap_{\alpha\in I}\overline{A_\alpha}$ (2) $\left(\bigcup_{\alpha\in I}A_\alpha\right)^\circ\supset\bigcup_{\alpha\in I}A_\alpha^\circ$. 举例真包含.

(1) $A_\alpha\subset\overline{A_\alpha}\Rightarrow\bigcap_{\alpha\in I}A_\alpha\subset\bigcap_{\alpha\in I}\overline{A_\alpha}$

$\Rightarrow\overline{\bigcap_{\alpha\in I}A_\alpha}\subset\overline{\bigcap_{\alpha\in I}\overline{A_\alpha}}\overset{\text{闭集}}{=}\bigcap_{\alpha\in I}\overline{A_\alpha}$

$A_1=(0,1).\ A_2=(1,2)$.

$(\overline{A_1\cap A_2}=\phi)\subsetneqq(\overline{A_1}\cap\overline{A_2}=\{1\})$

(2) $A_\alpha^\circ\subset A_\alpha\Rightarrow\bigcup_{\alpha\in I}A_\alpha^\circ\subset\bigcup_{\alpha\in I}A_\alpha$

$\Rightarrow\bigcup_{\alpha\in I}A_\alpha^\circ\overset{\text{开集}}{=}\left(\bigcup_{\alpha\in I}A_\alpha^\circ\right)^\circ\subset\left(\bigcup_{\alpha\in I}A_\alpha\right)^\circ$

$A_1=[0,1].\ A_2=[1,2]$.

$(A_1^\circ\cup A_2^\circ=(0,2)\setminus\{1\})\subsetneqq((A_1\cup A_2)^\circ=(0,2))$.

8.3-10. 设$E\subset\mathbb{R}^n$. 求证: ∂E闭集.

书上定义$\partial E=\mathbb{R}^n\setminus(\underset{\text{开}}{\underbrace{E^\circ\cup(E^c)^\circ}})\Rightarrow\partial E$闭.

另一种定义$x\in\partial E\Leftrightarrow B_r(x)\cap E\neq\phi$ 且 $B_r(x)\cap E^c\neq\phi\ \ \forall r$.

$\Rightarrow x\in(\partial E)^c\Leftrightarrow\ \exists r>0.\ B_r(x)\cap E=\phi$ 或 $B_r(x)\cap E^c=\phi$.

$\Rightarrow\forall y\in B_r(x)$. 取$r'=r-\|x-y\|$. $B_{r'}(y)\subset B_r(x)\Rightarrow B_{r'}(y)\cap E=\phi$ 或 $B_{r'}(y)\cap E^c=\phi$

$\Rightarrow y\in(\partial E)^c\ \Rightarrow\ B_r(x)\subset(\partial E)^c$.

8.3-12 P投影算子. $E\subset\mathbb{R}^2$开$\Rightarrow P(E)$开. 举例说明$A\subset\mathbb{R}^2$闭. $P(A)$不一定闭.

$\forall x\in P(E)\quad\exists(x,y)\in E\Rightarrow\exists r$ s.t. $B_r((x,y))\subset E$ $A=\{(x,\frac{1}{x}):x>0\}$.闭.

$\Rightarrow(x-r,x+r)\subset P(E)\Rightarrow P(E)$开. $P(A)=(0,+\infty)$非闭

8.3-13. E闭 $\Leftrightarrow \partial E\subset E$.

"$\Rightarrow$": $x\in\partial E \Leftrightarrow \forall r>0.\ B_r(x)\cap E\neq\phi$ 且 $B_r(x)\cap E^c\neq\phi$.

$\Rightarrow B_r(x)\not\subset E^c.\ \forall r.$ E闭 $\Rightarrow E^c$开

$\Rightarrow x\notin E^c$ (否则 $\exists r>0.\ B_r(x)\subset E^c$).

$\Rightarrow x\in E$.

"$\Leftarrow$": $\forall x\in E^c.\ x\notin\partial E \Rightarrow \exists r>0\ B_r(x)\cap E=\phi$ 或 $B_r(x)\cap E^c=\phi$.

$x\in E^c \Rightarrow B_r(x)\cap E=\phi \Rightarrow B_r(x)\subset E^c \Rightarrow E^c$开 $\Rightarrow E$闭.

8.4-1 P投影算子. $A\subset\mathbb{R}^2$ 紧致集 $\Rightarrow P(A)$紧致集.

① 紧致$\Leftrightarrow$列紧.

$\forall \{x_n\}_{n=1}^{\infty}\subset P(A)$. $\exists\{(x_n,y_n)\}_{n=1}^{\infty}\subset A$.

A列紧 $\Rightarrow$ 有收敛子列 $\{(x_{n_i},y_{n_i})\}_{i=1}^{\infty}$ 收敛到 $(x,y)\in A$.

收敛$\Leftrightarrow$按分量收敛. $\therefore\{x_{n_i}\}_{i=1}^{\infty}\to x\in P(A)$.

$\therefore P(A)$列紧.

② 直接应用定义.

要证$P(A)$的任意开覆盖有有限子覆盖.

设$\{G_\alpha\}_{\alpha\in I}$为$P(A)$的任意一个开覆盖.

记 $\tilde{G}_\alpha=\{(x,y): x\in G_\alpha, y\in\mathbb{R}\}$.

$\tilde{G}_\alpha$开集: $\forall(\tilde{x},\tilde{y})\in\tilde{G}_\alpha$. 由于$G_\alpha$是开集. $\exists r_0.$ s.t. $(\tilde{x}-r_0,\tilde{x}+r_0)\subset G_\alpha$.

$\therefore B_{r_0}((\tilde{x},\tilde{y}))\subset(\tilde{x}-r_0,\tilde{x}+r_0)\times(\tilde{y}-r_0,\tilde{y}+r_0)\subset G_\alpha\times\mathbb{R}=\tilde{G}_\alpha$

$\therefore\{\tilde{G}_\alpha\}_{\alpha\in I}$ 是A的开覆盖. $\Rightarrow \exists A$的有限覆盖$\{\tilde{G}_i\}_{i=1}^{m}$

则 $\forall x\in P(A)$. $\exists(x,y)\in A \Rightarrow \exists\ 1\le i\le m.$ s.t. $(x,y)\in\tilde{G}_i$

$\Rightarrow x\in G_i \Rightarrow \{G_i\}_{i=1}^{m}$ 是$P(A)$的有限覆盖.

8.4-2. $A,B\subset\mathbb{R}$. 证明: $A\times B$紧致$\Leftrightarrow A,B$紧致.

"$\Rightarrow$": 由上题.

"$\Leftarrow$": 紧致$\Leftrightarrow$列紧. 同上题.

8.4-3 $A\subset\mathbb{R}^n$紧致 $\Leftrightarrow$ 若$\mathcal{F}=\{A_\alpha\}$是$\mathbb{R}^n$中闭集族且 $A\cap(\bigcap_\alpha A_\alpha)=\phi$. 则$\exists A_1,\cdots,A_k\in\mathcal{F}$ 使 $A\cap(\bigcap_{i=1}^{k}A_i)=\phi$.

A紧 $\overset{定义}{\Leftrightarrow}$ 任意A的有限开覆盖$\{G_\alpha\}$. $\exists G_1,\cdots,G_k.\ A\subset\bigcup_{i=1}^{k}G_i$.

"$\Rightarrow$": 任意满足 $A\cap(\bigcap_\alpha A_\alpha)=\phi$ 的闭集族$\{A_\alpha\}$. 有 $A\subset(\bigcap_\alpha A_\alpha)^c=\bigcup_\alpha A_\alpha^c$. $\{A_\alpha^c\}$为A的开覆盖.

$\Rightarrow \exists A_1,\cdots,A_k.$ $A\subset\bigcup_{i=1}^{k}A_i^c=(\bigcap_{i=1}^{k}A_i)^c \Rightarrow A\cap\bigcap_{i=1}^{k}A_i=\phi$.

"$\Leftarrow$": 设$\{G_\alpha\}$是A的开覆盖 $A\subset\bigcup_\alpha G_\alpha \Rightarrow \phi=A\cap(\bigcup_\alpha G_\alpha)^c=A\cap(\bigcap_\alpha G_\alpha^c)$. G_α^c闭.

故$\{G_\alpha^c\}$是满足 $A\cap(\bigcap_\alpha G_\alpha^c)=\phi$的闭集族 $\Rightarrow\exists G_1,\cdots,G_k$ s.t. $A\cap(\bigcap_{i=1}^{k}G_i^c)=\phi$

$\Rightarrow A\subset(\bigcap_{i=1}^{k}G_i^c)^c=\bigcup_{i=1}^{k}G_i \Rightarrow$ 有有限覆盖$\{G_i\}_{i=1}^{k} \Rightarrow A$紧致.

8.4-4. Def: Fréchet 紧：$A\subset\mathbb{R}^n$ 的~~每~~一个无穷子集在A中有一个凝聚点.

证明：Fréchet紧 $\Leftrightarrow$ 列紧.

"$\Rightarrow$"：列紧 $\Leftrightarrow$ A中任一点列有收敛子列.

设 $\{x_n\}_{n=1}^{\infty}\subset A$ 为任意点列.

则 $\{x_n\}$ 为A的子集 $\begin{cases}\{x_n\}\text{为有限集合(不同项只有有限个)：显然有收敛子列.}\\ \{x_n\}\text{为无穷子集：由Fréchet紧. 在A中有凝聚点}x\Rightarrow\exists\text{子列收敛到}x.\end{cases}$

"$\Leftarrow$" 任取A的子集E. 将其看成A中点列.

则由列紧. E中必有收敛子列 $\{x_n\}\to x\in A$. ← 注意这里不能直接得到x为凝聚点.

由于E是无穷子集. $\not\exists x_n$. s.t. $x_n=x$. $\Rightarrow$ $\forall r>0$. $\exists N$. s.t. $x_N\in B_r(\check{x})\neq\phi$.

$\Rightarrow$ x是凝聚点. $\Rightarrow$ Fréchet紧.

8.4-5. 设 $F_1,\cdots,F_k,\cdots$ 是 $\mathbb{R}^n$ 中 $\begin{cases}\text{非空闭集}\\ \text{非空紧致集}\end{cases}$. 满足 $F_k\supset F_{k+1}(k\geq1)$. 是否一定 $\bigcap_{k=1}^{\infty}F_k\neq\phi$?

闭：不一定 $\mathbb{R}$：$F_k=[k,+\infty)$. $\bigcap_{k=1}^{\infty}F_k=\phi$.

紧：一定. 任取 $x_k\in F_k$. F_k为非空紧集 $\Rightarrow$ $\{x_k\}\subset F_1$ 是有界列.

由 Bolzano-Weierstraß Thm. $\exists$子列 $\{x_{k_i}\}\to x$. as $i\to\infty$.

且 $x_{k_i}\in\bigcap_{k=1}^{k_i}F_k\overset{i\to\infty}{\Rightarrow}x\in\bigcap_{k=1}^{\infty}F_k$.

8.5-1. 区域是道路连通的.

区域 $\Leftrightarrow$ 非空连通开集. (老师上课讲了).

反证. $\exists p.q\in D$. p.q间不存在D中道路.

记 $A=\{a\in D.\ a\text{与}p\text{之间有D中道路相连}\}$. $B=\{b\in D.\ b\text{与}p\text{之间无D中道路相连}\}$.

D开 $\Rightarrow B_r(p)\subset D$. 且 $B_r(p)$道路连通 $\Rightarrow B_r(p)\subset A\Rightarrow A\neq\phi$.

$q\in B\Rightarrow B\neq\phi$. 而 $A\cap B=\phi$. $D=A\cup B$.

下证A.B均为开集. 从而导出矛盾.

A开：$\forall x\in A\subset D$. $\exists r>0$. $B_r(x)\subset D$. $B_r(x)$连通 $\Rightarrow B_r(x)\subset A$.

B开：$\forall x\in B\subset D$. $\exists r>0$. $B_r(x)\subset D$. $B_r(x)$连通 $\Rightarrow B_r(x)\subset B$.

8.5-2. $A\subset\mathbb{R}^n$. 若A既开又闭. 求证 $A=\mathbb{R}^n$ 或 $A=\phi$.

反证. 假设A既开又闭. 且 $A\neq\mathbb{R}^n$. $A\neq\phi$. 则 $A^c=\mathbb{R}^n\setminus A\neq\phi$.

$\mathbb{R}^n=A\cup A^c$ 为 $\mathbb{R}^n$ 的剖分. $A\neq\phi$. $A^c\neq\phi$. A为开集

A闭 $\Rightarrow A^c$开 $\Rightarrow\mathbb{R}^n$不连通. 矛盾.

8.6-3. (2) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = 1$.

$\left|x^2y^2\log(x^2+y^2)\right| \leq (x^2+y^2)^2\left|\log(x^2+y^2)\right| \to 0$. $(\lim\limits_{t\to 0} t^2\log t = 0)$

(4) $\lim\limits_{\substack{x\to+\infty\\y\to+\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2} = 0$.

$0 < \frac{xy}{x^2+y^2} \leq \frac{\frac{x^2+y^2}{2}}{x^2+y^2} = \frac{1}{2} < 1$

$0 < \left(\frac{xy}{x^2+y^2}\right)^{x^2} \leq \left(\frac{1}{2}\right)^{x^2} \to 0$ as $x\to+\infty$.

(6) $\lim\limits_{\substack{x\to+\infty\\y\to+\infty}} (x^2+y^2)e^{-(x+y)} = 0$.

$(x^2+y^2)e^{-(x+y)} < (x+y)^2 e^{-(x+y)} \to 0 \qquad (\lim\limits_{t\to+\infty} t^2e^{-t} = 0)$.

8.6-5. (1) $f(x,y) = \frac{x^2y}{x^4+y^2}$ 在$(0,0)$的两个累次极限.

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} \frac{x^2y}{x^4+y^2} = 0$. $\lim\limits_{y\to 0}\lim\limits_{x\to 0} \frac{x^2y}{x^4+y^2} = 0$.

(2) 计算 $\lim\limits_{x\to\infty}\underbrace{\lim\limits_{y\to\infty} \sin\frac{\pi x}{2x+y}}_{=0}$. $\lim\limits_{y\to\infty}\underbrace{\lim\limits_{x\to\infty} \sin\frac{\pi x}{2x+y}}_{=1}$.

(3) 计算 $\lim\limits_{x\to+\infty}\underbrace{\lim\limits_{y\to 0^+} \frac{x^y}{1+xy}}_{=\frac{1}{2}}$. $\lim\limits_{y\to 0^+}\underbrace{\lim\limits_{x\to+\infty} \frac{x^y}{1+xy}}_{=1}$.

8.6-6. $f(x,y) = (x+y)\sin\frac{1}{x}\sin\frac{1}{y}$. 证明: 2个累次极限均不存在. 但 $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

$\lim\limits_{y\to 0}(x+y)\sin\frac{1}{x}\sin\frac{1}{y} = \lim\limits_{y\to 0} x\sin\frac{1}{x}\sin\frac{1}{y}$ 不存在. x同理 $\Rightarrow$ 累次极限不存在.

$0 < \left|(x+y)\sin\frac{1}{x}\sin\frac{1}{y}\right| \leq |x+y| \to 0$ as $(x,y)\to(0,0)$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

8.6-7. $\lim\limits_{\substack{x\to x_0\\y\to y_0}} f(x,y) = a$存在. 又对$y_0$近旁的每一个$y$. 极限$\lim\limits_{x\to x_0} f(x,y) = h(y)$存在. 证明: $\lim\limits_{y\to y_0} h(y) = a$.

$\lim\limits_{y\to y_0} h(y) = a \Leftrightarrow \quad \forall\varepsilon>0.\ \exists\delta>0.\ \|y-y_0\|<\delta$时 $|h(y)-a|<\varepsilon$.

已知: $\forall\varepsilon>0.\ \exists\delta>0.\ \|(x,y)-(x_0,y_0)\|<\delta$ 时. $|f(x,y)-a|<\varepsilon$.

$\overline{\lim\limits_{x\to x_0}}|f(x,y)-a| \leq \varepsilon \Rightarrow |h(y)-a| \leq \varepsilon$ 对$\|y-y_0\|<\delta$成立.

$\Rightarrow \lim\limits_{y\to y_0} h(y) = a$.

$\forall\varepsilon>0.\ \exists\delta_1>0.\ \|(x,y)-(x_0,y_0)\|<\delta_1.\ |f(x,y)-a|<\varepsilon$.

$\exists\delta_2(y)>0.\ \|x-x_0\|<\delta_2(y).\ |f(x,y)-h(y)|<\varepsilon(y)$.

取$\delta = \inf\limits_{y\in B_r(y_0)}\{\delta_1,\delta_2(y)\}$可能不存在.

$\to$ y_0近旁的每一个y.

8.6-8. $f(x,y)$在某点处2个累次极限和极限都存在. 则3个值相等

($\Rightarrow$ 2个累次极限存在且不等 $\Rightarrow$ 极限不存在).

由累次极限定义和8.6-7可直接得出.

2021.3.26 习题课.

开集 $\xrightarrow{\text{余集}}$ 闭集

	开集	闭集
判定	$\forall x\in A, \exists r>0, B_r(x)\subset A$.	A中任意收敛点列的极限在A中. (注意这不是定义).
反面	$\exists x_0\in A, \forall r>0, B_r(x_0)\cap A^c\neq\phi$.	$\exists\{x_n\}_{n=1}^{\infty}\subset A$, $\lim_{n\to\infty} x_n\notin A$.
性质	任意并, 有限交为开集.	任意交, 有限并为闭集.
等价关系	$A^\circ=A$, $\partial A\cap A=\phi$	$\bar{A}=A$, $A'\subset A$, $\partial A\subset A$.

点: 内点: $x\in A^\circ \Leftrightarrow \exists r>0. B_r(x)\subset A$.

$x\notin A^\circ \Leftrightarrow \forall r>0. B_r(x)\cap A^c\neq\phi$.

边界: $\partial A=\mathbb{R}^n\setminus(A^\circ\cup(A^c)^\circ)$.

边界点: $x\in\partial A \Leftrightarrow \forall r>0. B_r(x)\cap A\neq\phi. B_r(x)\cap A^c\neq\phi$.

$x\notin\partial A \Leftrightarrow \exists r_0>0. B_{r_0}(x)\cap A=\phi$ 或 $B_r(x)\cap A^c=\phi$

凝聚点: $x\in A' \Leftrightarrow \forall r>0. B_r(\mathring{x})\cap A\neq\phi$. (注意要去心.) $x\in A'\Rightarrow\exists\{x_n\}\subset A. x_n\to x$.

$x\notin A' \Leftrightarrow \exists r>0. B_r(\mathring{x})\cap A=\phi$.

孤立点: $\forall r>0. B_r(x)\cap A=\{x\}$. 一定是边界点.

Thm (闭集套定理) $\mathbb{R}^n$. $\{D_i\}_{i=1}^{\infty}$ $\forall i. D_i\subset\mathbb{R}^n$.

(1) $\forall i$. D_i为$\mathbb{R}^n$中闭集

(2) $\forall i$. $D_i\supset D_{i+1}$.

(3) $\operatorname{diam} D_i\to 0$ $(i\to+\infty)$. ←注意这个条件.

则 $\exists!\, x\in\mathbb{R}^n$. $\bigcap_{i=1}^{\infty}D_i=\{x\}$.

关于有界闭集. 等价性证明!

列紧集: D中任一点列$\{x_n\}$必有收敛子列. 且极限在D中.

紧致集: D的任意开覆盖$\{E_i\}_{i\in A}$. 必定存在有限个开集$E_1,\cdots,E_k$. $\bigcup_{i=1}^{k}E_i\supset D$.

连通性: $D\neq\phi$. (不讨论空集的连通性.)

$D\subset\mathbb{R}^n$ 连通 $\Leftrightarrow$ 对于D的任意剖分. $D=A\cup B$. $A.B\neq\phi$. $A\cap B=\phi$. 则 $A\cap B'\neq\phi$ 或 $A'\cap B\neq\phi$ 至少有一个成立.

$D\subset\mathbb{R}^n$ 不连通 $\Leftrightarrow$ $\exists A.B\neq\phi$. $D=A\cup B$ $A\cap B=\phi$. 且 $A\cap B'=\phi$. $A'\cap B=\phi$.

区域$\Leftrightarrow$连通开集.

闭区域$\Leftrightarrow$区域的闭包$\neq$连通闭集.

多变量函数的极限 ← 单变量.

	道路连通		连通
$\mathbb{R}$		$\Leftrightarrow$	$\Leftrightarrow$ 区间
$\mathbb{R}^n(n>1)$		$\Rightarrow$	参考问题8.5-2.
$\mathbb{R}^n$开集		$\Leftrightarrow$	

第七周作业主要问题

@rosefantasie

2021 年 4 月 30 日

1 计算结果化简。但 Taylor 展开按照定义式不应该化简 (**9.10-1,2,3**)。

2 **9.9-6** 注意 $u = u(x, y, z)$.

3 判断是否极值：判断是极值只需按照课本定理，$\det Hf = 0$ 得到的结论是无法判断而非不是极值，需要通过别的途径（比如极值的定义）证明其非极值。

4 **9.11-2** $Jf = (0, 0)$ 的解一共有 9 组，不要漏解。

5 求极值的题要把极值求出来，而不是只写出极值点。

6 说明：这次作业扣分比较严格，尤其是对 Taylor 展开定义的理解不准确之处，因为 Taylor 展开比较重要.

5

9.12-1. 求条件极值.

(3) $u=x-2y+2z$, $x^2+y^2+z^2=1$.

$F(x,y,z)=x-2y+2z-\lambda(x^2+y^2+z^2-1)$

$$\begin{cases}\frac{\partial F}{\partial x}=1-2\lambda x=0\\ \frac{\partial F}{\partial y}=-1-2\lambda y=0\\ \frac{\partial F}{\partial z}=2-2\lambda z=0\\ x^2+y^2+z^2=1.\end{cases}\Rightarrow\begin{cases}x=\frac{1}{2\lambda}\\ y=-\frac{1}{\lambda}.\\ z=\frac{1}{\lambda}.\end{cases}\quad \frac{1}{4\lambda^2}+\frac{1}{\lambda^2}+\frac{1}{\lambda^2}=1$$

$\Rightarrow \frac{9}{4}\cdot\frac{1}{\lambda^2}=1 \Rightarrow \lambda=\pm\frac{3}{2}$.

$$\Rightarrow\begin{cases}\lambda=\frac{3}{2}\\ x=\frac{1}{3}\\ y=-\frac{2}{3}\\ z=\frac{2}{3}\end{cases}\text{ or }\begin{cases}\lambda=-\frac{3}{2}\\ x=-\frac{1}{3}\\ y=\frac{2}{3}\\ z=-\frac{2}{3}\end{cases}$$

$u(\frac{1}{3},-\frac{2}{3},\frac{2}{3})=3$. $u(-\frac{1}{3},\frac{2}{3},-\frac{2}{3})=-3$.

(4) $u=3x^2+3y^2+z^2$, $x+y+z=1$.

$F(x,y,z)=3x^2+3y^2+z^2-\lambda(x+y+z-1)$.

$$\begin{cases}\frac{\partial F}{\partial x}=6x-\lambda.\\ \frac{\partial F}{\partial y}=6y-\lambda.\\ \frac{\partial F}{\partial z}=2z-\lambda.\\ x+y+z=1\end{cases}\Rightarrow\begin{cases}x=\frac{\lambda}{6}\\ y=\frac{\lambda}{6}\\ z=\frac{\lambda}{2}.\end{cases}\quad \frac{\lambda}{6}+\frac{\lambda}{6}+\frac{\lambda}{2}=1\Rightarrow\lambda=\frac{6}{5}.$$

$\Rightarrow x=\frac{1}{5}.\ y=\frac{1}{5}.\ z=\frac{3}{5}$.

$u(\frac{1}{5},\frac{1}{5},\frac{3}{5})=\frac{3}{5}$ 极小.

9.12-2. 计算

(1) 原点到 $\begin{cases}2x+2y+z+9=0\\ 2x-y-2z-18=0.\end{cases}$ 的距离.

$d^2=x^2+y^2+z^2$.

$F(x,y,z)=x^2+y^2+z^2-\lambda_1(2x+2y+z+9)-\lambda_2(2x-y-2z-18)$

$$\begin{cases}\frac{\partial F}{\partial x}=2x-2\lambda_1-2\lambda_2.\\ \frac{\partial F}{\partial y}=2y-2\lambda_1+\lambda_2.\\ \frac{\partial F}{\partial z}=2z-\lambda_1+2\lambda_2.\\ 2x+2y+z+9=0.\\ 2x-y-2z-18=0.\end{cases}\quad\begin{cases}x=\lambda_1+\lambda_2\\ y=\lambda_1-\frac{\lambda_2}{2}.\\ z=\frac{\lambda_1}{2}-\lambda_2.\end{cases}$$

$$\begin{cases}2\lambda_1+2\lambda_2+2\lambda_1-\lambda_2+\frac{\lambda_1}{2}-\lambda_2+9=0.\\ 2\lambda_1+2\lambda_2-\lambda_1+\frac{\lambda_2}{2}-\lambda_1+2\lambda_2-18=0.\end{cases}\Rightarrow\begin{cases}\frac{9}{2}\lambda_1+9=0.\\ \frac{9}{2}\lambda_2-18=0\end{cases}\Rightarrow\begin{cases}\lambda_1=-2\\ \lambda_2=4.\end{cases}$$

$$\Rightarrow\begin{cases}x=2\\ y=-4\\ z=-5\end{cases}\quad d_{min}=\sqrt{2^2+4^2+5^2}=3\sqrt{5}.$$

(2) 原点到 $x+2y+3z+4=0$. 的距离.

$F(x,y,z)=x^2+y^2+z^2-\lambda(x+2y+3z+4)$.

$$\begin{cases}\frac{\partial F}{\partial x}=2x-\lambda.\\ \frac{\partial F}{\partial y}=2y-2\lambda.\\ \frac{\partial F}{\partial z}=2z-3\lambda.\\ x+2y+3z+4=0.\end{cases}\Rightarrow\begin{cases}x=\frac{\lambda}{2}\\ y=\lambda.\\ z=\frac{3}{2}\lambda.\end{cases}\quad(\frac{1}{2}+2+\frac{9}{2})\lambda+4=0\Rightarrow\lambda=-\frac{4}{7}$$

$$\Rightarrow\begin{cases}x=-\frac{2}{7}\\ y=-\frac{4}{7}\\ z=-\frac{6}{7}\end{cases}\quad d_{min}=\frac{2}{7}\sqrt{1+2^2+3^2}=\frac{2}{7}\sqrt{14}.$$

9.12-4. 设 $a>0$. 求 $\begin{cases} x^2+y^2=2az \\ x^2+y^2+xy=a^2 \end{cases}$ 上的点到 Oxy 平面的最小距离和最大距离.

$$F(x,y,z)=z^2+\lambda_1(x^2+y^2-2az)-\lambda_2(x^2+y^2+xy-a^2).$$

$$\begin{cases} \frac{\partial F}{\partial x}=-2\lambda_1 x-2\lambda_2 x-\lambda_2 y=0 \\ \frac{\partial F}{\partial y}=-2\lambda_1 y-2\lambda_2 y-\lambda_2 x=0 \\ \frac{\partial F}{\partial z}=2z+2a\lambda_1=0. \\ x^2+y^2=2az \\ x^2+y^2+xy=a^2 \end{cases}$$

$(x,y)\neq 0$.

$2(\lambda_1+\lambda_2)\cdot 2(\lambda_1+\lambda_2)-\lambda_2^2=0.$

$\Rightarrow \pm\lambda_2=2(\lambda_1+\lambda_2) \Rightarrow 2\lambda_1+(2\pm 1)\lambda_2=0.$

① $2\lambda_1+3\lambda_2=0 \Rightarrow x=y$

$\begin{cases} x^2+y^2=2az \\ 3x^2=a^2 \end{cases} \Rightarrow z=\frac{a}{3}$ 极小

② $2\lambda_1+\lambda_2=0 \Rightarrow x=-y.$

$\begin{cases} 2x^2=2az \\ x^2=a^2. \end{cases} \Rightarrow z=a.$ 极大.

9.12-6. 设 $a_i\geq 0\ (i=1,2,\cdots,n)$. $p>1$. 证明: $\frac{a_1+a_2+\cdots+a_n}{n}\leq\left(\frac{a_1^p+\cdots+a_n^p}{n}\right)^{\frac{1}{p}}$.

$a_1+\cdots+a_n=c.$

$f(a_1,\cdots,a_n)=a_1^p+\cdots+a_n^p-\lambda(a_1+\cdots+a_n-c).$

$\frac{\partial f}{\partial a_i}=pa_i^{p-1}-\lambda=0 \Rightarrow a_i^{p-1}=\frac{\lambda}{p} \Rightarrow a_i=\left(\frac{\lambda}{p}\right)^{\frac{1}{p-1}}$

$c=\lambda^{\frac{1}{p-1}}\cdot\frac{n}{p^{\frac{1}{p-1}}} \Rightarrow c^{p-1}=\lambda\cdot\frac{n^{p-1}}{p} \Rightarrow \lambda=\left(\frac{c}{n}\right)^{p-1}\cdot p.$

$a_i^{p-1}=\left(\frac{c}{n}\right)^{p-1} \Rightarrow a_i=\frac{c}{n}.$

$a_1^p+\cdots+a_n^p\geq\left(\frac{c}{n}\right)^p\cdot n \Rightarrow \frac{a_1^p+\cdots+a_n^p}{n}\geq\left(\frac{a_1+\cdots+a_n}{n}\right)^p.$

9.12-7. 证明 Hölder ineq: 设 $a_i\geq 0$. $x_i\geq 0\ (i=1,2,\cdots,n)$. $p>1$. $\frac{1}{p}+\frac{1}{q}=1$.

则 $\sum_{i=1}^n a_i x_i\leq\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}}\cdot\left(\sum_{i=1}^n x_i^q\right)^{\frac{1}{q}}$.

$\sum_{i=1}^n a_i x_i=c.$ $f(x_1,\cdots,x_n)=x_1^q+\cdots+x_n^q-\lambda\left(\sum_{i=1}^n a_i x_i-c\right)$

$\frac{\partial f}{\partial x_i}=qx_i^{q-1}-\lambda a_i=0 \Rightarrow x_i^{q-1}=\frac{\lambda a_i}{q}.$

$\sum_{i=1}^n a_i\cdot\left(\frac{\lambda}{q}\right)^{\frac{1}{q-1}}\cdot a_i^{\frac{1}{q-1}}=c \Rightarrow \left(\frac{\lambda}{q}\right)^{\frac{1}{q-1}}\cdot\sum_{i=1}^n a_i^{\frac{q}{q-1}=p}=c \Rightarrow \frac{\lambda}{q}=\frac{c^{q-1}}{\left(\sum_{i=1}^n a_i^p\right)^{q-1}}$

$\Rightarrow x_i=\left(\frac{\lambda}{q}\right)^{\frac{1}{q-1}}a_i^{\frac{1}{q-1}}$

$\sum_{i=1}^n x_i^q\geq\left(\frac{\lambda}{q}\right)^{\frac{q}{q-1}}\cdot a_i^{\frac{q}{q-1}}=\left(\frac{\lambda}{q}\right)^p\cdot\sum a_i^p=\left(\frac{c}{\sum_{i=1}^n a_i^p}\right)^q\cdot\sum_{i=1}^n a_i^p=c^q\cdot\left(\sum_{i=1}^n a_i^p\right)^{1-q}$

$\left(\sum_{i=1}^n x_i^q\right)^{\frac{1}{q}}\geq\sum_{i=1}^n a_i x_i\cdot\left(\sum_{i=1}^n a_i^p\right)^{\frac{1-q}{q}}=\sum_{i=1}^n a_i x_i\left(\sum_{i=1}^n a_i^p\right)^{-\frac{1}{p}}$

①

Weed 9 HW.

§ 10.1

1. 设 $\pi = \pi_x \times \pi_y$ $\pi_x: 0 = x_0 < \cdots < x_n = 1$, $\pi_y: 0 = y_0 < \cdots < y_m = 1$.

$I_{x_i} := [x_{i-1}, x_i]$, $I_{y_j} = [y_{j-1}, y_j]$, $I_{i,j} := I_{x_i} \times I_{y_j}$. $1 \le i \le n$; $1 \le j \le m$.

(special case)

case 1. 若有 $f(x), g(x) \ge 0$ $\forall x \in [0, 1]$.

$a_i := \inf f(I_{x_i})$, $A_i := \sup f(I_{x_i})$.

则 $S(fg, \pi) = \sum_{i,j} f(\tilde{x}_{i,j}) g(\tilde{y}_{i,j}) \sigma(I_{i,j})$

在 $I_{i,j}$ 中任取的点 $(\tilde{x}_{ij}, \tilde{y}_{ij})$.
注意与 i, j 均有关,
不能写成 $(\tilde{x}_i, \tilde{y}_j)$.

$$\le \sum_{i,j} A_i \sigma(I_{x_i}) g(\tilde{y}_{i,j}) \sigma(I_{y_j})$$

$$\le \sum_{i,j} A_i \sigma(I_{x_i}) \overline{S}(g, \pi_y)$$

$$= \overline{S}(f, \pi_x) \overline{S}(g, \pi_y).$$

同理, $S(fg, \pi) \ge \underline{S}(f, \pi_x) \underline{S}(g, \pi_y)$

令 $\|\pi\| \to 0$. 则 $\|\pi_x\| \le \|\pi\| \to 0$, $\|\pi_y\| \le \|\pi\| \to 0$.

夹逼 $\Rightarrow \lim_{\|\pi\| \to 0} S(fg, \pi) = \int_0^1 f(x)\,dx \int_0^1 g(y)\,dy$.

(general case).

case 2. f, g 可积, 故有界.

设 $f \ge m$, $g \ge M$.

由 case 1 $\Rightarrow$ $(f-m)(g-M)$ 可积

$\Rightarrow$ $fg - Mf - mg + mM$ 可积

又 mM, mg, Mf 均可积 (可用定义说明)

$\Rightarrow$ fg 可积: 且 $\iint fg$

$= \iint [(f-m)(g-M) + mg + Mf + Mm) = \cdots = \checkmark$

$f(x)(g(y)+M)$ 可积. 又 $Mf(x)$ 可积. 故 $fg = f(g+M)-Mf$ 也可积 ✓.

且 $\iint fg = \iint (f(g+M)-Mf) = \int f \int (g+M) - M\iint f$
$= \int f \int g + M\int f - M\int f$
$= \int f dx \int g dy$. 囗.

2. $(e-1)^2$.

3. 几何上: I 关于原点对称, 且 $\sin(-x-y) = -\sin(x+y)$.
积分值相抵.

5. 利用定理 10.1.8 中等价条件 (2).
$f(x,y)$ 在 $[a,b]\times[c,d]$ 上连续.
~~故~~ $\cancel{\lim_{\|\pi\|\to 0} \omega_i = 0 \quad \forall i}$

有 $\sum_{i=1}^{k} \omega_i \sigma(I_i) \le \max_i\{\omega_i\} \sum_i \sigma(I_i) = (d-c)(b-a)\max_{1\le i\le k}\omega_i$.
$\omega_i \ge 0$

$\left.\begin{array}{l} f \text{ 连续} \\ [a,b]\times[c,d] \text{ 紧} \end{array}\right\} \Rightarrow f$ 一致连续

$\Rightarrow |f(x_1,y_1) - f(x_2,y_2)| < \varepsilon$
$\forall\ (x_1,y_1), (x_2,y_2)$ satisfies
$\|(x_1,y_1) - (x_2,y_2)\| < \delta = \delta(\varepsilon)$.

取 $\|\pi\| < \delta$
$\Rightarrow \max_{1\le i\le k} \omega_i \le \varepsilon$ for $\|\pi\| < \delta$

ε 任意小
$\Rightarrow 0 \le \lim_{\|\pi\|\to 0} \sum_i \omega_i \sigma(I_i)$
$\le \lim_{\|\pi\|\to 0} (d-c)(b-a)\max_{1\le i\le k}\omega_i$
$\le (d-c)(b-a)\varepsilon, \quad \forall \varepsilon$

$\Rightarrow \lim_{\|\pi\|\to 0} \sum_i \omega_i \sigma(I_i) = 0$ 囗.

③

DATE

§ 10.2

2. B 有界，故 $\overline{B}$ 紧. B' 为零面积集，故 对 $\varepsilon>0$，∃ 开矩形 $I_1, \cdots, I_n$

s.t. $B' \subset \bigcup_{i=1}^{n} I_i$，$\sum_{i=1}^{n} \sigma(I_i) < \frac{\varepsilon}{2}$.

而 $\overline{B} \setminus \bigcup_{i=1}^{n} I_i = \overline{B} \cap \left(\bigcap_{i=1}^{n} I_i^c\right)$ 为紧集

其中又只有 B 的孤立点. 故 $\overline{B} \setminus \bigcup_{i=1}^{n} I_i$ 为有限点集.

(否则 ∃ $\{x_k\} \subset \overline{B} \setminus \bigcup_{i=1}^{n} I_i$，$x_k \to x_0 \in \overline{B} \setminus \bigcup_{i=1}^{n} I_i$

$\Rightarrow x_0 \in B' \subset \bigcup_{i=1}^{n} I_i$. 矛盾.)

~~故可取有限个~~ 故 $\overline{B} \subset \underbrace{\left(\overline{B} \setminus \bigcup_{i=1}^{n} I_i\right)}_{\text{零面积集}} \cup \underbrace{\left(\bigcup_{i=1}^{n} I_i\right)}_{\text{零面积集}}$ 为零面积集.

4. 由 Lebesgue 定理，f 在 I 上全体不连续点所成的集合 $D(f)$ 零测. 于是 $D(f|_J) \subset D(f)$ 也零测. 故 $f|_J$ 可积. (故已经蕴含了有界条件) ☒

5. f 可积 $\Rightarrow D(f)$ 零测

$\Rightarrow \exists\, x \in I^\circ \setminus D(f)$, i.e. x 为连续点，且 ∃ δ_x, $B_{\delta_x}(x) \subset I$

$\Rightarrow$ 对 $\varepsilon = f(x)/2$，∃ δ，$|f(y) - f(x)| < \varepsilon$ for $|y-x| < \delta$.

$\Rightarrow \int_I f\, d\sigma$

$\geq \int_{B_{\min\{\delta, \delta_x\}}(x)} f\, d\sigma \geq \frac{f(x)}{2} \cdot \pi \left(\min\{\delta, \delta_x\}\right)^2$

> 0. ☒

6. $D(f) = \{(x, y): x=0$ 或 $y=0\}$ 零测.

且 f 有界. 故 f 可积. ☒

7. $D(f) = B$ 零测 (应具体说明 why $D(f) = B$)

且 f 有界. 故 f 可积. ☒

8. $D(fg) \overset{\text{why}}{\subset} D(f) \cup D(g)$ 零测

$\Rightarrow$ fg 可积. (since f, g 均有界，故 fg 也有界. 故可应用 Lebesgue 定理).

$D(f/g) \subset D(f) \cup D(g)$ 零测

又 f/g 有界 $\Rightarrow f/g$ 可积. ☒

④

§10.3

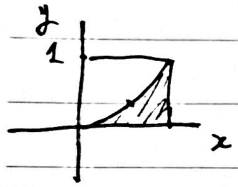
1. (1) $\frac{1}{3}\cdot(\arctan 1 - 0) = \pi/12$

(2) $\int_0^{\pi/2}(\sin(xy)\big|_{y=0}^{1})dx = \int_0^{\pi/2}\sin x\,dx = 1$

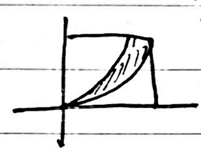
(3) $\int_0^{\pi}(-\cos(x+y)\big|_{y=0}^{\pi})\,dx = -\sin(x+\pi)\big|_0^{\pi} + \sin(x)\big|_0^{\pi}$
$= 0$

2. $\int_c^d \left(\frac{\partial f}{\partial y}(b,y) - \frac{\partial f}{\partial y}(a,y)\right)dy$
$= f(b,d) - f(b,c) - f(a,d) + f(a,c)$

3. (1) $\iint_I f\,d\sigma = \int_0^1\left(\int_0^{x^2} ds\right)dx = 1/3.$



(2) $\iint_I f\,d\sigma = \int_0^1\left(\int_{x^2}^{2x^2}(x+s)ds\right)dx$



$= \int_0^1 dx\left(x^3 + \frac{1}{2}\cdot 3x^4\right)$
$= \frac{1}{4} + \frac{3}{2}\frac{1}{5} = \frac{5+6}{20} = 11/20.$

4. 取 $[a,b]\times[c,d] = [0,n]\times[0,2]$.
$f([m,m+1)\times[0,1]) = a_{m+1}$.
$f([m,m+1)\times(1,2]) = b_{m+1}$. $0 \le m \le n-1$.

Minkowski
$\Rightarrow$ $\left(\int_0^n\left(\int_0^2 f(x,y)\,dy\right)^p dx\right)^{1/p} \le \int_c^d\left(\int_a^b f^p(x,y)\,dx\right)^{1/p}dy$

$\left(\sum_{k=1}^n (a_k+b_k)^p\right)^{1/p}$ $\qquad$ $\left(\sum_{k=1}^n a_k^p\right)^{1/p} + \left(\sum_{k=1}^n b_k^p\right)^{1/p}$ ▨.

问题：但 f 并不连续，无法应用 thm10.3.4.
但从定理证明中可以看出，f 的连续性只在积分换序（P21 第二行文字中的 thm 10.3.3）中用到，故 f 的条件可以放弱为分段连续.（事实上只要对 x, y 均可积）
（因为这在 thm 10.3.3 的证明中是不影响的）

(5)

DATE

5. $\iint_{[a,b]^2} \left(f(x)\, g(y) - f(y)\, g(x)\right)^2 dx\, dy \geq 0$

$\|$

$2 \int_a^b f^2 \int_a^b g^2 - 2\left(\int_a^b f(x) g(x)\, dx\right)^2$ 日.

13.

10.4-2. 证明: $1.96<\iint\limits_{|x|+|y|\le 10}\frac{dxdy}{100+\cos^2 x+\cos^2 y}<2$.

$E:=\{(x,y): |x|+|y|\le 10\}$

$\sigma(E)=\frac{1}{2}\times 20\times 20=200$.

$\frac{1}{100}\le\frac{1}{100+\cos^2 x+\cos^2 y}\le\frac{1}{102}$.

$1.96\approx\frac{200}{102}\le\iint\limits_{|x|+|y|\le 10}\frac{dxdy}{100+\cos^2 x+\cos^2 y}\le 2$

由于最大值或最小值不是几乎处处取到. 故不取等.

10.4-3. $B\subset\mathbb{R}^2$有界集. B有面积$\Leftrightarrow$ 对$I\supset B$的矩形I的任何矩形分割π. 均有$\lim\limits_{\|\pi\|\to 0}\sum\limits_{I_j\cap B\ne\phi}\sigma(I_j)=\lim\limits_{\|\pi\|\to 0}\sum\limits_{I_j\subset B}\sigma(I_j)$.

B有面积 $\Leftrightarrow$ ∂B是零面积集.

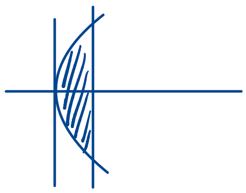
对任何矩形分割 $\lim\limits_{\|\pi\|\to 0}\sum\limits_{I_j\cap B\ne\phi}\sigma(I_j)=\lim\limits_{\|\pi\|\to 0}\sum\limits_{I_j\subset B}\sigma(I_j)+\lim\limits_{\|\pi\|\to 0}\sum\limits_{\substack{I_j\cap B\ne\phi\\ 且I_j\cap B^c\ne\phi}}\sigma(I_j)$.

$\lim\limits_{\|\pi\|\to 0}\sum\limits_{I_j\cap B\ne\phi}\sigma(I_j)=\lim\limits_{\|\pi\|\to 0}\sum\limits_{I_j\subset B}\sigma(I_j)\Leftrightarrow\lim\limits_{\|\pi\|\to 0}\sum\limits_{\substack{I_j\cap B\ne\phi\\ 且I_j\cap B^c\ne\phi}}\sigma(I_j)=0$

$\overset{定义}{\Leftrightarrow}$ ∂B是零测集

$\overset{\partial B有界且闭}{\Leftrightarrow}$ ∂B是零面积集.

10.5-1. (2) $\iint\limits_D xy^2\,dxdy$. D由 $y^2=4x$ 和 $x=1$ 围成.



$=\int_0^1\int_{-2\sqrt{x}}^{2\sqrt{x}}xy^2\,dy\,dx$

$=\int_0^1 x\,dx\int_{-2\sqrt{x}}^{2\sqrt{x}}y^2\,dy$.

$=\int_0^1\frac{1}{3}x\cdot 16x\sqrt{x}\,dx$

$=\frac{16}{3}\int_0^1 x^2\sqrt{x}\,dx$.

$=\frac{32}{3}\int_0^1 y^6\,dy$

$=\frac{32}{21}$

(4) $\iint_D|xy|\,dxdy$. $D=\{(x,y): x^2+y^2\le a^2\}$

$x=r\cos\theta$. $y=r\sin\theta$.

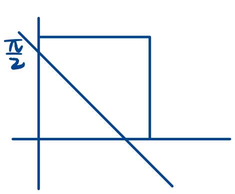
$=\int_0^a\int_0^{2\pi}r^2|\cos\theta\sin\theta|\cdot r\,d\theta\,dr$.

$=\int_0^a r^3dr\int_0^{2\pi}\frac{1}{2}|\sin 2\theta|\,d\theta$.

$=\frac{1}{4}a^4\cdot 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2\theta\,d\theta$

$=\frac{1}{4}a^4\cdot 2=\frac{a^4}{2}$.

(6) $\iint\limits_D|\cos(x+y)|\,dxdy$. $D=[0,1]^2$.



$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}-x}\cos(x+y)\,dy\,dx+\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}-x}^{1}-\cos(x+y)\,dy\,dx+\int_{\frac{\pi}{2}}^{1}\int_0^1-\cos(x+y)\,dy\,dx$.

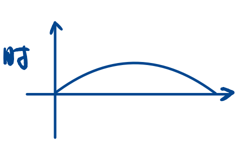
$=3-\pi+2\cos 1+\cos 2$.

(7) $\iint\limits_D y^2\,dxdy$. D由$\begin{cases}x=a(t-\sin t)\\ y=a(1-\cos t)\end{cases}$. $0\le t\le 2\pi$ 与 $y=0$ 围成.

$y=0$时 $1-\cos t=0\Rightarrow t=0$ or $t=2\pi$.

$x=0$, $x=2\pi a$

$a>0$时



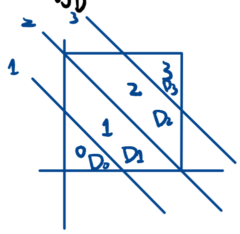
$=\int_0^{2\pi a}\int_0^{a(1-\cos t)}y^2\,dy\,dx$. $x=a(t-\sin t)$.

$=\int_0^{2\pi}\int_0^{a(1-\cos t)}y^2dy\cdot a(1-\cos t)\,dt$

$=\int_0^{2\pi}\frac{1}{3}a^4(1-\cos t)^4\,dt$

$=\frac{35}{12}\pi a^4$.

(8) $\iint_D [x+y]\,dxdy$. $D=[0,2]^2$.



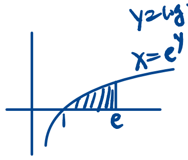
$= \sum_{i=0}^{3} i\cdot\sigma(D_i)$. $= \frac{3}{2}+3\times\frac{3}{2}=6$.

$\sigma(D_0)=\sigma(D_3)=\frac{1}{2}$.

$\sigma(D_1)=\sigma(D_2)=\frac{1}{2}\times(2^2-1^2)=\frac{3}{2}$.

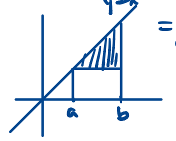
10.5-2. 改变积分次序.

(2) $\int_1^e dx\int_0^{\log x} f(x,y)dy$.



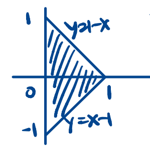
$=\int_0^1 dy\int_{e^y}^{e} f(x,y)dx$.

(4) $\int_a^b dx\int_a^x f(x,y)dy$.



$=\int_a^b dy\int_y^b f(x,y)dx$

(6) $\int_0^1 dx\int_{x-1}^{1-x} f(x,y)dy$.



$=\int_{-1}^0 dy\int_0^{y+1} f(x,y)dx$

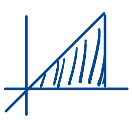
$+\int_0^1 dy\int_0^{1-y} f(x,y)dx$

$=\int_{-1}^1 dy\int_0^{1-|y|} f(x,y)dx$.

10.5-4. $f\in C(\mathbb{R}^2)$. $\int_0^a dx\int_0^x f(y)dy=\int_0^a (a-t)f(t)dt$.

$LHS = \int_0^a dy\int_y^a f(y)dx$

$=\int_0^a f(y)dy\cdot\int_y^a dx$



$=\int_0^a (a-y)f(y)dy = RHS$.

10.5-5. $f\in C(\mathbb{R}^3)$. $\int_a^b dx\int_a^x dy\int_a^y f(x,y,z)dz=\int_a^b dz\int_z^b dy\int_y^b f(x,y,z)dx$.

$LHS = \int_a^b dx\int_a^x\left(\int_a^y f(x,y,z)dz\right)dy = \int_a^b dy\int_y^b\left(\int_a^y f(x,y,z)dz\right)dx$

$=\int_a^b\left(\int_y^b dx\int_a^y f(x,y,z)\,dz\right)dy = \int_a^b\left(\int_a^y dz\int_y^b f(x,y,z)dx\right)dy$

$=\int_a^b dy\int_a^b\left(\int_y^b f(x,y,z)dx\right)dz = \int_a^b dz\int_z^b dy\int_y^b f(x,y,z)dx = RHS$.

10.5-6. $f\in C(\mathbb{R}^3)$. $\int_0^a dx\int_0^x dy\int_0^y f(x)f(y)f(z)dz=\frac{1}{3!}\left(\int_0^a f(t)dt\right)^3$.

设 $F(u)=\int_0^u f(t)dt$. $F(0)=0$.

$LHS = \int_0^a dx\int_0^x dy\int_0^y f(x)f(y)F'(z)dz = \int_0^a dx\int_0^x f(x)f(y)F(y)dy$.

$=\int_0^a f(x)dx\int_0^x F(y)\cdot F'(y)dy = \int_0^a f(x)dx\cdot \frac{1}{2}F(y)^2\Big|_0^x$

$=\frac{1}{2}\int_0^a f(x)\cdot F(x)^2dx = \frac{1}{2}\int_0^a F'(x)F(x)^2dx$.

$=\frac{1}{6}\,F(x)^3\Big|_{x=0}^{a} = \frac{1}{6}F(a)^3 = RHS$.

10.5-7. $f\in C(\mathbb{R}^3)$ $\int_0^a dx\int_0^x dy\int_0^y f(z)dz=\frac{1}{2}\int_0^a (a-t)^2 f(t)dt$.

$LHS = \int_0^a dx\int_0^x (x-t)f(t)dt = \int_0^a dt\int_t^a (x-t)f(t)dx$

$=\int_0^a f(t)dt\cdot\frac{1}{2}(x-t)^2\Big|_{x=t}^{x=a} = \frac{1}{2}\int_0^a (a-t)^2 f(t)dt$.

10.6-1. (1) $\iint_D (x-y)^2\sin(x+y)\,dxdy$. D由 $(\pi,0),(2\pi,\pi),(\pi,2\pi),(0,\pi)$ 顺次连成正方形.

$$= \iint_D (x-y)^2\sin(2\pi-x-y)\,dxdy$$

$$= -\iint (x-y)^2\sin(x+y)\,dxdy \Rightarrow \ = 0.$$

(2) $\iint_D (x^2+y^2)\,dxdy$. D由 $x^2-y^2=1$. $x^2-y^2=2$. $xy=1$, $xy=2$ 围成.

$u=x^2-y^2$. $v=xy$. $x^2+y^2=\sqrt{u^2+4v^2}$ $\quad J\varphi = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}$. $\frac{\partial(u,v)}{\partial(x,y)} = 2(x^2+y^2)$

$$= \int_1^2\int_1^2 \frac{1}{2}\,dudv = \frac{1}{2}$$

10.6-2. 求面积

(1) $(\underbrace{a_1x+b_1y+c_1}_{u})^2 + (\underbrace{a_2x+b_2y+c_2}_{v})^2 = 1$. $a_1b_2 \neq a_2b_1$

$J\varphi = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$. $\frac{\partial(u,v)}{\partial(x,y)} = a_1b_2 - a_2b_1$.

$$\sigma = \iint_{u^2+v^2\le 1} |a_1b_2-a_2b_1|^{-1}\,dudv = \pi|a_1b_2-a_2b_1|^{-1}.$$

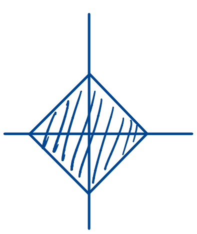
(2) $\underbrace{\sqrt{x}}_{u}+\underbrace{\sqrt{y}}_{v} = \sqrt{a}$. $x=0$. $y=0$ 围成.

$(u,v)=\Phi(x,y)$.

$J\Phi^{-1} = \begin{pmatrix} 2u & 0 \\ 0 & 2v \end{pmatrix}$. $\frac{\partial(x,y)}{\partial(u,v)} = 4uv$.

$$\sigma = \iint_{\substack{u+v\le\sqrt{a}\\ u\ge 0\\ v\ge 0}} 4uv\,dudv = \frac{1}{6}a^2.$$

10.6-3. $\iint_{|x|+|y|\le 1} f(x+y)\,dxdy = \int_{-1}^1 f(t)\,dt$.



$u=x+y$, $v=x-y$.

$J\varphi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. $\frac{\partial(u,v)}{\partial(x,y)} = -2$.

$$\text{LHS} = \int_{-1}^1\int_{-1}^1 \frac{1}{2}f(u)\,dudv$$

$$= \int_{-1}^1 f(u)\,du.$$

10.6-5 (2) $\iint_{x^2+y^2\le Rx} \sqrt{R^2-x^2-y^2}\,dxdy$.

$x=r\cos\theta$. $y=r\sin\theta$.

$$r^2 \le Rr\cos\theta \Rightarrow \begin{cases} 0\le r\le R\cos\theta \\ \cos\theta\ge 0 \Rightarrow \theta\in[0,\frac{\pi}{2}]\cup[\frac{3}{2}\pi,2\pi] \end{cases}$$

$$= \left(\int_0^{\frac{\pi}{2}} + \int_{\frac{3\pi}{2}}^{2\pi}\right)\int_0^{R\cos\theta} r\sqrt{R^2-r^2}\,drd\theta = \left(\frac{\pi}{3}-\frac{4}{9}\right)R^3.$$

(4) $\iint_{x^2+y^2\le x+y} \sqrt{x^2+y^2}\,dxdy$.

$x=r\cos\theta$, $y=r\sin\theta$

$$r^2 \le r(\cos\theta+\sin\theta) \Rightarrow \begin{cases} 0\le r\le\sqrt{2}\sin(\theta+\frac{\pi}{4}) \\ \sin(\theta+\frac{\pi}{4})\ge 0 \Rightarrow \theta\in[0,\frac{3}{4}\pi]\cup[\frac{7}{4}\pi,2\pi] \end{cases}$$

$$= \left(\int_0^{\frac{3}{4}\pi} + \int_{\frac{7}{4}\pi}^{2\pi}\right)\int_0^{\sqrt{2}\sin(\theta+\frac{\pi}{4})} r^2\,drd\theta$$

$$= \left(\int_0^{\frac{3}{4}\pi} + \int_{\frac{7}{4}\pi}^{2\pi}\right)\frac{2\sqrt{2}}{3}\sin^3(\theta+\frac{\pi}{4})\,d\theta$$

$$= \frac{8\sqrt{2}}{9}.$$

10.6-7. $a,b>0$. $D=\{(x,y): \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ 且 $x\ge y\ge 0\}$. 计算 $\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,dxdy$.

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta. \end{cases} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{pmatrix} = abr$$

$x\ge y\ge 0 \Rightarrow ar\cos\theta \ge br\sin\theta \ge 0 \Rightarrow \tan\theta \le \frac{a}{b}$. 且 $\theta\in[0,\frac{\pi}{2}]$. $\theta\in[0,\arctan\frac{a}{b}]$

$$\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,dxdy = \int_0^1\int_0^{\arctan\frac{a}{b}} r\cdot abr\,d\theta dr$$
$$= \int_0^1 abr^2\arctan\frac{a}{b}\,dr$$
$$= \frac{1}{3}ab\arctan\frac{a}{b}.$$

补:

问题 10.6-1. $\int_0^1\int_0^1 (xy)^{xy}\,dxdy = \int_0^1 t^t dt$.

$$\text{LHS} \overset{t=xy}{\underset{dt=xdy}{=}} \int_0^1 \frac{1}{x}\int_0^x t^t\,dtdx$$

$f(x) = \int_0^x t^t dt$.

$$\int_0^1 \frac{1}{x}f(x)dx = \int_0^1 f(x)\,d\log x = -\int_0^1 x^x\log x\,dx.$$
$$= -\int_0^1 e^{x\log x}\cdot\log x\,dx.$$
$$= -\int_0^1 e^{x\log x}(\log x+1) - e^{x\log x}\,dx.$$
$$= -\int_0^1 \underbrace{e^{x\log x}\cdot(\log x+1)}_{(e^{x\log x})'}dx + \int_0^1 t^t dt$$
$$= -e^{x\log x}\Big|_0^1 + \int_0^1 t^t dt.$$
$$= \int_0^1 t^t dt = \text{RHS}.$$

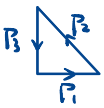
13

11.1-1. $\int_\Gamma (x^2+y^2)^n ds$. $\Gamma: \begin{cases} x=a\cos t \\ y=a\sin t \end{cases}$ $(0\le t\le 2\pi)$

$dx=-a\sin t\,dt$. $dy=a\cos t\,dt$.

$= \int_0^{2\pi} a^{2n}\cdot|a|\,dt = 2\pi\cdot|a|^{2n+1}$

11.1-2. $\int_\Gamma (x+y)ds$. Γ: (0,0). (1,0). (0,1) 三角形边界.



$= \int_{\Gamma_1} x\,dx + \int_{\Gamma_2} 1\,ds + \int_{\Gamma_3} y\,dy$

$= \int_0^1 x\,dx + \int_1^0 -1\cdot\sqrt{2}\,dx + \int_1^0 y\,|dy|$

$= \frac{1}{2}+\sqrt{2}+\frac{1}{2} = 1+\sqrt{2}$.

11.1-3. $\int_\Gamma z\,ds$. $\Gamma: \begin{cases} x=t\cos t. \\ y=t\sin t \\ z=t \end{cases}$ $0\le t\le 2\pi$.

$\begin{cases} dx=(\cos t - t\sin t)\,dt \\ dy=(\sin t + t\cos t)\,dt \\ dz=dt. \end{cases}$

$ds=\sqrt{1+t^2+1}\,dt=\sqrt{2+t^2}\,dt$

$\int_\Gamma z\,ds = \int_0^{2\pi} t(2+t^2)^{\frac{1}{2}}dt$

$= \frac{1}{3}(2+t^2)^{\frac{3}{2}}\Big|_0^{2\pi} = \frac{1}{3}\left[(2+4\pi^2)^{\frac{3}{2}}-2\sqrt{2}\right]$

11.1-4. $\int_\Gamma x^2 ds$. $\Gamma: \begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0. \end{cases}$

$=\frac{1}{3}\int_\Gamma (x^2+y^2+z^2)ds = \frac{a^2}{3}\int_\Gamma ds = \frac{a^2}{3}\cdot 2\pi a = \frac{2}{3}\pi a^3$.

11.1-5. $\int_\Gamma y^2 ds$. $\Gamma: \begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$ $0\le t\le 2\pi$.

$\begin{cases} dx=a(1-\cos t)\,dt \\ dy=a\sin t\,dt. \end{cases}$

$\int_\Gamma y^2 ds = \int_\Gamma a^2(1-\cos t)^2\,ds = \int_0^{2\pi} a^2\cdot 4\sin^4\frac{t}{2}\cdot 2|a|\left|\sin\frac{t}{2}\right|dt$

$= 8|a|^3\int_0^{2\pi}\sin^5\frac{t}{2}\,dt = \frac{256}{15}|a|^3$.

$ds = |a|\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt = |a|\sqrt{2-2\cos t} = 2|a|\cdot\left|\sin\frac{t}{2}\right|$.

11.2-1.(1) $\int_\Gamma \frac{x\,dy-y\,dx}{x^2+y^2}$. Γ表示逆时针方向 $x^2+y^2=a^2$.

$\begin{cases} x=a\cos t & dx=-a\sin t\,dt \\ y=a\sin t. & dy=a\cos t\,dt \end{cases}$

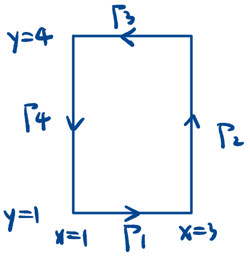
$= \int_0^{2\pi}\frac{a^2\cos^2 t\,dt+a^2\sin^2 t\,dt}{a^2} = 2\pi$.

(3) $\int_\Gamma (x^2-2xy)dx+(y^2-2xy)dy$. Γ: $x=y^2$. $-1\le y\le 1$. $y\uparrow$.

$= \int_{-1}^1 (y^4-2y^3)\cdot 2y\,dy + (y^2-2y^3)dy = \int_{-1}^1 (2y^5-4y^4-2y^3+y^2)\,dy$

$= -\frac{8}{5}+\frac{2}{3} = -\frac{14}{15}$

(5) $\int_\Gamma (x^2+y^2)\,dy$. Γ: $x=1,3$. $y=1,4$ 矩形. 逆时针.



$$= \int_{\Gamma_2} (9+y^2)\,dy + \int_{\Gamma_4} (1+y^2)\,dy$$
$$= \int_1^4 (9+y^2)\,dy + \int_4^1 (1+y^2)\,dy$$
$$= 27-3 = 24.$$

11.2-2. $ac-b^2>0$. $\int_\Gamma \frac{x\,dy-y\,dx}{ax^2+2bxy+cy^2}$. Γ: 逆时针单位圆周.

$$\begin{cases} x=\cos t & dx=-\sin t\,dt \\ y=\sin t. & dy=\cos t\,dt \end{cases}$$

$$原式 = \int_0^{2\pi} \frac{dt}{a\cos^2 t + 2b\cos t\sin t + c\sin^2 t}$$
$$= 2\int_{-\pi/2}^{\pi/2} \frac{1}{a+2b\cdot\tan t + c\tan^2 t}\cdot \frac{dt}{\cos^2 t} \qquad \frac{dt}{\cos^2 t} = d\tan t.$$
$$= 2\int_{-\infty}^{+\infty} \frac{1}{a+2bt+ct^2}\,dt.$$
$$= 2\int_{-\infty}^{+\infty} \frac{1}{c(t+\frac{b}{c})^2 - \frac{b^2}{c}+a}\,dt$$
$$= 2\,\frac{c}{ac-b^2}\int_{-\infty}^{+\infty} \frac{1}{1+\frac{c^2}{ac-b^2}(t+\frac{b}{c})^2}\,dt.$$

$u=\frac{|c|}{\sqrt{ac-b^2}}(t+\frac{b}{c})$

$$= 2\,\frac{c}{ac-b^2}\cdot\frac{\sqrt{ac-b^2}}{|c|}\int_{-\infty}^{+\infty} \frac{1}{1+u^2}\,du$$
$$= \frac{2}{\sqrt{ac-b^2}}\,\frac{c}{|c|}\arctan u\Big|_{-\infty}^{+\infty} = \frac{2\pi c}{|c|\sqrt{ac-b^2}}.$$

11.2-3. (1) $\int_\Gamma xz^2\,dx + yx^2\,dy + zy^2\,dz$. Γ: $\begin{cases} x=t \\ y=t^2 \\ z=t^3 \end{cases}$ $t: 0\to 1$.

$$= \int_0^1 t^7\,dt + t^4\cdot 2t\,dt + t^7\cdot 3t^2\,dt$$
$$= \int_0^1 t^7 + 2t^5 + 3t^9\,dt$$
$$= \frac{1}{8}+\frac{1}{3}+\frac{3}{10} = \frac{91}{120}.$$

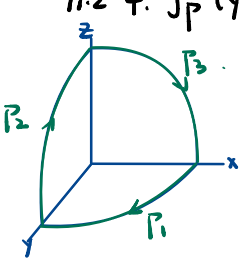
$a\sin 2t$
$2a\cos 2t$

(2) $\int_\Gamma (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$. Γ: $\begin{cases} x=a\sin^2 t \\ y=2a\sin t\cos t \\ z=a\cos^2 t. \end{cases}$ $t: 0\to\pi$.

$$\begin{cases} dx = 2a\sin t\cos t\,dt \\ dy = 2a(\cos^2 t-\sin^2 t). \\ dz = -2a\sin t\cos t. \end{cases}$$

$$= a^2\int_0^\pi (2\sin t\cos t+\cos^2 t)\cdot 2\sin t\cos t\,dt + 2(\cos^2 t-\sin^2 t)\,dt + (\sin^2 t+2\sin t\cos t)\cdot(-2\sin t\cos t)\,dt$$
$$= a^2\int_0^\pi (\cos^2 t-\sin^2 t)\cdot(2\sin t\cos t+2)\,dt$$
$$= a^2\int_0^\pi \cos 2t\,(2+\sin 2t)\,dt.$$
$$= a^2\int_0^\pi \tfrac{1}{2}\sin 4t + 2\cos 2t\,dt = 0.$$

11.2-4. $\int_\Gamma \underbrace{(y^2-z^2)\,dx + (z^2-x^2)\,dy + (x^2-y^2)\,dz}_{A}$. Γ: $x^2+y^2+z^2=1$. $x\geq 0$. $y\geq 0$. $z\geq 0$ 边界. 方向 $(1,0,0)\to(0,1,0)\to(0,0,1)\to(1,0,0)$.



Γ_1: $dz=0$. $z=0$. Γ_2: $x=0$. $dx=0$. Γ_3: $y=0$. $dy=0$.

$$\int_{\Gamma_1} A = \int_{\Gamma_1} y^2\,dx - x^2\,dy = \int_0^{\frac{\pi}{2}} -\sin^3 t - \cos^3 t\,dt = -\frac{4}{3}. \quad\Rightarrow \int_\Gamma A = -4.$$
$$\int_{\Gamma_2} A = \int_{\Gamma_2} z^2\,dy - y^2\,dz = \int_0^{\frac{\pi}{2}} -\sin^3 t - \cos^3 t\,dt = -\frac{4}{3}$$
$$\int_{\Gamma_3} A = \int_{\Gamma_3} -z^2\,dx + x^2\,dz = -\frac{4}{3}$$

11.2.5. $\left|\int_\Gamma \vec{F}\cdot d\vec{p}\right| \leq \int_\Gamma \|\vec{F}\|\, ds$.

$$\text{LHS} = \left|\int_\Gamma \sum_{i=1}^{n} F_i \cdot dp_i\right|$$

$$\leq \int_\Gamma \sum_{i=1}^{n} |F_i||dp_i| .$$

$$\overset{\text{Cauchy}}{\leq} \int_\Gamma \left(\sum_{i=1}^{n}|F_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}|dp_i|^2\right)^{\frac{1}{2}}$$

$$= \int_\Gamma \|\vec{F}\|\cdot ds = \text{RHS}.$$

11-3-1. 用 Green 公式计算. $\int_{\partial\Omega} Pdx + Qdy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$.

(1) $\int_\Gamma xy^2dy - x^2y\,dx$. $\Gamma: x^2+y^2=a^2$. 逆时针.

$P=-x^2y$. $Q=xy^2$. $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y^2+x^2$

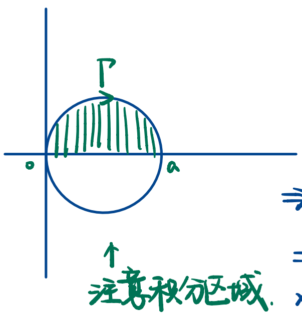
原式 $=\iint_{B_a(0)} x^2+y^2\,dxdy = \int_0^{2\pi}\int_0^a r^2\cdot r\,drd\theta = 2\pi\cdot\frac{1}{4}a^4=\frac{\pi a^4}{2}$.

(2) $\int_\Gamma (x+y)dx-(x-y)dy$. $\Gamma: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 逆时针.

$P=x+y$. $Q=-(x-y)$. $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-1-1=-2$.

原式 $=\iint_\Omega -2\,dxdy = -2\pi ab$.

(3) $\int_\Gamma e^x\sin y\,dx + e^x\cos y\,dy$. Γ: $x^2+y^2=ax$ 的上半圆周.沿x增加方向.



Γ: $(x-\frac{a}{2})^2+y^2=\frac{a^2}{4}$.　　$P=e^x\sin y$. $Q=e^x\cos y$

$\Gamma_1=-\Gamma$. $\Gamma_2=\{(x,0): x:0\to a\}$.　　$\frac{\partial P}{\partial y}=e^x\cos y$. $\frac{\partial Q}{\partial x}=e^x\cos y$.

$\int_{\Gamma_2} e^x\sin y\,dx+e^x\cos y\,dy=0$.　$(y=0, dy=0)$.

$\Rightarrow \int_{\Gamma_1} e^x\sin y\,dx+e^x\cos y\,dy = \int_{\Gamma_1+\Gamma_2} e^x\sin y\,dx+e^x\cos y\,dy$

$= \iint_{\substack{x^2+y^2\le ax\\ y\ge 0}} \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy = 0$.

$\Rightarrow \int_\Gamma e^x\sin y\,dx+e^x\cos y\,dy = -\int_{\Gamma_1} e^x\sin y\,dx+e^x\cos y\,dy=0$.

11-3-2. 用Green计算面积. $\sigma(D)=\int_{\partial D} x\,dy = -\int_{\partial D} y\,dx = \frac{1}{2}\int_{\partial D} x\,dy-y\,dx$.

(1) 星形线 $\begin{cases} x=a\cos^3 t & 0\le t\le 2\pi. \\ y=a\sin^3 t. \end{cases}$

$dy=3a\sin^2 t\cos t\,dt \Rightarrow \sigma(D)=\int_0^{2\pi} a\cos^3 t\cdot 3a\sin^2 t\cos t\,dt = 3a^2\int_0^{2\pi}\cos^4 t-\cos^6 t\,dt$.

$\cos^4 t=\left(\frac{1+\cos 2t}{2}\right)^2=\frac{1}{4}\cos^2 2t+\frac{1}{2}\cos 2t+\frac{1}{4}=\frac{1}{4}\cdot\frac{1+\cos 4t}{2}+\frac{1}{2}\cos 2t+\frac{1}{4}.=\frac{3}{8}+\frac{1}{2}\cos 2t+\frac{1}{2}\cos 4t$.

$\cos^6 t=\left(\frac{1+\cos 2t}{2}\right)^3=\frac{1}{8}\cos^3 2t+\frac{3}{8}\cos^2 2t+\frac{3}{8}\cos 2t+\frac{1}{8}$

$=\frac{1}{8}\cdot\frac{\cos 6t+3\cos 2t}{4}+\frac{3}{8}\cdot\frac{1+\cos 4t}{2}+\frac{3}{8}\cos 2t+\frac{1}{8}$

$=\frac{5}{16}+\frac{15}{32}\cos 2t+\frac{3}{16}\cos 4t+\frac{1}{32}\cos 6t$.

$\Rightarrow \sigma(D)=3a^2\cdot 2\pi\left(\frac{3}{8}-\frac{5}{16}\right)=\frac{3}{8}\pi a^2$.

(2) 双纽线. $(x^2+y^2)^2=a^2(x^2-y^2)$.

令 $y=x\tan\theta$. $[x^2(1+\tan^2\theta)]^2=a^2(1-\tan^2\theta)x^2$.

$\Rightarrow x^2(1+\tan^2\theta)^2=a^2(1-\tan^2\theta)$.

$x^2=a^2\cos^4\theta(1-\tan^2\theta)=a^2\cos^2\theta(\cos^2\theta-\sin^2\theta)\geq 0$.　$\theta\in[0,\frac{\pi}{4})$.

第I象限: $x=a|\cos\theta|\sqrt{\cos 2\theta}=a\cos\theta\sqrt{\cos 2\theta}$. $y=a\sin\theta\sqrt{\cos 2\theta}$

$dy=\left(a\cos\theta\sqrt{\cos 2\theta}+a\sin\theta\frac{-2\sin 2\theta}{2\sqrt{\cos 2\theta}}\right)d\theta=\left(a\cos\theta\sqrt{\cos 2\theta}-\frac{a\sin\theta\sin 2\theta}{\sqrt{\cos 2\theta}}\right)d\theta$.

$x\,dy=(a^2\cos^2\theta\cos 2\theta-a^2\sin\theta\cos\theta\sin 2\theta)\,d\theta$.　　$\frac{1}{2}\sin 2\theta+\frac{1}{4}\sin 4\theta\Big|_0^{\frac{\pi}{4}}$

$=a^2\cos\theta\cos 3\theta\,d\theta=\frac{a^2}{2}(\cos 2\theta+\cos 4\theta)\,d\theta$.

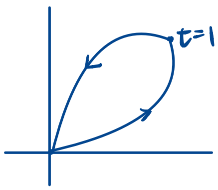
由对称性及坐标轴上积分为0. $\sigma(D)=4\int_{\Gamma_1}x\,dy=4\int_0^{\frac{\pi}{4}}\frac{a^2}{2}(\cos 2\theta+\cos 4\theta)d\theta=a^2$.

(3) 叶形线 $x^3+y^3=3axy\ (a>0)$

$y=xt$. $\quad x^3(1+t^3)=3ax^2t$. $\quad t\in(0,+\infty)$.

$\Rightarrow x=\frac{3at}{1+t^3}$. $\quad y=\frac{3at^2}{1+t^3}$. $\quad dx=\frac{3a[(1+t^3)-t\cdot 3t^2]}{(1+t^3)^2}dt=\frac{3a(1-2t^3)}{(1+t^3)^2}dt$.

$\sigma(D)=\int_{\partial D}-ydx=9a^2\int_0^{+\infty}\frac{t^2(2t^3-1)}{(1+t^3)^3}dt\overset{u=t^3}{=\!=}3a^2\int_0^{+\infty}\frac{2u-1}{(1+u)^3}du$



$\frac{2u-1}{(1+u)^3}=\frac{2}{(1+u)^2}-\frac{3}{(1+u)^3}$. $\quad =3a^2\int_0^{+\infty}\frac{2}{(1+u)^2}-\frac{3}{(1+u)^3}du$

$=3a^2\left(-\frac{2}{1+u}+\frac{3}{2(1+u)^2}\right)\Big|_0^{+\infty}$.

$=3a^2\cdot\frac{1}{2}=\frac{3a^2}{2}$.

11.3-3. 封闭曲线 Γ: $\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$, $\alpha\le t\le\beta$. 参数增加为正向. 证明 Γ 围成的面积 $A=\frac{1}{2}\int_\alpha^\beta\begin{vmatrix}\varphi(t)&\psi(t)\\ \varphi'(t)&\psi'(t)\end{vmatrix}dt$.

$dx=\varphi'(t)dt$. $\quad dy=\psi'(t)dt$

$A=\frac{1}{2}\int_\Gamma xdy-ydx=\frac{1}{2}\int_\alpha^\beta\varphi(t)\psi'(t)dt-\psi(t)\varphi'(t)\,dt=\frac{1}{2}\int_\alpha^\beta\begin{vmatrix}\varphi(t)&\psi(t)\\ \varphi'(t)&\psi'(t)\end{vmatrix}dt$.

11.3-4. $f\in C^1$. Γ 分段光滑封闭. 证明.

(1) $\int_\Gamma f(xy)(ydx+xdy)=0$.

(2) $\int_\Gamma f(x^2+y^2)(xdx+ydy)=0$.

(3) $\int_\Gamma f(x^n+y^n)(x^{n-1}dx+y^{n-1}dy)=0$.

(1) $P=f(xy)y$. $\quad Q=f(xy)x$.

$\frac{\partial P}{\partial y}=f(xy)+y\cdot f'(xy)\cdot x$. $\quad \frac{\partial Q}{\partial x}=f(xy)+x\cdot f'(xy)\cdot y$

$\Rightarrow \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0\overset{\text{Green}}{\Longrightarrow}$ 原式 $=0$.

(3) $P=f(x^n+y^n)x^{n-1}$. $\quad Q=f(x^n+y^n)y^{n-1}$.

$\frac{\partial P}{\partial y}=x^{n-1}f'(x^n+y^n)\cdot ny^{n-1}$. $\quad \frac{\partial Q}{\partial x}=y^{n-1}f'(x^n+y^n)\cdot nx^{n-1}$.

$\Rightarrow \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0\overset{\text{Green}}{\Longrightarrow}=0$.

11.3-5. Γ: $\mathbb{R}^2$ 光滑封闭曲线. $\vec{n}$: 单位外法向. $\vec{a}$ 是固定单位向量. $\int_\Gamma\cos(\vec{a},\vec{n})ds=0$.

$\text{LHS}=\int_\Gamma\vec{a}\cdot\vec{n}\,ds \qquad \vec{n}\perp\vec{t}(s) \quad \vec{t}\cdot ds=(dx,dy).\Rightarrow\vec{n}\cdot ds=(dy,-dx)$.

$=\int_\Gamma a_1dy-a_2dx$.

$\overset{\text{Green}}{=\!=}\int_\Omega\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}\,dxdy=0$

11.3-6. 计算. $\int_\Gamma x\cos(\vec{n},\vec{i})+y\cos(\vec{n},\vec{j})\,ds$. Γ 光滑封闭曲线.

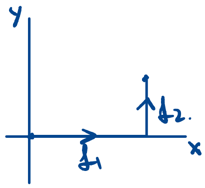
$=\int_\Gamma x\cdot\vec{n}\cdot\vec{i}+y\cdot\vec{n}\cdot\vec{j}\,ds \qquad \vec{n}\cdot\vec{i}ds=dy.\quad \vec{n}\cdot\vec{j}ds=-dx$.

$=\int_\Gamma \underset{Q}{x}dy-\underset{P}{y}dx=\iint_\Omega 1+1\,dxdy=2\sigma(\Omega)$

11.3-7. (1) $\int_L (x^2+2xy-y^2)dx + (x^2-2xy-y^2)dy$. L 连接 $A=(0,0)$. $B=(2,1)$ 的任意光滑线段.

$P=x^2+2xy-y^2$. $Q=x^2-2xy-y^2$.

$\frac{\partial P}{\partial y}=2x-2y$. $\frac{\partial Q}{\partial x}=2x-2y$. $\Rightarrow$ 积分与路径无关.



L_1: $y=0$. $x: 0\to 2$. L_2: $x=2$, $y: 0\to 1$.

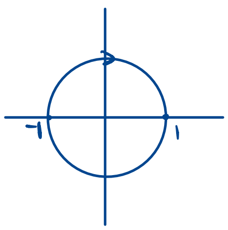
$$原式 = \int_0^2 x^2 dx + \int_0^1 (4-4y-y^2)dy$$
$$= \frac{8}{3}+4-2-\frac{1}{3} = \frac{13}{3}$$

(2) $\int_L \frac{(x+y)dx-(x-y)dy}{x^2+y^2}$. L: 上半平面 $(-1,0)\to(1,0)$ 任意光滑线段.

$P=\frac{x+y}{x^2+y^2}$. $Q=\frac{y-x}{x^2+y^2}$.

$$\frac{\partial P}{\partial y}=\frac{x^2+y^2-2y(x+y)}{(x^2+y^2)^2}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2} \qquad \frac{\partial Q}{\partial x}=\frac{-(x^2+y^2)-2x(y-x)}{(x^2+y^2)^2}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

$\Rightarrow$ 积分与路径无关.



$\begin{cases} x=\cos\theta \\ y=\sin\theta \end{cases}$ $\theta: -\frac{\pi}{2}\to\frac{\pi}{2}$.

$$原式 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta+\sin\theta)\cdot -\sin\theta d\theta - (\cos\theta-\sin\theta)\cos\theta d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi.$$

11.3-8. 计算 $\int_\Gamma \frac{e^x}{x^2+y^2}[(x\sin y-y\cos y)dx+(x\cos y+y\sin y)dy]$. Γ: 原点在内部的封闭曲线.

Ω: Γ 围成区域. $\tilde\Omega=\Omega\setminus B_\varepsilon(0)$. $\tilde\Gamma=\partial\tilde\Omega$ 有向.

$P=\frac{e^x}{x^2+y^2}(x\sin y-y\cos y)$. $\frac{\partial P}{\partial y}=\frac{-2ye^x}{(x^2+y^2)^2}(x\sin y-y\cos y)+\frac{e^x}{x^2+y^2}(x\cos y-\cos y+y\sin y)$.

$Q=\frac{e^x}{x^2+y^2}(x\cos y+y\sin y)$. $\frac{\partial Q}{\partial x}=\frac{e^x(x^2+y^2)-2xe^x}{(x^2+y^2)^2}(x\cos y+y\sin y)+\frac{e^x}{x^2+y^2}\cos y$

$$=\frac{e^x}{x^2+y^2}(x\cos y+y\sin y+\cos y)-\frac{2xe^x}{(x^2+y^2)^2}(x\cos y+y\sin y).$$

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$
$$=\frac{e^x}{x^2+y^2}\cdot 2\cos y-\frac{e^x}{(x^2+y^2)^2}(2x^2\cos y+2xy\sin y-2xy\sin y+2y^2\cos y)=0.$$

$\Rightarrow \int_{\tilde\Gamma} Pdx+Qdy=0$.

$$\int_{\partial B_\varepsilon(0)} Pdx+Qdy=\frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon(0)} e^x[(x\sin y-y\cos y)dx+(x\cos y+y\sin y)dy]$$

$P'=e^x(x\sin y-y\cos y)$ $\quad \frac{\partial P'}{\partial y}=e^x(x\cos y-\cos y+y\sin y)$.

$Q'=e^x(x\cos y+y\sin y)$ $\quad \frac{\partial Q'}{\partial x}=e^x(x\cos y+y\sin y+\cos y)$

$$\int_{\partial B_\varepsilon(0)} Pdx+Qdy=\frac{1}{\varepsilon^2}\iint_{B_\varepsilon(0)} e^x\cdot 2\cos y\, dxdy \xrightarrow{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\cdot 2\pi\varepsilon^2=2\pi.$$

$$\therefore 原式=\left(\int_{\tilde\Gamma}+\int_{\partial B_\varepsilon(0)}\right)(Pdx+Qdy)=0+2\pi=2\pi.$$

11-3-1. 用Green公式计算. $\int_{\partial\Omega} Pdx+Qdy = \iint_{\Omega}(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$.

(1) $\int_\Gamma xy^2dy - x^2y\,dx$. $\Gamma: x^2+y^2=a^2$. 逆时针.

$P=-x^2y$. $Q=xy^2$. $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y^2+x^2$

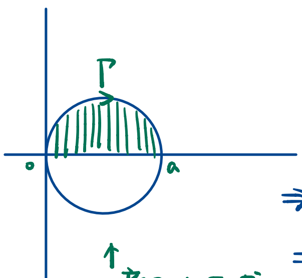
原式 $=\iint_{B_a(0)} x^2+y^2\,dxdy = \int_0^{2\pi}\int_0^a r^2\cdot r\,drd\theta = 2\pi\cdot\frac14 a^4=\frac{\pi a^4}{2}$.

(2) $\int_\Gamma (x+y)dx-(x-y)dy$. $\Gamma: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 逆时针.

$P=x+y$. $Q=-(x-y)$. $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-1-1=-2$.

原式 $=\iint_\Omega -2\,dxdy = -2\pi ab$.

(3) $\int_\Gamma e^x\sin y\,dx + e^x\cos y\,dy$. Γ: $x^2+y^2=ax$ 的上半圆周. 沿x增加方向.



注意积分区域.

$\Gamma: (x-\frac a2)^2+y^2=\frac{a^2}{4}$. $P=e^x\sin y$. $Q=e^x\cos y$

$\Gamma_1=-\Gamma$. $\Gamma_2=\{(x,0): x:0\to a\}$. $\frac{\partial P}{\partial y}=e^x\cos y$. $\frac{\partial Q}{\partial x}=e^x\cos y$.

$\int_{\Gamma_2} e^x\sin y\,dx+e^x\cos y\,dy=0$. $(y=0, dy=0)$.

$\Rightarrow \int_{\Gamma_1} e^x\sin y\,dx+e^x\cos y\,dy = \int_{\Gamma_1+\Gamma_2} e^x\sin y\,dx+e^x\cos y\,dy$

$=\iint_{\substack{x^2+y^2\le ax\\ y\ge0}} (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=0$.

$\Rightarrow \int_\Gamma e^x\sin y\,dx+e^x\cos y\,dy = -\int_{\Gamma_1}e^x\sin y\,dx+e^x\cos y\,dy=0$.

11-3-2. 用Green计算面积. $\sigma(D)=\int_{\partial D}x\,dy = -\int_{\partial D}y\,dx=\frac12\int_{\partial D}x\,dy-y\,dx$.

(1) 星形线 $\begin{cases}x=a\cos^3t\\ y=a\sin^3t\end{cases}$. $0\le t\le 2\pi$.

$dy=3a\sin^2t\cos t\,dt \Rightarrow \sigma(D)=\int_0^{2\pi} a\cos^3t\cdot 3a\sin^2t\cos t\,dt = 3a^2\int_0^{2\pi}\cos^4t-\cos^6t\,dt$.

$\cos^4t = (\frac{1+\cos2t}{2})^2=\frac14\cos^22t+\frac12\cos2t+\frac14=\frac14\cdot\frac{1+\cos4t}{2}+\frac12\cos2t+\frac14=\frac38+\frac12\cos2t+\frac12\cos4t$.

$\cos^6t = (\frac{1+\cos2t}{2})^3=\frac18\cos^32t+\frac38\cos^22t+\frac38\cos2t+\frac18$

$=\frac18\cdot\frac{\cos6t+3\cos2t}{4}+\frac38\cdot\frac{1+\cos4t}{2}+\frac38\cos2t+\frac18$

$=\frac{5}{16}+\frac{15}{32}\cos2t+\frac{3}{16}\cos4t+\frac{1}{32}\cos6t$.

$\Rightarrow\sigma(D)=3a^2\cdot2\pi(\frac38-\frac{5}{16})=\frac38\pi a^2$.

(2) 双纽线. $(x^2+y^2)^2=a^2(x^2-y^2)$.

令 $y=x\tan\theta$. $[x^2(1+\tan^2\theta)]^2=a^2(1-\tan^2\theta)x^2$.

$\Rightarrow x^2(1+\tan^2\theta)^2=a^2(1-\tan^2\theta)$.

$x^2=a^2\cos^4\theta(1-\tan^2\theta)=a^2\cos^2\theta(\cos^2\theta-\sin^2\theta)\ge0$. $\theta\in[0,\frac\pi4)$.

第I象限: $x=a|\cos\theta|\sqrt{\cos2\theta}=a\cos\theta\sqrt{\cos2\theta}$. $y=a\sin\theta\sqrt{\cos2\theta}$

$dy=(a\cos\theta\sqrt{\cos2\theta}+a\sin\theta\frac{-2\sin2\theta}{2\sqrt{\cos2\theta}})d\theta=(a\cos\theta\sqrt{\cos2\theta}-\frac{a\sin\theta\sin2\theta}{\sqrt{\cos2\theta}})d\theta$.

$x\,dy=(a^2\cos^2\theta\cos2\theta-a^2\sin\theta\cos\theta\sin2\theta)d\theta$. $\frac12\sin2\theta+\frac14\sin4\theta\Big|_0^{\frac\pi4}$

$=a^2\cos\theta\cos3\theta\,d\theta=\frac{a^2}{2}(\cos2\theta+\cos4\theta)d\theta$.

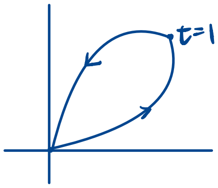
由对称性及坐标轴上积分为0. $\sigma(D)=4\int_{\Gamma_I}x\,dy=4\int_0^{\frac\pi4}\frac{a^2}{2}(\cos2\theta+\cos4\theta)d\theta=a^2$.

(3) 叶形线 $x^3+y^3=3axy\ (a>0)$

$y=xt$. $\quad x^3(1+t^3)=3ax^2t$. $\quad t\in(0,+\infty)$.

$\Rightarrow x=\frac{3at}{1+t^3}$. $\quad y=\frac{3at^2}{1+t^3}$. $\quad dx=\frac{3a[(1+t^3)-t\cdot 3t^2]}{(1+t^3)^2}dt=\frac{3a(1-2t^3)}{(1+t^3)^2}dt$.

$$\sigma(D)=\int_{\partial D}-ydx=9a^2\int_0^{+\infty}\frac{t^2(2t^3-1)}{(1+t^3)^3}dt \overset{u=t^3}{=} 3a^2\int_0^{+\infty}\frac{2u-1}{(1+u)^3}du$$



$\frac{2u-1}{(1+u)^3}=\frac{2}{(1+u)^2}-\frac{3}{(1+u)^3}$. $\quad =3a^2\int_0^{+\infty}\frac{2}{(1+u)^2}-\frac{3}{(1+u)^3}du$

$$=3a^2\left(-\frac{2}{1+u}+\frac{3}{2(1+u)^2}\right)\Big|_0^{+\infty}.$$

$$=3a^2\cdot\frac{1}{2}=\frac{3a^2}{2}.$$

11.3-3. 封闭曲线Γ: $\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$. $\alpha\le t\le\beta$. 参数增加为正向. 证明Γ围成的面积 $A=\frac{1}{2}\int_\alpha^\beta\begin{vmatrix}\varphi(t) & \psi(t)\\ \varphi'(t) & \psi'(t)\end{vmatrix}dt$.

$dx=\varphi'(t)dt$. $\quad dy=\psi'(t)dt$

$$A=\frac{1}{2}\int_\Gamma xdy-ydx=\frac{1}{2}\int_\alpha^\beta\varphi(t)\psi'(t)dt-\psi(t)\varphi'(t)dt=\frac{1}{2}\int_\alpha^\beta\begin{vmatrix}\varphi(t) & \psi(t)\\ \varphi'(t) & \psi'(t)\end{vmatrix}dt.$$

11.3-4. $f\in C^1$. Γ分段光滑封闭. 证明.

(1) $\int_\Gamma f(xy)(ydx+xdy)=0$.

(2) $\int_\Gamma f(x^2+y^2)(xdx+ydy)=0$.

(3) $\int_\Gamma f(x^n+y^n)(x^{n-1}dx+y^{n-1}dy)=0$.

(1) $P=f(xy)y$. $\quad Q=f(xy)x$.

$\frac{\partial P}{\partial y}=f(xy)+y\cdot f'(xy)\cdot x$. $\quad \frac{\partial Q}{\partial x}=f(xy)+x\cdot f'(xy)\cdot y$

$\Rightarrow \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0 \overset{\text{Green}}{\Longrightarrow}$ 原式$=0$.

(3) $P=f(x^n+y^n)x^{n-1}$. $\quad Q=f(x^n+y^n)y^{n-1}$.

$\frac{\partial P}{\partial y}=x^{n-1}f'(x^n+y^n)\cdot ny^{n-1}$. $\quad \frac{\partial Q}{\partial x}=y^{n-1}f'(x^n+y^n)\cdot nx^{n-1}$.

$\Rightarrow \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0 \overset{\text{Green}}{\Longrightarrow} =0$.

11.3-5. Γ: $\mathbb{R}^2$光滑封闭曲线. $\vec{n}$: 单位外法向. $\vec{a}$是固定单位向量. $\int_\Gamma\cos(\vec{a},\vec{n})ds=0$.

LHS $=\int_\Gamma\vec{a}\cdot\vec{n}\,ds \qquad \vec{n}\perp\vec{t}(s) \qquad \vec{t}\cdot ds=(dx,dy)$. $\Rightarrow \vec{n}\cdot ds=(dy,-dx)$.

$=\int_\Gamma a_1dy-a_2dx$.

$\overset{\text{Green}}{=}\iint_\Omega\frac{\partial a_1}{\partial x}+\frac{\partial a_2}{\partial y}\,dxdy=0$

11.3-6. 计算. $\int_\Gamma x\cos(\vec{n},\vec{i})+y\cos(\vec{n},\vec{j})\,ds$. Γ光滑封闭曲线.

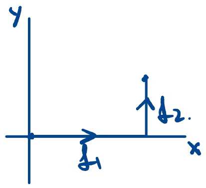
$=\int_\Gamma x\cdot\vec{n}\cdot\vec{i}+y\cdot\vec{n}\cdot\vec{j}\,ds \qquad \vec{n}\cdot\vec{i}\,ds=dy$. $\quad\vec{n}\cdot\vec{j}\,ds=-dx$.

$=\int_\Gamma \underset{Q}{x}dy-\underset{P}{y}dx=\iint_\Omega 1+1\,dxdy=2\sigma(\Omega)$

11.3-7. (1) $\int_L (x^2+2xy-y^2)dx + (x^2-2xy-y^2)dy$. L连接 $A=(0,0)$. $B=(2,1)$的任意光滑线段.

$P=x^2+2xy-y^2$. $Q=x^2-2xy-y^2$.

$\frac{\partial P}{\partial y}=2x-2y$. $\frac{\partial Q}{\partial x}=2x-2y$. $\Rightarrow$ 积分与路径无关.



L_1: $y=0$. $x: 0\to 2$. L_2: $x=2$, $y: 0\to 1$.

原式 $=\int_0^2 x^2dx + \int_0^1(4-4y-y^2)dy$

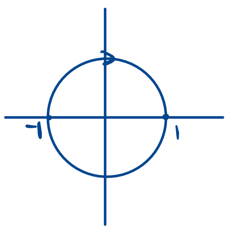
$=\frac{8}{3}+4-2-\frac{1}{3}=\frac{13}{3}$

(2) $\int_L \frac{(x+y)dx-(x-y)dy}{x^2+y^2}$. L: 上半平面 $(-1,0)\to(1,0)$ 任意光滑线段.

$P=\frac{x+y}{x^2+y^2}$. $Q=\frac{y-x}{x^2+y^2}$.

$\frac{\partial P}{\partial y}=\frac{x^2+y^2-2y(x+y)}{(x^2+y^2)^2}$ $\quad \frac{\partial Q}{\partial x}=\frac{-(x^2+y^2)-2x(y-x)}{(x^2+y^2)^2}$

$=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$ $\quad =\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$ $\Rightarrow$ 积分与路径无关.



$\begin{cases} x=\cos\theta. \\ y=\sin\theta. \end{cases}$ $\quad \theta: -\frac{\pi}{2}\to\frac{\pi}{2}$.

原式 $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta+\sin\theta)\cdot -\sin\theta d\theta - (\cos\theta-\sin\theta)\cos\theta d\theta$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi$.

11.3-8. 计算 $\int_\Gamma \frac{e^x}{x^2+y^2}[(x\sin y-y\cos y)dx+(x\cos y+y\sin y)dy]$. Γ: 原点在内部的封闭曲线.

Ω: Γ围成区域. $\tilde{\Omega}=\Omega\setminus B_\varepsilon(0)$. $\tilde{\Gamma}=\partial\tilde{\Omega}$ 有向.

$P=\frac{e^x}{x^2+y^2}(x\sin y-y\cos y)$. $\quad \frac{\partial P}{\partial y}=\frac{-2ye^x}{(x^2+y^2)^2}(x\sin y-y\cos y)+\frac{e^x}{x^2+y^2}(x\cos y-\cos y+y\sin y)$.

$Q=\frac{e^x}{x^2+y^2}(x\cos y+y\sin y)$. $\quad \frac{\partial Q}{\partial x}=\frac{e^x(x^2+y^2)-2xe^x}{(x^2+y^2)^2}(x\cos y+y\sin y)+\frac{e^x}{x^2+y^2}\cos y$

$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$ $\quad =\frac{e^x}{x^2+y^2}(x\cos y+y\sin y+\cos y)-\frac{2xe^x}{(x^2+y^2)^2}(x\cos y+y\sin y)$.

$=\frac{e^x}{x^2+y^2}\cdot 2\cos y-\frac{e^x}{(x^2+y^2)^2}(2x^2\cos y+2xy\sin y-2xy\sin y+2y^2\cos y)=0$.

$\Rightarrow \int_{\tilde{\Gamma}} Pdx+Qdy=0$.

$\int_{\partial B_\varepsilon(0)} Pdx+Qdy=\frac{1}{\varepsilon^2}\int_{\partial B_\varepsilon(0)} e^x[(x\sin y-y\cos y)dx+(x\cos y+y\sin y)dy]$

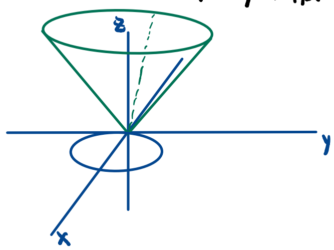
$P'=e^x(x\sin y-y\cos y)$ $\quad \frac{\partial P'}{\partial y}=e^x(x\cos y-\cos y+y\sin y)$.

$Q'=e^x(x\cos y+y\sin y)$ $\quad \frac{\partial Q'}{\partial x}=e^x(x\cos y+y\sin y+\cos y)$

$\int_{\partial B_\varepsilon(0)} Pdx+Qdy=\frac{1}{\varepsilon^2}\iint_{B_\varepsilon(0)} e^x\cdot 2\cos y\, dxdy \xrightarrow{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\cdot 2\pi\varepsilon^2=2\pi$.

$\therefore$ 原式 $=\left(\int_{\tilde{\Gamma}}+\int_{\partial B_\varepsilon(0)}\right)(Pdx+Qdy)=0+2\pi=2\pi$.

12.1-1. $z=\sqrt{x^2+y^2}$. 被 $x^2+y^2=2x$ 截.



$$\sigma(D)=\iint\limits_{\substack{x^2+y^2\le 2x\\ z=\sqrt{x^2+y^2}}} d\sigma. \qquad \frac{\partial z}{\partial x}=\frac{x}{z}.\ \frac{\partial z}{\partial y}=\frac{y}{z}.$$

$$d\sigma=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\,dxdy=\sqrt{2}\,dxdy.$$

$$\sigma(D)=\iint\limits_{x^2+y^2\le 2x}\sqrt{2}\,dxdy=\sqrt{2}\pi.$$

12.1-3. $x^2+y^2=a^2$. 介于平面 $x\pm z=0$ 之间

$\vec{r}=(a\cos\theta, a\sin\theta, z)$. 对 $x>0$. $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

$-a\cos\theta\le z\le a\cos\theta$ $\quad \vec{r}_\theta=(-a\sin\theta, a\cos\theta, 0)$ $\quad \vec{r}_z=(0,0,1)$.

$\vec{r}_\theta\times\vec{r}_z=(a\cos\theta, a\sin\theta, 0)$ $\quad \|\vec{r}_\theta\times\vec{r}_z\|=a$.

$$\frac{1}{2}\sigma(D)=\int_{-\pi/2}^{\pi/2}\int_{-a\cos\theta}^{a\cos\theta}a\,dz\,d\theta=\int_{-\pi/2}^{\pi/2}2a^2\cos\theta\,d\theta=4a^2$$

$\therefore\ \sigma(D)=8a^2$

12.1-5. $az=xy$ 被 $x^2+y^2=a^2$ 截.

$\vec{r}=(x,y,\frac{xy}{a})$. $x^2+y^2\le a^2$.

$\vec{r}_x=(1,0,\frac{y}{a})$. $\vec{r}_y=(0,1,\frac{x}{a})$. $\quad \vec{r}_x\times\vec{r}_y=(-\frac{y}{a},-\frac{x}{a},1)$

$\|\vec{r}_x\times\vec{r}_y\|=\sqrt{1+\frac{y^2}{a^2}+\frac{x^2}{a^2}}$

$$\sigma(D)=\iint\limits_{x^2+y^2\le a^2}\sqrt{1+\frac{x^2+y^2}{a^2}}\,dxdy=\int_0^a\int_0^{2\pi}\sqrt{1+\frac{r^2}{a^2}}\cdot r\,d\theta\,dr=2\pi\cdot\frac{a^2}{3}\left(1+\frac{r^2}{a^2}\right)^{\frac{3}{2}}\Big|_0^a=\frac{2\pi}{3}a^2(2\sqrt{2}-1).$$

12.1-7. 螺旋面 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=h\theta.\end{cases}$ $\quad 0<r<a.\ 0\le\theta\le 2\pi.$

$\vec{R}(r,\theta)=(r\cos\theta, r\sin\theta, h\theta)$.

$\vec{R}_r=(\cos\theta,\sin\theta,0)$. $\quad \vec{R}_\theta=(-r\sin\theta, r\cos\theta, h)$.

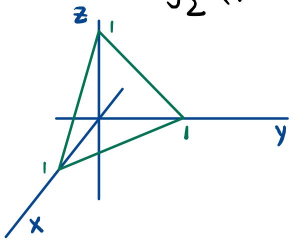
$\vec{R}_r\times\vec{R}_\theta=(h\sin\theta, -h\cos\theta, r)$ $\quad \|\vec{R}_r\times\vec{R}_\theta\|=\sqrt{h^2+r^2}$.

$$\sigma(D)=\iint\limits_{\substack{0<r<a\\ 0\le\theta\le 2\pi}}\sqrt{h^2+r^2}\,dr\,d\theta=2\pi\int_0^a\sqrt{h^2+r^2}\,dr\overset{t=\frac{r}{h}}{=}2\pi\int_0^{\frac{a}{h}}h^2\sqrt{1+t^2}\,dt.$$

$$\int_0^x\sqrt{1+t^2}\,dt\ \underset{u=\log(t+\sqrt{1+t^2})}{\overset{t=\sinh u}{=}}\ \int_0^{u_0}\cosh u\,d\sinh u=\int_0^{u_0}\cosh^2u\,du=\int_0^{u_0}\frac{1}{2}\cosh 2u+\frac{1}{2}\,du=\underbrace{\frac{1}{4}\sinh 2u_0}_{\frac{1}{2}\sinh u_0\cosh u_0=\frac{1}{2}x\sqrt{1+x^2}}+\frac{1}{2}u_0=\frac{1}{2}x\sqrt{1+x^2}+\frac{1}{2}\log(x+\sqrt{1+x^2}).$$

$$\Rightarrow\ \sigma(D)=2\pi h^2\left[\frac{1}{2}\cdot\frac{a}{h}\sqrt{1+\frac{a^2}{h^2}}+\frac{1}{2}\log\left(\frac{a}{h}+\sqrt{1+\frac{a^2}{h^2}}\right)\right]=\pi a\sqrt{h^2+a^2}+\pi h^2\log\left(\frac{a}{h}+\sqrt{1+\frac{a^2}{h^2}}\right).$$

12.2-1. $\int_\Sigma \frac{d\sigma}{(1+x+y)^2}$. Σ: $x+y+z\le 1$. $x\ge 0$. $y\ge 0$. $z\ge 0$ 边界.



$\Sigma_1: \{x+y+z=1\}\cap\Sigma$.

$\Sigma_2: \{x=0\}\cap\Sigma$. $\Sigma_3: \{y=0\}\cap\Sigma$. $\Sigma_4: \{z=0\}\cap\Sigma$.

Σ_1: $r_x=(1,0,-1)$. $r_y=(0,1,-1)$. $r_x\times r_y=(1,1,1)$. $\|r_x\times r_y\|=\sqrt{3}$.

$$\int_{\Sigma_1}\frac{d\sigma}{(1+x+y)^2}=\iint_{\substack{0\le x+y\le 1\\0\le x\le 1\\0\le y\le 1}}\frac{\sqrt{3}}{(1+x+y)^2}dxdy=\int_0^1\int_0^{1-x}\frac{\sqrt{3}}{(1+x+y)^2}dydx=\sqrt{3}\int_0^1\frac{1}{1+x}-\frac{1}{2}dx=\sqrt{3}\log(1+x)\Big|_0^1-\frac{\sqrt{3}}{2}.$$
$$=\sqrt{3}\left(\log 2-\frac{1}{2}\right).$$

Σ_2: $x=0$; $0\le y+z\le 1$; $y\ge 0$. $z\ge 0$. $\|r_y\times r_z\|=1$.

$$\int_{\Sigma_2}\frac{d\sigma}{(1+x+y)^2}=\iint_{\substack{0\le y+z\le 1\\y\ge 0\\z\ge 0}}\frac{1}{(1+y)^2}dydz=\int_0^1\int_0^{1-z}\frac{1}{(1+y)^2}dydz=\int_0^1 1-\frac{1}{2-z}dz=1-\log 2.$$

Σ_3: y 与 x 对称 $\Rightarrow \int_{\Sigma_3}=\int_{\Sigma_2}$.

Σ_4: $z=0$, $0\le x+y\le 1$, $x\ge 0$. $y\ge 0$. $\|r_x\times r_y\|=1$.

$$\int_{\Sigma_4}\frac{d\sigma}{(1+x+y)^2}=\iint_{\substack{0\le x+y\le 1\\x\ge 0\\y\ge 0}}\frac{1}{(1+x+y)^2}dxdy=\log 2-\frac{1}{2}\quad(\text{计算同}\Sigma_1).$$

$$\Rightarrow\int_\Sigma\frac{1}{(1+x+y)^2}d\sigma=(\sqrt{3}+1)\left(\log 2-\frac{1}{2}\right)+2(1-\log 2).$$
$$=(\sqrt{3}-1)\log 2+\frac{3-\sqrt{3}}{2}.$$

12.2-2. $\int_\Sigma|xyz|\,d\sigma$. Σ: $z=x^2+y^2$. $z\le 1$.

$r_x=(1,0,2x)$. $r_y=(0,1,2y)$. $r_x\times r_y=(-2x,-2y,1)$ $\|r_x\times r_y\|=\sqrt{4(x^2+y^2)+1}$.

$$\int_\Sigma|xyz|\,d\sigma=\iint_{0\le x^2+y^2\le 1}|xy|\cdot(x^2+y^2)\sqrt{4(x^2+y^2)+1}\,dxdy.$$
$$=\int_0^{2\pi}\int_0^1 r^2|\cos\theta\sin\theta|\cdot r^2\sqrt{4r^2+1}\cdot r\,drd\theta$$
$$=\frac{1}{2}\int_0^{2\pi}|\sin 2\theta|\,d\theta\int_0^1 r^5\sqrt{4r^2+1}\,dr.$$
$$=2\int_0^1 r^5\sqrt{4r^2+1}\,dr$$

$u=\sqrt{4r^2+1}$. $du=\frac{4r}{\sqrt{4r^2+1}}dr$. $r^2=\frac{u^2-1}{4}$.

$$=\frac{1}{2}\int_0^1 r^4(4r^2+1)\cdot\frac{4r}{\sqrt{4r^2+1}}dr=\int_1^{\sqrt{5}}\left(\frac{u^2-1}{4}\right)^2\cdot u^2\cdot du$$
$$=\frac{1}{32}\int_1^{\sqrt{5}}u^6-2u^4+u^2\,du=\frac{1}{32}\left[\left(\frac{125\sqrt{5}}{7}-\frac{2}{5}\cdot 25\sqrt{5}+\frac{5\sqrt{5}}{3}\right)-\left(\frac{1}{7}-\frac{2}{5}+\frac{1}{3}\right)\right].$$
$$=\frac{1}{32}\cdot\frac{8}{105}(125\sqrt{5}-1)$$
$$=\frac{125\sqrt{5}-1}{420}.$$

12.2-3. $\int_\Sigma(xy+yz+zx)\,d\sigma$. $\Sigma: z=\sqrt{x^2+y^2}$ 被 $x^2+y^2=2x$ 截.

$r_x=(1,0,\frac{x}{z})$. $r_y=(0,1,\frac{y}{z})$. $r_x\times r_y=(-\frac{x}{z},-\frac{y}{z},1)$ $\|r_x\times r_y\|=\sqrt{1+\frac{x^2+y^2}{z^2}}=\sqrt{2}$.

$$=\iint_{x^2+y^2\le 2x}\left[xy+(x+y)\sqrt{x^2+y^2}\right]\cdot\sqrt{2}\,dxdy$$

$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$. $r^2\le 2r\cos\theta\Rightarrow 0\le r\le 2\cos\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\left[r^2\cos\theta\sin\theta+r(\cos\theta+\sin\theta)\cdot r\right]r\,drd\theta=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}r^3(\cos\theta\sin\theta+\cos\theta+\sin\theta)\,drd\theta.$$
$$=\sqrt{2}\int_{-\pi/2}^{\pi/2}4\cos^4\theta(\cos\theta\sin\theta+\cos\theta+\sin\theta)\,d\theta.$$
$$=4\sqrt{2}\int_{-\pi/2}^{\pi/2}-(\cos^5\theta+\cos^4\theta)\,d\cos\theta+4\sqrt{2}\int_{-\pi/2}^{\pi/2}\cos^5\theta\,d\theta.\quad=4\sqrt{2}\int_{-\pi/2}^{\pi/2}\cos^5\theta\,d\theta.$$

$\cos^5\theta = \cos^2\theta\cdot\cos^3\theta = \frac{\cos2\theta+1}{2}\cdot\frac{\cos3\theta+3\cos\theta}{4} = \frac18\cos3\theta+\frac38\cos\theta+\frac18\cos2\theta\cos3\theta+\frac38\cos2\theta\cos\theta.$

$= \frac18\cos3\theta+\frac38\cos\theta+\frac1{16}(\cos5\theta+\cos\theta)+\frac3{16}(\cos\theta+\cos3\theta).$

$= \frac58\cos\theta+\frac5{16}\cos3\theta+\frac1{16}\cos5\theta.$

$= 4\sqrt2\int_{-\pi/2}^{\pi/2}\frac58\cos\theta+\frac5{16}\cos3\theta+\frac1{16}\cos5\theta\, d\theta.$

$= 4\sqrt2\left(\frac58\cdot2-\frac5{16}\cdot\frac13\cdot2+\frac1{16}\cdot\frac15\cdot2\right).$

$= \frac{64}{15}\sqrt2.$

12.3-1. (1) $\iint_\Sigma x^4dydz+y^4dzdx+z^4dxdy$. $\Sigma: x^2+y^2+z^2=a^2$. 内侧.

$\iint_\Sigma x^4dydz = \underset{-}{\overset{x>0}{\iint_{\Sigma_1}}} x^4dydz + \underset{+}{\overset{x<0}{\iint_{\Sigma_2}}} x^4dydz$ $\quad\vec n=-\frac1a(x,y,z)$.

$= -\iint_{y^2+z^2\le a^2}(a^2-y^2-z^2)^2dydz + \iint_{y^2+z^2\le a^2}(a^2-y^2-z^2)^2dydz = 0.$

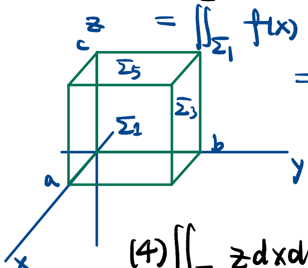
由对称性. 原式 $=0$.

(2) $\iint_\Sigma xzdydz+yzdzdx+x^2dxdy$. $\Sigma: x^2+y^2+z^2=a^2$. 外侧.

$\vec F=(xz,yz,x^2)\quad \vec n=\frac1a(x,y,z)$.

$= \iint_\Sigma\frac1a(x^2z+y^2z+x^2z)\,d\sigma. = \frac1a\iint_\Sigma(2x^2+y^2)z\,d\sigma \overset{关于z奇}{=} 0.$

(3) $\iint_\Sigma f(x)dydz+g(y)dzdx+h(z)dxdy$. $\Sigma: [0,a]\times[0,b]\times[0,c]$边界. 外侧.



$= \iint_{\Sigma_1}f(x)\,d\sigma+\iint_{\Sigma_2}-f(x)\,d\sigma+\iint_{\Sigma_3}g(y)\,d\sigma+\iint_{\Sigma_4}-g(y)\,d\sigma+\iint_{\Sigma_5}h(z)\,d\sigma+\iint_{\Sigma_5}-h(z)\,d\sigma.$

$= [f(a)-f(0)]bc+[g(b)-g(0)]ac+[h(c)-h(0)]ab.$

(4) $\iint_\Sigma zdxdy$. $\Sigma: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. 外侧.

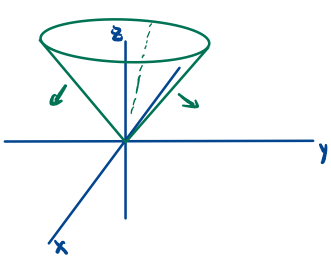
$= \underset{+}{\overset{z>0}{\iint_{\Sigma_1}}} zdxdy + \underset{-}{\overset{z<0}{\iint_{\Sigma_2}}} zdxdy.$

$= \iint_{0\le\frac{x^2}{a^2}+\frac{y^2}{b^2}\le1} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy - \iint_{0\le\frac{x^2}{a^2}+\frac{y^2}{b^2}\le1} c\cdot-\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy$

$= 2c\iint_{0\le\frac{x^2}{a^2}+\frac{y^2}{b^2}\le1}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy \qquad \begin{cases}x=ar\cos\theta\\ y=br\sin\theta\end{cases} \quad \frac{\partial(x,y)}{\partial(r,\theta)}=abr.$

$= 2c\int_0^1\int_0^{2\pi}abr\sqrt{1-r^2}\,d\theta dr = 2abc\cdot2\pi\int_0^1 r\sqrt{1-r^2}\,dr = \frac{4\pi}{3}abc$

(5) $\iint_\Sigma(y-z)dydz+(z-x)dzdx+(x-y)dxdy$. $\Sigma: z=\sqrt{x^2+y^2}$. $z\le h$. 外侧.



$\iint_\Sigma(y-z)dydz = \underset{+}{\iint_{\Sigma\cap\{x>0\}}}(y-z)dydz + \underset{-}{\iint_{\Sigma\cap\{x<0\}}}(y-z)dydz = \iint_{D_{yz}}(y-z)dydz - \iint_{D_{yz}}(y-z)dydz$ ← 在yz平面的投影.

同理 $\iint_\Sigma(z-x)dzdx=0$.

$\underbrace{\iint_\Sigma(x-y)\,dxdy}_{第II型曲面积分} = -\underbrace{\overset{z>0}{\iint_\Sigma}(x-y)\,dxdy}_{第I型曲面积分} = -\iint_{x^2+y^2\le h^2}(x-y)dxdy = 0$ ← 积分关于x,y对称. ($\iint xdxdy=\iint ydxdy$).

12.3-2. 流速场 $\vec{F}=(y,z,x)$. 封闭曲面 $x^2+y^2=R^2$. $z=0$. $z=h$. 计算 $\vec{F}$ 流向曲面之外的流量.

流出曲面的流量即 $\iint_\Sigma \vec{F}\cdot\vec{n}\,d\sigma$ 　$\vec{n}$ 为外法向.

Σ_1: $x^2+y^2=R^2$. $0\le z\le h$.
$\vec{n}=\frac{1}{R}(x,y,0)$.

$\iint_{\Sigma_1}\vec{F}\cdot\vec{n}\,d\sigma=\iint_{\Sigma_1}\frac{1}{R}(xy+yz)\,d\sigma$.

$=\iint_{\substack{\Sigma_1\\(y>0)}}+\frac{1}{R}y(x+z)\,d\sigma+\iint_{\substack{\Sigma_1\\(y<0)}}-\frac{1}{R}y(x+z)\,d\sigma$.

$=0$.

Σ_2: $z=0$. $x^2+y^2\le R^2$.
$\vec{n}=(0,0,-1)$.

$\iint_{\Sigma_2}\vec{F}\cdot\vec{n}\,d\sigma=\iint_{\Sigma_2}-x\,d\sigma$

$=0$.

Σ_3: $z=h$. $x^2+y^2\le R^2$.
$\vec{n}=(0,0,1)$.

$\iint_{\Sigma_3}\vec{F}\cdot\vec{n}\,d\sigma=\iint_{\Sigma_3}x\,d\sigma$

$=0$.

$\therefore$ 总流出量为 0.

Homework of Mathematical Analysis(A2)

Week15

12.3

3

直接仿照本节式(10)推导即可。正负号的确定应予说明：我们所取的是向上的法向量，所以法向量与 z 轴夹角为锐角，符号取正。

12.4

1

(1)(2)

直接利用 ***Gauss*** 公式计算即可。

(3)(4)

注意曲面并不封闭。需要补上一个曲面

$$\tilde{\Sigma}:\ z=1,\ x^2+y^2\le 1$$

构成封闭曲面才可利用 ***Gauss*** 公式。计算时注意方向是否正确。（一个简单的判断结果正负的方式：如果积分号内表达式处处非负，即与法向量夹角始终为锐角，那么结果肯定为正。）

2

利用

$$\int_{\partial\Omega}\cos(\mathbf{e},\mathbf{n})\mathrm{d}\sigma=\int_{\partial\Omega}\mathbf{e}\cdot\mathbf{n}\,\mathrm{d}\sigma$$

再运用 ***Gauss*** 公式。

3

与**2**类似。利用

$$\cos(\mathbf{p},\mathbf{n})=\frac{\mathbf{p}\cdot\mathbf{n}}{p}$$

4

(1)

(2)

(3)

$$\begin{aligned}\nabla\times(y^2,z^2,x^2)&=(-2z,-2x,-2y)\\ \int_\Gamma y^2\mathrm{d}x+z^2\mathrm{d}y+x^2\mathrm{d}z&=\int_S(-2z,-2x,-2y)\cdot\mathbf{n}\mathrm{d}\sigma\\ &=\frac{1}{\sqrt{3}}\int_S(2z+2x+2y)\mathrm{d}\sigma\\ &=\frac{2a}{\sqrt{3}}\int_S\mathrm{d}\sigma\\ &=\frac{2a}{\sqrt{3}}\cdot\pi(\frac{\sqrt{2}a}{\sqrt{3}})^2\\ &=\frac{4\pi a^3}{3\sqrt{3}}\end{aligned}$$

5

拆开验算即可。计算得出：

$$\nabla\times(\mathbf{a}\times\mathbf{p})=\nabla\times(a_2z-a_3y,a_3x-a_1z,a_1y-a_2x)=2\mathbf{a}.$$

证毕。

6

7

8

12.5

很容易。Omit.

13.1

1

$$\begin{aligned}\partial_x(\frac{f}{g})&=\frac{f_xg-g_xf}{g^2}\\ \nabla(\frac{f}{g})=(\frac{f_xg-g_xf}{g^2},\frac{f_yg-g_yf}{g^2},\frac{f_zg-g_zf}{g^2})&=\frac{g\nabla f-f\nabla g}{g^2}\end{aligned}$$

2, 3

Nabla 算子的运算法则，除了旋度的部分公式记忆起来相对费力一点外，梯度和散度的部分都是容易记忆的（也更常用）。后者应该熟记。

4

f 沿 $\mathbf{u}$ 方向的变化率为

$$\frac{\partial f}{\partial\mathbf{u}}=\nabla f\cdot\frac{\mathbf{u}}{||\mathbf{u}||}$$

故为

$$\nabla f\cdot\nabla g/||\nabla g||$$

5

对于任意的方向 **e**，有

$$\begin{aligned}
&(\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}\iint_{\partial\Omega}u\mathbf{n}\mathrm{d}\sigma)\cdot\mathbf{e}\\
&=\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}\iint_{\partial\Omega}u\mathbf{e}\cdot\mathbf{n}\,\mathrm{d}\sigma\\
(Gauss)&=\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}\iiint_{\Omega}\nabla\cdot(u\mathbf{e})\,\mathrm{d}V\\
&=\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}\iiint_{\Omega}\nabla u\cdot\mathbf{e}\,\mathrm{d}V\\
&=\nabla u(\mathbf{p})\cdot\mathbf{e}
\end{aligned}$$

故

$$\nabla u(\mathbf{p})=\lim_{\Omega\to\mathbf{p}}\frac{1}{\mu(\Omega)}\iint_{\partial\Omega}u\mathbf{n}\,\mathrm{d}\sigma$$

13.2

1

直接计算即可。（跳步情况严重。计算确实很基本，不过希望大家是确实会算才跳的步。）

2

直接计算即可。

$$\begin{aligned}
&\triangle(fg)=\nabla\cdot\nabla(fg)=\nabla\cdot(f\nabla g+g\nabla f)\\
&=f\nabla\cdot\nabla g+\nabla f\cdot\nabla g+g\nabla\cdot\nabla f+\nabla g\cdot\nabla f\\
&=f\triangle g+g\triangle f+2\nabla f\cdot\nabla g
\end{aligned}$$

3

(1)

只需注意到

$$\frac{\partial u}{\partial\mathbf{n}}=\nabla u\cdot\mathbf{n},\ \ \text{for }||\mathbf{n}||=1.$$

再利用 ***Gauss*** 公式。

(2)(3)

同**(1)**。

4

(1)(2)

直接利用 **3** 的结论。

Week16

13.3

1

只计算 *x* 分量。

$$
\nabla\times(\nabla\times F)\cdot\mathbf{e}_x=\partial_y(\partial_x F_y-\partial_y F_x)-\partial_z(\partial_z F_x-\partial_x F_z)
$$
$$
(\nabla(\nabla\cdot F)-\nabla^2 F)\cdot e_x=\partial_x(\partial_x F_x+\partial_y F_y+\partial_z F_z)-(\partial_x^2+\partial_y^2+\partial_z^2)F_x
$$

化简即可。

2

与13.1.5类似，注意到

$$
(\mathbf{n}\times\mathbf{F})\cdot\mathbf{e}=(\mathbf{F}\times\mathbf{e})\cdot\mathbf{n}
$$

3

$$
\int_{\partial\Omega}\frac{\partial f}{\partial\mathbf{n}}\mathrm{d}\sigma=\int_\Omega\nabla\cdot\nabla f\mathrm{d}V=\int_\Omega(\partial_x^2 f+\partial_y^2 f+\partial_z^2 f)\mathrm{d}V
$$

我们已知：

$$
af=div(f\nabla f)=(\partial_x f)^2+(\partial_y f)^2+(\partial_z f)^2+f(\partial_x^2 f+\partial_y^2 f+\partial_z^2 f)
$$
$$
bf=||\nabla f||^2=(\partial_x f)^2+(\partial_y f)^2+(\partial_z f)^2
$$

故

$$
\partial_x^2 f+\partial_y^2 f+\partial_z^2 f=a-b
$$
$$
\int_{\partial\Omega}\frac{\partial f}{\partial\mathbf{n}}\mathrm{d}\sigma=\int_\Omega(\partial_x^2 f+\partial_y^2 f+\partial_z^2 f)\mathrm{d}V=(a-b)\mu(\Omega)
$$

13.4

1

(1)

先验证：

$$
\nabla\times\mathbf{F}=0
$$

再任取起点计算。（注意要避开无定义的点）

$$
\begin{aligned}
\phi&=\int_{(1,1,1)}^{(x,y,z)}\mathbf{F}\cdot\mathrm{d}\mathbf{p}=\int_1^x(1,2x',-x')\cdot\mathbf{e}_x\mathrm{d}x'+\int_1^y(1-\frac{1}{y'}+y',x+\frac{x}{y'^2},-xy')\cdot\mathbf{e}_y\mathrm{d}y'\\
&+\int_1^z(1-\frac{1}{y}+\frac{y}{z'},\frac{x}{z'}+\frac{x}{y^2},-\frac{xy}{z'^2})\cdot\mathbf{e}_z\mathrm{d}z'\\
&=\int_1^x 1\,\mathrm{d}x'+\int_1^y(x+\frac{x}{y'^2})\,\mathrm{d}y'+\int_1^z(-\frac{xy}{z'^2})\,\mathrm{d}z'\\
&=(x-1)+[x(y-1)-x(\frac{1}{y}-1)]+xy(\frac{1}{z}-1)\\
&=\frac{xy}{z}-\frac{x}{y}+x-1
\end{aligned}
$$

全部的势函数还需要加上一个任意常数 *C*。

(2)

计算繁琐一些，但是方法没有本质区别。

2

(1)(2)(3)

当然，容易验证它们都是无旋的，所以只需要选任意一条路径积分即可。不过，前两题的势函数是容易看出的（分别是 ***(x^2)/2+(y^3)/3-(z^3)/3, xyz***），直接带入初末值相减即可。第三题则要选取一个合适的路径。

4

没什么好说的。不过题目的意思有点模糊，最后需要化成什么样的形式?

6

(1)

$$\mathrm{d}(\frac{1}{2}(x^2+y^2))=0$$

(3)

$$\mathrm{d}(\frac{1}{2}(x^2+y^2)+2xy)=0$$

(5)

$$\mathrm{d}(xe^y-y^2)=0$$

(7)

$$\mathrm{d}(\arctan(\frac{y}{x}))=\mathrm{d}(\frac{1}{2}(x^2+y^2))$$