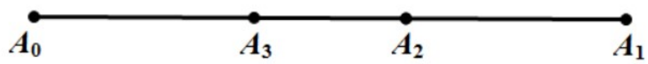
第一次作业答案

第一题

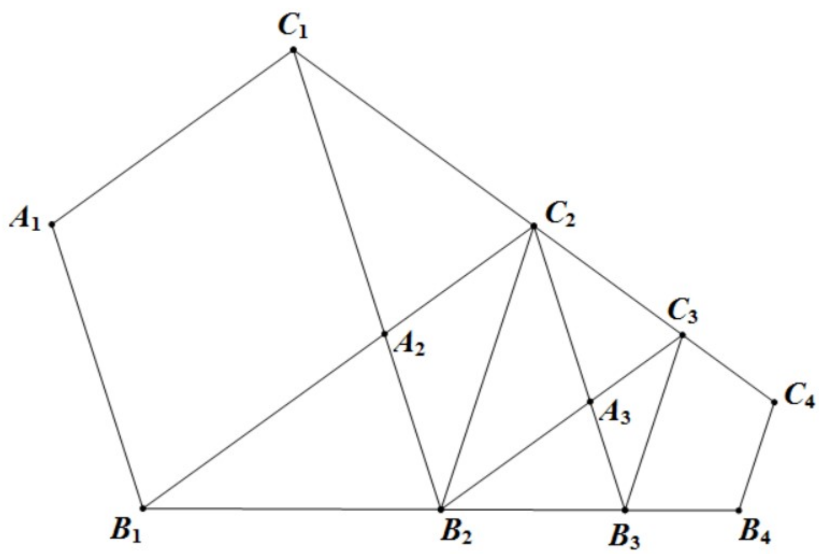


（1）如图，在线段 A_0A_1 上，取点 A_2 使得 $\dfrac{A_0A_1}{A_0A_2}=\dfrac{A_0A_2}{A_2A_1}$，再取点 A_3 使得 $A_0A_3=A_2A_1$。请验证：$\dfrac{A_0A_2}{A_0A_3}=\dfrac{A_0A_3}{A_3A_2}$。
（2）由自相似性，应该如何选取 A_4？
解：（1）首先，$\dfrac{A_0A_1}{A_0A_2}=\dfrac{A_0A_2+A_2A_1}{A_0A_3+A_3A_2}=\dfrac{A_0A_2+A_0A_3}{A_0A_3+A_3A_2}$；其次，$\dfrac{A_0A_2}{A_2A_1}=\dfrac{A_0A_2}{A_0A_3}$；最后，由于 $\dfrac{A_0A_1}{A_0A_2}=\dfrac{A_0A_2}{A_2A_1}$，故 $\dfrac{A_0A_2+A_0A_3}{A_0A_3+A_3A_2}=\dfrac{A_0A_2}{A_0A_3}$，因此 $\dfrac{A_0A_3}{A_3A_2}=\dfrac{A_0A_2}{A_0A_3}$。
（2）观察，A_2 的位置满足 $\dfrac{A_0A_1}{A_0A_2}=\dfrac{A_0A_2}{A_2A_1}$，$A_3$ 的位置满足 $\dfrac{A_0A_2}{A_0A_3}=\dfrac{A_0A_3}{A_3A_2}$，因此继续下去，$A_4$ 的位置应满足 $\dfrac{A_0A_3}{A_0A_4}=\dfrac{A_0A_4}{A_4A_3}$。类比 A_3 的位置选取（$A_0A_3=A_2A_1$），就知道 A_4 的位置应为 $A_0A_4=A_3A_2$。用（1）的思路可以证明这样选取的 A_4 满足 $\dfrac{A_0A_3}{A_0A_4}=\dfrac{A_0A_4}{A_4A_3}$。

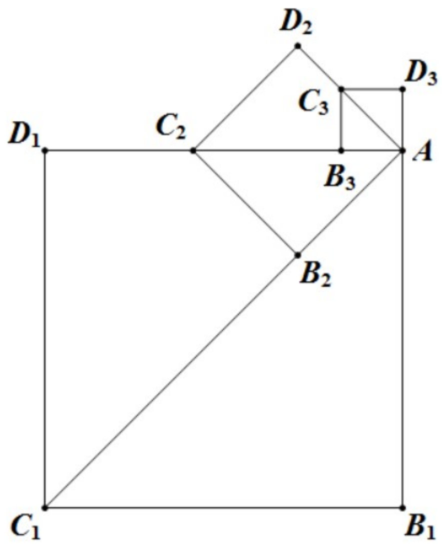
第二题　称两个长度为 a,b 的线段是**可公度的**，如果存在一个长度为 c 的线段使得 $a=mc,b=nc$，其中 m,n 均为整数。
（1）证明：a,b 是可公度的，当且仅当 a/b 是有理数；
（2）以下为辗转丈量法的过程：
不妨设 $a>b$（不然 $a=b$，则令 $c=a$），则有 $a=n_1b+a_1$，其中 n_1 为正整数，$0\le a_1<b$。若 $a_1=0$，则令 $c=b$；否则，$a>b>a_1$，对 (b,a_1) 重复上述操作，得 $b=n_2a_1+b_1$，其中 n_2 为正整数，$0\le b_1<a_1$。若 $b_1=0$，则令 $c=a_1$，有 $b_2=n_2c$，$a=(n_1n_2+1)c$；否则 $a>b>a_1>b_1$，再对 (a_1,b_1) 重复上述操作，从而给出一个序列 $\{a,b,a_1,b_1,\cdots\}$，直到为 0。
证明：a,b 是可公度的，当且仅当经过有限步辗转丈量后某个 a_n 或 b_n 等于 0。
解：
（1）若 a,b 可公度，则存在 c 使得 $a=mc,b=nc$，m,n 均为正整数，则 $a/b=m/n$ 为有理数；若 a/b 为有理数，则存在正整数 m,n 使得 $a/b=m/n$，令 $c=a/m$，则 $c=b/n$，因此 $a=mc,b=nc$，故 a,b 可公度。

（2）若 a,b 可公度，则存在 c 使得 $a=pc,b=qc$，不妨 $p>q$。则根据辗转丈量的过程，我们能得到序列 $\{a,b,a_1,b_1,\cdots\}$，其中 $a_i=p_ic,b_i=q_ic$，$i\geq 1$，且非负整数序列 $\{p,q,p_1,q_1,\cdots\}$ 是严格单调减的，故必在有限项 p_k 或 q_k 变为 0，故对应的 a_k 或 b_k 就是 0，因此辗转丈量比在有限步结束；
若 a,b 的辗转丈量在有限步结束，则有（只证明 a_k 先变为 0 的情况，b_k 先变为 0 的情况同理可证）$a=n_1b+a_1,b=n_2a_1+b_1,a_1=n_3b_1+a_2,b_1=n_4a_2+b_2,\cdots,a_k=n_{2k+1}b_k+a_{k+1}$，其中 $a_{k+1}=0$。则取 $c=b_k$，有 $a_k=n_{2k+1}c$，$b_{k-1}=(n_{2k}n_{2k+1}+1)c$，以此类推，最后得到 $a=mc,b=nc$，因此 a,b 可公度。

第三题



（1）如图构建无穷正五边形序列。其中 $B_1C_2=a_1$，$B_1B_2=b_1$，$B_2B_3=a_2$，$B_3B_4=b_2$，以此类推，证明这个序列是公比小于 1 的等比序列，从而 $\{a_1,b_1,a_2,b_2,\cdots\}$ 严格递减趋于 0。
（2）在边长为 1 的正方形中作出类似的图，证明 $\sqrt{2}$ 是无理数。
解：
（1）由 $\triangle B_1B_2C_2\sim\triangle B_2B_3C_3\sim\triangle B_3B_4C_4$，得 $\dfrac{B_1C_2}{B_1B_2}=\dfrac{B_2C_3}{B_2B_3}=\dfrac{B_3B_4}{B_4C_4}$，因此 $a_1/b_1=b_1/a_2=a_2/b_2$，以此类推，知序列 $\{a_1,b_1,\cdots\}$ 是等比序列。又 $a_1>b_1$，故公比 $q=b_1/a_1<1$，因此它是公比小于 1 的等比序列。
（2）如图。

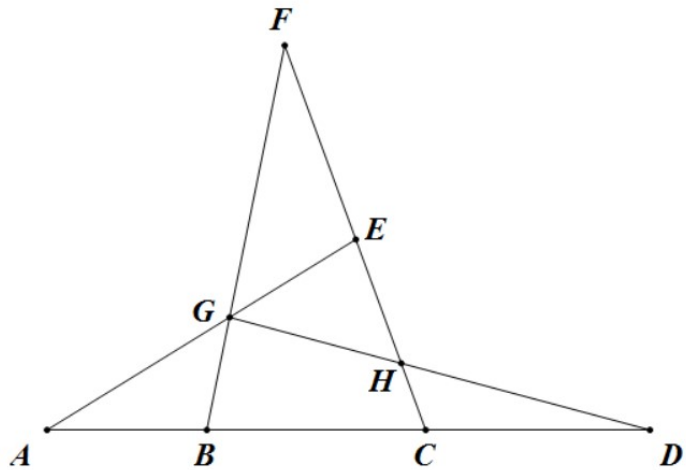


作正方形 $AB_1C_1D_1$，令 $AB_1 = b_1 = 1, AC_1 = a_1 = \sqrt{2}$。要证 $\sqrt{2} = a_1/b_1$ 不可公度。在 AC_1 上取 B_2 使得 $B_1C_1 = C_1B_2$，作正方形 $AB_2C_2D_2$，令 $AC_2 = a_2, AB_2 = b_2$,以此类推。则 $\{a_1, b_1, a_2, b_2, \cdots\}$ 满足 $a_1 - b_1 = b_2, b_1 - b_2 = a_2, \cdots$。而由于 $\{a_1, a_2, \cdots\}$ 与 $\{b_1, b_2, \cdots\}$ 均为公比小于 1 的等比序列，因此这个过程可以无限进行下去。假设 a_1, b_1 可公度，即存在 c 使得 $a_1 = m_1c, b_1 = n_1c$（$m_1 > n_1$），则有 $a_k = m_kc, b_k = n_kc$，且 $m_k > m_{k+1}, n_k > n_{k+1}$，因此这个过程必在有限步内结束，矛盾。因此 a_1, b_1 不可公度，即 $\sqrt{2}$ 是无理数。

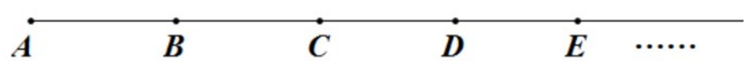
第二次作业答案

第一题　利用关联公理与顺序公理证明：直线上存在无穷个点。

解：先证明如下引理：**一条直线上有 A, B, C, D 四点。若 B 在线段 AC 内，C 在线段 BD 内，则 C 在线段 AD 内**。如图。根据公理 I_3，取直线外一点 E，又根据公理 II_2，取一点 F，使 E 在 C, F 之间。则运用公理 II_3，II_4，可知线段 BF 和线段 AE 有一交点 G，且直线 CF 交线段 DG 于一点 H，而 H 在线段 DG 内。而从公理 II_3，知 E 不在线段 AG 内，再应用公理 II_4，可知直线 EH 通过线段 AD 内的一点，即 C 在 AD 内。



下面证明原命题。如下图。首先，由 I_3，直线上存在 A, B 两点。由 II_2，直线上有一点 C 使得 B 在 A, C 之间。再由 II_2，直线上有一点 D 使得 C 在 B, D 之间，由 II_3 知 D 不能与 A 重合，因此 D 是一个新的点。再由 II_2，直线上有一点 E 使得 D 在 C, E 之间，由 II_3 和引理知 E 不能与 A, B 重合，因此 E 是一个新的点。以此类推，可在直线上找到无穷多个新的点。



第二题　证明下列命题：

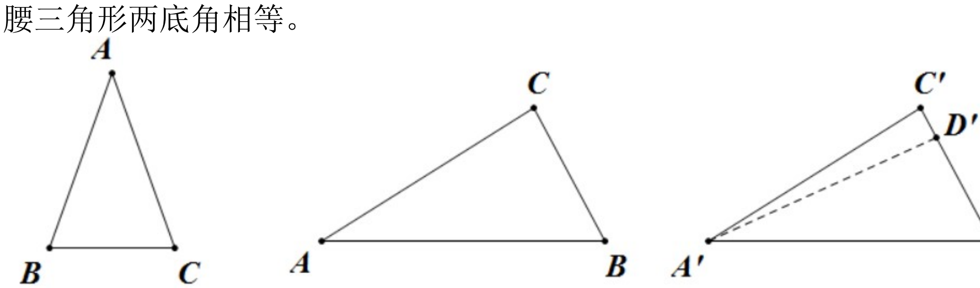
（1）利用合同公理证明等腰三角形两底角相等。

（2）定义两个三角形 $\triangle ABC$ 与 $\triangle A'B'C'$ 全等如果对应有下列合同式成立：

$$\begin{cases} AB \equiv A'B', AC \equiv A'C', BC \equiv B'C' & (1) \\ \angle A \equiv \angle A', \angle B \equiv \angle B', \angle C \equiv \angle C' & (2) \end{cases}$$

求证：若两三角形 $\triangle ABC$ 与 $\triangle A'B'C'$ 满足 $AB \equiv A'B', AC \equiv A'C', \angle A \equiv \angle A'$，则两个三角形全等。

解：（1）如左图。设 $\triangle ABC$ 是等腰三角形，$AB \equiv AC$，则我们有合同式：$AB \equiv AC, AC \equiv AB, \angle BAC \equiv \angle CAB$，故由 III_5，$\angle ABC \equiv \angle CAB$，即等腰三角形两底角相等。



（2）如右图。由 III_5，知 $\angle B \equiv \angle B', \angle C \equiv \angle C'$。故只需证明 $BC \equiv B'C'$。用反证法。假设 BC 不合同于 $B'C'$，在从 B' 起始的射线 $B'C'$ 取一点 D' 使 $BC \equiv B'D'$。则根据 III_5，$\angle BAC \equiv \angle B'A'D'$，因此 $\angle B'A'C' \equiv \angle B'A'D'$，但这与 III_4 中角合同的唯一性矛盾。故假设不成立，因此 $\triangle ABC$ 与 $\triangle A'B'C'$ 全等。

第三题　证明下列命题：

（1）反射不是平移。

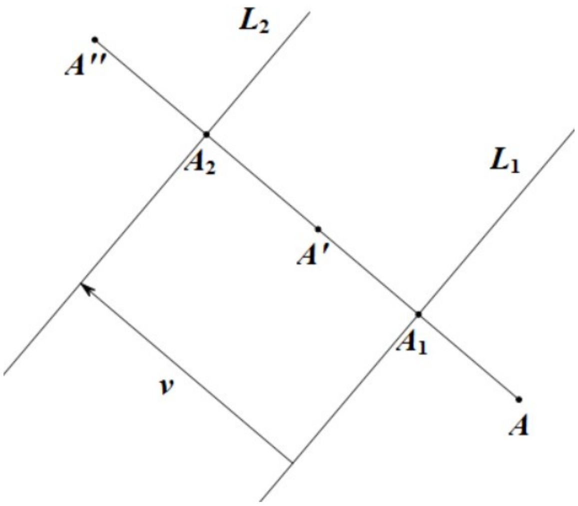
（2）两个反射的复合是平移当且仅当两个反射面平行。

（3）三个反射的复合有可能是平移吗？

解：我们介绍三种思路。

第一种思路：利用不动点的观点。

（1）反射的不动点集是一个平面，而平移的不动点集是空集。故反射不是平移。



（2）一方面，若两个反射面平行，则设这两个反射为 $\mathcal{R}_{L_2} \circ \mathcal{R}_{L_1}$。如图。设连接 L_1, L_2 的垂直于它们的向量为 $\mathbf{v}$。在空间里任取一点 A，它对 L_1 的反射的像点为 A'，令 $A_1 = AA' \cap L_1$；再令 A' 对 L_2 的反射的像点为 A''，令 $A_2 = A'A'' \cap L_2$。则由反射的基本性质，知 $\overrightarrow{AA'} = 2\overrightarrow{AA_1} = 2\overrightarrow{A_1A'}, \overrightarrow{A'A''} = 2\overrightarrow{A'A_2} = 2\overrightarrow{A_2A''}$，以及 $\overrightarrow{A_1A_2} = \mathbf{v}$。于是两个反射复合对 A 点的作用为 $\overrightarrow{AA''} = 2(\overrightarrow{A_1A'} + \overrightarrow{A'A_2}) = 2\overrightarrow{A_1A_2} = 2\mathbf{v}$，对任意的 A 都成立。故是一个平移。

另一方面，若两个反射面不平行，则它们有一条交线，这条线是两个反射的复合的不动点集，但平移的不动点集是空集，故则两个反射的复合不是平移。

（3）设三个反射的复合为 $\mathcal{R}_{L_1} \circ \mathcal{R}_{L_2} \circ \mathcal{R}_{L_3}$，其中 L_1, L_2, L_3 是不同的三个平面。则分情况讨论：

(a) 如果 L_1, L_2 平行。假设此时这三个反射的复合为平移 $\tau_{\mathbf{v}}$，则由（2），$\mathcal{R}_{L_1} \circ$

$\mathcal{R}_{L_2}=\tau_{\mathbf{u}}$ 也是平移，因此有 $\tau_{\mathbf{u}}\circ\mathcal{R}_{L_3}=\tau_{\mathbf{v}}$，即 $\mathcal{R}_{L_3}=\tau_{\mathbf{-u}}\circ\tau_{\mathbf{v}}$，也就是某个反射是某个平移，由（1）这是不可能的；

(b) 如果 L_1,L_2 不平行。假设此时这三个反射的复合是平移 τ，则 $\mathcal{R}_{L_1}\circ\mathcal{R}_{L_2}=\tau\circ\mathcal{R}_{L_3}$，而 $\mathcal{R}_{L_1}\circ\mathcal{R}_{L_2}$ 的不动点集是一条直线，$\tau\circ\mathcal{R}_{L_3}$ 的不动点集要么是一个平面，要么是空集。故这两个变换不可能相等。

综上，三个反射的复合不可能是平移。

第二种思路：利用角定向的观点。

（1）反射使角反向，而平移不改变角的定向，故反射不是平移。

（3）三个反射的复合使角反向，而平移不改变角的定向，故三个反射的复合不是平移。

第三种思路：考察特殊点。

（1）设反射是 $\mathcal{R}_{L_1}$，取平面 L_1 上的一点 P，则它关于反射不变，因此若这个反射是平移，则必有平移向量 $\mathbf{v}=\overrightarrow{PP}=\overrightarrow{0}$，因此它是恒等变换。而取不在 L_1 上的一点 Q，它关于反射的像 $Q'\neq Q$，矛盾。因此反射不是平移。

（2）对反射面不平行的情况，设两个反射为 $\mathcal{R}_{L_2}\circ\mathcal{R}_{L_1}$，$L_1\cap L_2=l$，$l$ 为一条直线。取直线 l 上的一点 P，则它关于两个反射不变，因此若两个反射的复合是平移，则必为恒等变换。而取 L_1 上不在 l 上的点 Q，它关于两个反射的像 $Q'\neq Q$（否则 $L_2=L_1$），矛盾。因此反射面不平行时，反射的复合不是平移。

第四题　（1）设 O,P,Q,R 为平面上不同的四点，则 P,Q,R 三点共线，当且仅当存在不全为 0 的实数 α,β,γ 使得

$$\begin{cases}\alpha\cdot\overrightarrow{OP}+\beta\cdot\overrightarrow{OQ}+\gamma\cdot\overrightarrow{OR}=0 & (3)\\ \alpha+\beta+\gamma=0 & (4)\end{cases}$$

（2）试用（1）证明如下命题：设 $\overrightarrow{OP},\overrightarrow{OQ}$ 为平面上不平行的两个向量，则对于平面上任意的向量 $\overrightarrow{OR}$，存在唯一的实数组 (α,β) 使得 $\overrightarrow{OR}=\alpha\cdot\overrightarrow{OP}+\beta\cdot\overrightarrow{OQ}$。

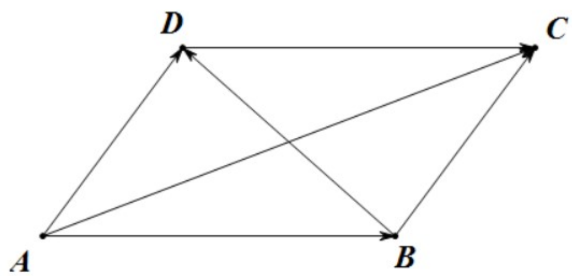
解：（1）P,Q,R 三点共线，当且仅当存在实数 t 使得 $\overrightarrow{PQ}=t\overrightarrow{PR}$，当且仅当 $\overrightarrow{OQ}-\overrightarrow{OP}=t(\overrightarrow{OR}-\overrightarrow{OP})$，当且仅当 $(t-1)\overrightarrow{OP}+\overrightarrow{OQ}+(-t)\overrightarrow{OR}=0$（注意到 $(t-1)+1+(-t)=0$），当且仅当存在不全为 0 的实数 α,β,γ 使得 $\alpha\cdot\overrightarrow{OP}+\beta\cdot\overrightarrow{OQ}+\gamma\cdot\overrightarrow{OR}=0$ 且 $\alpha+\beta+\gamma=0$。

（2）先证明唯一性：若存在两个实数组 $(\alpha,\beta),(\alpha',\beta')$，均满足 $\overrightarrow{OR}=\alpha\cdot\overrightarrow{OP}+\beta\cdot\overrightarrow{OQ}=\alpha'\cdot\overrightarrow{OP}+\beta'\cdot\overrightarrow{OQ}$，则 $(\alpha-\alpha')\cdot\overrightarrow{OP}=(\beta'-\beta)\cdot\overrightarrow{OQ}$。由于 $\overrightarrow{OP}$ 与 $\overrightarrow{OQ}$ 不平行，则只能 $\alpha-\alpha'=\beta'-\beta=0$，故 $(\alpha,\beta)=(\alpha',\beta')$，故唯一性得证。
下面证明存在性。若 $\overrightarrow{OR}$ 平行于 $\overrightarrow{PQ}$，则存在实数 t 使得 $\overrightarrow{OR}=t\overrightarrow{PQ}=$

$t\overrightarrow{OQ}-t\overrightarrow{OP}$；若 $\overrightarrow{OR}$ 与 $\overrightarrow{PQ}$ 不平行，则设它们所在直线的交点为 S，那么存在非 0 实数 s 使得 $\overrightarrow{OR}=s\overrightarrow{OS}$，而由（1），存在不为 0 的实数 t_1,t_2，使得 $\overrightarrow{OS}=t_1\overrightarrow{OP}+t_2\overrightarrow{OQ}$，故 $\overrightarrow{OR}=st_1\overrightarrow{OP}+st_2\overrightarrow{OQ}$。证毕。

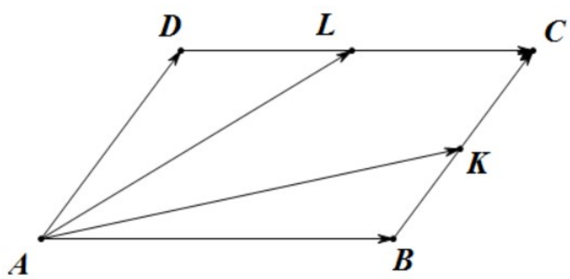
第三次作业答案

1. 已知平行四边的对角线 $\overrightarrow{AC}=\mathbf{a},\overrightarrow{BD}=\mathbf{b}$，求 $\overrightarrow{AB},\overrightarrow{BC},\overrightarrow{CD},\overrightarrow{DA}$。



解： 如图，有 $\begin{cases}\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{AD}\\ \overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}\end{cases}$，故得 $\begin{cases}\overrightarrow{AB}=-\overrightarrow{CD}=\frac{1}{2}\overrightarrow{AC}-\frac{1}{2}\overrightarrow{BD}\\ \overrightarrow{BC}=-\overrightarrow{DA}=\frac{1}{2}\overrightarrow{AC}+\frac{1}{2}\overrightarrow{BD}\end{cases}$。

2. 已知平行四边形的边 BC,CD 的中点分别是 K,L，且 $\overrightarrow{AK}=\mathbf{k},\overrightarrow{AL}=\mathbf{l}$，求 $\overrightarrow{BC}$ 和 $\overrightarrow{CD}$。



解： 如图，有 $\begin{cases}\mathbf{k}=\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}=\frac{1}{2}\overrightarrow{BC}-\overrightarrow{CD}\\ \mathbf{l}=\overrightarrow{AD}+\frac{1}{2}\overrightarrow{DC}=\overrightarrow{BC}-\frac{1}{2}\overrightarrow{CD}\end{cases}$，故得 $\begin{cases}\overrightarrow{BC}=\frac{4}{3}\mathbf{l}-\frac{2}{3}\mathbf{k}\\ \overrightarrow{CD}=\frac{2}{3}\mathbf{l}-\frac{4}{3}\mathbf{k}\end{cases}$。

3. 设 $\overrightarrow{AM}=\overrightarrow{MB}$，证明：对任意一点 O，$\overrightarrow{OM}=\dfrac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})$。

解： 由题，$\overrightarrow{OM}=\overrightarrow{OA}+\overrightarrow{AM}=\overrightarrow{OB}+\overrightarrow{BM}=\dfrac{1}{2}(\overrightarrow{OA}+\overrightarrow{AM}+\overrightarrow{OB}+\overrightarrow{BM})=\dfrac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{AM}-\overrightarrow{MB})=\dfrac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})$。

4. 设 M 是平行四边形 $ABCD$ 的对角线的交点，证明：对任意一点 O，$\overrightarrow{OM}=\frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$。

解：由题，$\overrightarrow{OM}=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OC})=\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OD})=\frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$。

5. 设 $P_1,P_2,\cdots,P_n$ 是以 O 为中心的圆周上的 n 等分点，证明：$\overrightarrow{OP_1}+\overrightarrow{OP_2}+\cdots+\overrightarrow{OP_n}=\mathbf{0}$。

解：我们首先约定：若 $i>n$，则 $P_i=P_{i-n}$。$n=2$ 时显然。对 $n\geq 3$，我们知道，存在实数 $\lambda\in(1,2)$，使得对任意的 $1\leq i\leq n$，$\overrightarrow{OP_i}+\overrightarrow{OP_{i+2}}=\lambda\overrightarrow{OP_{i+1}}$，因此，我们有 $2\sum_{i=1}^{n}\overrightarrow{OP_i}=\sum_{i=1}^{n}(\overrightarrow{OP_i}+\overrightarrow{OP_{i+2}})=\lambda\sum_{i=1}^{n}\overrightarrow{OP_{i+1}}=\lambda\sum_{i=1}^{n}\overrightarrow{OP_i}$，因此 $(2-\lambda)(\overrightarrow{OP_1}+\overrightarrow{OP_2}+\cdots+\overrightarrow{OP_n})=0$，故 $\overrightarrow{OP_1}+\overrightarrow{OP_2}+\cdots+\overrightarrow{OP_n}=0$。

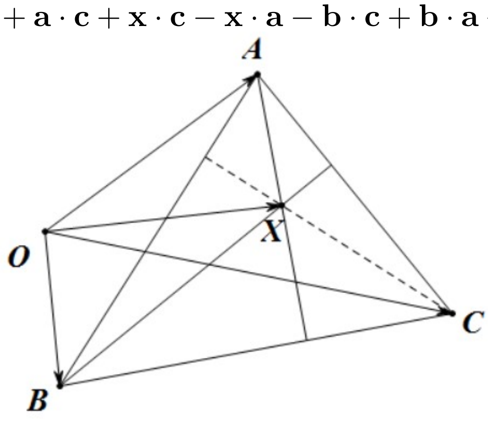
另一种思路：设 $\mathbf{t}=\overrightarrow{OP_1}+\cdots+\overrightarrow{OP_n}$。将平面绕原点旋转 $2\pi/n$，则 $\overrightarrow{OP_i}\to\overrightarrow{OP_{i+1}}$，故 $\mathbf{t}\to\mathbf{t}$，只能 $\mathbf{t}=\mathbf{0}$。

6. 设 O 是不共线三点 A,B,C 所在平面外一点。证明：A,B,C,D 四点共面当且仅当 $\overrightarrow{OD}=\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}+\nu\overrightarrow{OC},\lambda+\mu+\nu=1$。

解：由于 A,B,C 不共线，故 $\overrightarrow{AB},\overrightarrow{AC}$ 不共线。这样，A,B,C,D 四点共面，当且仅当 $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ 共面，当且仅当存在 t_1,t_2 使得 $\overrightarrow{AD}=t_1\overrightarrow{AB}+t_2\overrightarrow{AC}$，当且仅当存在 t_1,t_2 使得 $\overrightarrow{OD}-\overrightarrow{OA}=t_1(\overrightarrow{OB}-\overrightarrow{OA})+t_2(\overrightarrow{OC}-\overrightarrow{OA})$，当且仅当存在 t_1,t_2 使得 $\overrightarrow{OD}=(1-t_1-t_2)\overrightarrow{OA}+t_1\overrightarrow{OB}+t_2\overrightarrow{OC}$，当且仅当存在 $\lambda+\mu+\nu=1$ 使得 $\overrightarrow{OD}=\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}+\nu\overrightarrow{OC}$。

7. 设 $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{x}$ 为四个向量，试证：$(\mathbf{x}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})+(\mathbf{x}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a})+(\mathbf{x}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{b})=0$。并由此证明三角形三边上的高交于一点。

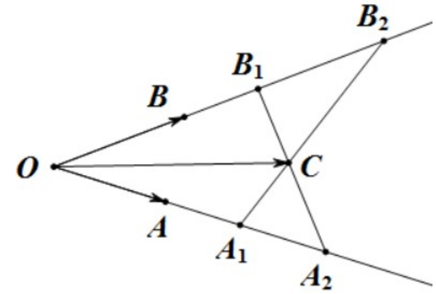
解：$(\mathbf{x}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})+(\mathbf{x}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a})+(\mathbf{x}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{x}\cdot\mathbf{b}-\mathbf{x}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}+\mathbf{x}\cdot\mathbf{c}-\mathbf{x}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}+\mathbf{x}\cdot\mathbf{a}-\mathbf{x}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b}=0$。



如图，设 $\triangle ABC$，$\overrightarrow{OA}=\mathbf{a},\overrightarrow{OB}=\mathbf{b},\overrightarrow{OC}=\mathbf{c}$。设 BC 和 CA 边上的高的交

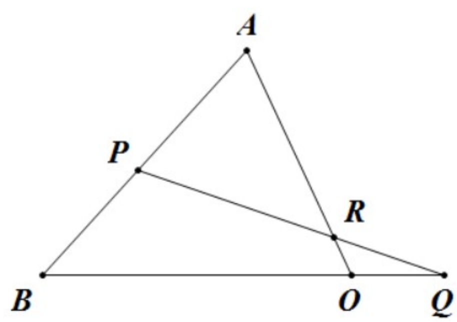
点为 X，$\overrightarrow{OX}=\mathbf{x}$。则 $(\mathbf{x}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})=0,(\mathbf{x}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a})=0$，故由所证，知 $(\mathbf{x}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{b})=0$，因此 $XC\perp AB$，因此三角形三边上的高交于一点。

8. 如图所示，$\mathbf{a}=\overrightarrow{OA},\mathbf{b}=\overrightarrow{OB}$ 是两个不共线的向量。A_1,A_2 是直线 OA 上相异两点，B_1,B_2 是直线 OB 上相异两点。设 $\overrightarrow{OA_1}=\alpha_1\mathbf{a},\overrightarrow{OA_2}=\alpha_2\mathbf{a},\overrightarrow{OB_1}=\beta_1\mathbf{b},\overrightarrow{OB_2}=\beta_2\mathbf{b}$（$\alpha_1,\alpha_2,\beta_1,\beta_2\neq 0$），直线 A_1B_2 和 A_2B_1 相交于点 C。试用 $\alpha_1,\alpha_2,\beta_1,\beta_2$ 表达 $\mathbf{c}=\overrightarrow{OC}=x\mathbf{a}+y\mathbf{b}$ 的系数 x,y。



解： 由题，$\mathbf{c}=x\mathbf{a}+y\mathbf{b}=\dfrac{x}{\alpha_1}\overrightarrow{OA_1}+\dfrac{y}{\beta_2}\overrightarrow{OB_2}=\dfrac{x}{\alpha_2}\overrightarrow{OA_2}+\dfrac{y}{\beta_1}\overrightarrow{OB_1}$，由于 A_1,C,B_2 共线，A_2,C,B_1 共线，故有方程组 $\begin{cases}\frac{x}{\alpha_1}+\frac{y}{\beta_2}=1\\ \frac{x}{\alpha_2}+\frac{y}{\beta_1}=1\end{cases}$。解，得 $x=\dfrac{(\beta_1-\beta_2)\alpha_1\alpha_2}{\alpha_1\beta_1-\alpha_2\beta_2},y=\dfrac{(\alpha_1-\alpha_2)\beta_1\beta_2}{\alpha_1\beta_1-\alpha_2\beta_2}$。

9. 如图，利用向量代数试证：在给定 $\triangle OAB$ 的三边所在直线 AB,OB,OA 上各取一点 P,Q,R，则三点 P,Q,R 共线的充要条件是 $\dfrac{\overrightarrow{OR}}{\overrightarrow{RA}}\cdot\dfrac{\overrightarrow{AP}}{\overrightarrow{PB}}\cdot\dfrac{\overrightarrow{BQ}}{\overrightarrow{QO}}=-1$。

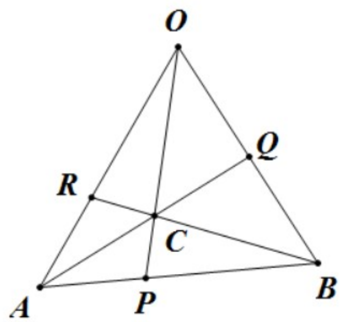


解： 设 $\overrightarrow{OR}=\alpha\overrightarrow{OA},\overrightarrow{OQ}=\beta\overrightarrow{OB}$。设 $\overrightarrow{AP}=\gamma\overrightarrow{AB}$，则 $\overrightarrow{OP}=\overrightarrow{OA}+\gamma\overrightarrow{AB}=(1-\gamma)\overrightarrow{OA}+\gamma\overrightarrow{OB}=\dfrac{1-\gamma}{\alpha}\overrightarrow{OR}+\dfrac{\gamma}{\beta}\overrightarrow{OQ}$。因此，利用第二次作业第四题的结论 (a)，$P,Q,R$ 三点共线，当且仅当 $\dfrac{1-\gamma}{\alpha}+\dfrac{\gamma}{\beta}=1$，即 $\gamma=\dfrac{\alpha\beta-\beta}{\alpha-\beta}$，即 $\dfrac{\alpha}{1-\alpha}\cdot\dfrac{\gamma}{1-\gamma}\cdot\dfrac{\beta-1}{\beta}=-1$，即 $\dfrac{\overrightarrow{OR}}{\overrightarrow{RA}}\cdot\dfrac{\overrightarrow{AP}}{\overrightarrow{PB}}\cdot\dfrac{\overrightarrow{BQ}}{\overrightarrow{QO}}=-1$。

第四次作业答案

1. 试证：在给定 $\triangle OAB$ 的三边上各取一点 P, Q, R，将它们和对顶点相连的三条直线 OP, AQ, BR 交于一点的充要条件是

$$\frac{OR}{RA} \cdot \frac{AP}{PB} \cdot \frac{BQ}{QO} = 1.$$



解：如图。设 AQ, BR 交于点 C。令 $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}$，设 $\overrightarrow{OR} = \alpha\mathbf{a}, \overrightarrow{OQ} = \beta\mathbf{b}$。则由第三次作业第 8 题，知 $\overrightarrow{OC} = \dfrac{\alpha\beta - \alpha}{\alpha\beta - 1}\mathbf{a} + \dfrac{\alpha\beta - \beta}{\alpha\beta - 1}\mathbf{b}$。再设 $\overrightarrow{AP} = \gamma\overrightarrow{AB} = -\gamma\mathbf{a} + \gamma\mathbf{b}$，则 $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = (1-\gamma)\mathbf{a} + \gamma\mathbf{b}$。因此，$OP, AQ, BR$ 三线共点，当且仅当 O, C, P 三点共线，当且仅当 $\dfrac{\alpha\beta - \alpha}{1-\gamma} = \dfrac{\alpha\beta - \beta}{\gamma}$，即 $\dfrac{\alpha}{1-\alpha} \cdot \dfrac{\gamma}{1-\gamma} \cdot \dfrac{1-\beta}{\beta} = 1$，即 $\dfrac{OR}{RA} \cdot \dfrac{AP}{PB} \cdot \dfrac{BQ}{QO} = 1$。

2. 利用向量代数证明：三角形的三条中线相交于一点。

解：在第 1 题的过程中令 $\alpha = \beta = \gamma = 1/2$，就得到这道题的过程。

3. 已知 $\mathbf{a} = (3, 5, 7), \mathbf{b} = (0, 4, 3), \mathbf{c} = (-1, 2, -4)$，对下面的各组 $(\mathbf{x}, \mathbf{y})$，求 $\langle \mathbf{x}, \mathbf{y} \rangle, |\mathbf{x}|, |\mathbf{y}|$ 和 $\angle(\mathbf{x}, \mathbf{y})$（即 $\mathbf{x}, \mathbf{y}$ 的夹角）：

（1）$\mathbf{x} = 3\mathbf{a} + 4\mathbf{b} - \mathbf{c}, \mathbf{y} = 2\mathbf{b} + \mathbf{c}$;

（2）$\mathbf{x} = 4\mathbf{a} + 3\mathbf{b} + 2\mathbf{c}, \mathbf{y} = \mathbf{a} + 2\mathbf{b} - \mathbf{c}$。

解：（1）由题，$\mathbf{x} = (10, 29, 37), \mathbf{y} = (-1, 10, 2)$，则 $\langle \mathbf{x}, \mathbf{y} \rangle = 354, |\mathbf{x}| = \sqrt{2310}, |\mathbf{y}| = \sqrt{105}$，$\cos\angle(\mathbf{x}, \mathbf{y}) = \dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{|\mathbf{x}| \cdot |\mathbf{y}|} = \dfrac{118}{35\sqrt{22}} \approx 0.72$，故 $\angle(\mathbf{x}, \mathbf{y}) = \arccos\dfrac{118}{35\sqrt{22}} = \arccos 0.72$。

（2）$\mathbf{x} = (10, 36, 29), \mathbf{y} = (4, 11, 17), \langle \mathbf{x}, \mathbf{y} \rangle = 929, |\mathbf{x}| = \sqrt{2237}, |\mathbf{y}| = \sqrt{426}, \angle(\mathbf{x}, \mathbf{y}) =$

$\arccos \dfrac{929}{\sqrt{952962}} \approx 0.95$。

4. 由外积和内积的对应，即由

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}, \quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

推导：$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$。

解：方法一：由第二个式子，$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot(\mathbf{b}\times(\mathbf{c}\times\mathbf{d}))$，而由第一个式子，$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{a}\cdot(\mathbf{c}(\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{b}\cdot\mathbf{c}))$，故 $\mathbf{a}\cdot(\mathbf{b}\times(\mathbf{c}\times\mathbf{d})) = \mathbf{a}\cdot(\mathbf{c}(\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{b}\cdot\mathbf{c}))$，即 $\mathbf{a}\cdot(\mathbf{b}\times(\mathbf{c}\times\mathbf{d}) - \mathbf{c}(\mathbf{b}\cdot\mathbf{d}) + \mathbf{d}(\mathbf{b}\cdot\mathbf{c})) = 0$。由 $\mathbf{a}$ 的任意性，取 $\mathbf{a} = \mathbf{b}\times(\mathbf{c}\times\mathbf{d}) - \mathbf{c}(\mathbf{b}\cdot\mathbf{d}) + \mathbf{d}(\mathbf{b}\cdot\mathbf{c})$，得 $|\mathbf{b}\times(\mathbf{c}\times\mathbf{d}) - \mathbf{c}(\mathbf{b}\cdot\mathbf{d}) + \mathbf{d}(\mathbf{b}\cdot\mathbf{c})| = 0$，于是 $\mathbf{b}\times(\mathbf{c}\times\mathbf{d}) - \mathbf{c}(\mathbf{b}\cdot\mathbf{d}) + \mathbf{d}(\mathbf{b}\cdot\mathbf{c}) = 0$，即 $\mathbf{b}\times(\mathbf{c}\times\mathbf{d}) = \mathbf{c}(\mathbf{b}\cdot\mathbf{d}) - \mathbf{d}(\mathbf{b}\cdot\mathbf{c})$。这样，$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = -(\mathbf{b}\cdot\mathbf{c})\mathbf{a} + (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$。

方法二：对任意的 $\mathbf{d}$，$((\mathbf{a}\times\mathbf{b})\times\mathbf{c} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\mathbf{a})\cdot\mathbf{d} = ((\mathbf{a}\times\mathbf{b})\times\mathbf{c})\cdot\mathbf{d} - (\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))\cdot\mathbf{d} - (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) + (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) - (\mathbf{d}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) + (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix} - \begin{vmatrix} \mathbf{b}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{d}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{c} \end{vmatrix} - (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) + (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) = 0$。故取 $\mathbf{d} = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$，得 $|(\mathbf{a}\times\mathbf{b})\times\mathbf{c} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\mathbf{a}| = 0$，即 $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} - \mathbf{a}\times(\mathbf{c}\times\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\mathbf{a} = 0$，故 $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$。

5. 利用直角坐标系证明二重外积展开式：

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

解：方法一：设 $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3), \mathbf{c} = (c_1, c_2, c_3)$，则 $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_1 & a_2 & a_3 \\ \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \end{vmatrix} = (a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1)\mathbf{e}_x + (-a_1b_1c_2 + a_1b_2c_1 + a_3b_2c_3 - a_3b_3c_2)\mathbf{e}_y + (-a_1b_1c_3 + a_1b_3c_1 - a_2b_2c_3 + a_2b_3c_2)\mathbf{e}_z = (a_1c_1 + a_2c_2 + a_3c_3)(b_1\mathbf{e}_x + b_2\mathbf{e}_y + b_3\mathbf{e}_z) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\mathbf{e}_x + c_2\mathbf{e}_y + c_3\mathbf{e}_z) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$。

方法二：易验证，等式左边的向量的 x 分量为 $a_2(b_1c_2 - b_2c_1) - a_3(-b_1c_3 + b_3c_1)$，而右边的向量的 x 分量为 $(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1$，二者

是相等的。同理可以验证 y,z 分量也相等。故 $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$。

6. 时空是四维的，Hamilton 赋予了时空一种代数结构：四元数，作为实数与复数的一种推广。四元数定义在四维空间 $\mathbb{R}\oplus E^3=\{(t,\mathbf{u});t\in\mathbb{R},\mathbf{u}\in E^3\}$ 上，其加法与乘法运算如下：

$$(t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2)=(t_1+t_2,\mathbf{u}_1+\mathbf{u}_2),$$

$$(t_1,\mathbf{u}_1)\cdot(t_2,\mathbf{u}_2)=(t_1t_2-\langle\mathbf{u}_1,\mathbf{u}_2\rangle,t_1\mathbf{u}_2+t_2\mathbf{u}_1+\mathbf{u}_1\times\mathbf{u}_2).$$

证明：四元数空间关于加法、乘法构成一个除环，即：

（1）四元数空间关于加法构成交换群，即：

(a) 满足加法结合律：$(t_1,\mathbf{u}_1)+((t_2,\mathbf{u}_2)+(t_3,\mathbf{u}_3))=((t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2))+(t_3,\mathbf{u}_3)$；

(b) 满足加法交换律：$(t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2)=(t_2,\mathbf{u}_2)+(t_1,\mathbf{u}_1)$；

(c) 存在零元：存在四元数 $(t_0,\mathbf{u}_0)$，使得对任意的四元数 $(t,\mathbf{u})$，有 $(t,\mathbf{u})+(t_0,\mathbf{u}_0)=(t,\mathbf{u})$；

(d) 存在负元：对任意的四元数 $(t_1,\mathbf{u}_1)$，存在四元数 $(t_2,\mathbf{u}_2)$ 使得 $(t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2)=(t_0,\mathbf{u}_0)$，其中 $(t_0,\mathbf{u}_0)$ 是零元。

（2）四元数空间除去加法零元关于乘法构成群，即：

(a) 满足乘法结合律：$(t_1,\mathbf{u}_1)\cdot((t_2,\mathbf{u}_2)\cdot(t_3,\mathbf{u}_3))=((t_1,\mathbf{u}_1)\cdot(t_2,\mathbf{u}_2))\cdot(t_3,\mathbf{u}_3)$；

(b) 存在单位元：存在四元数 $(t',\mathbf{u}')$，使得对任意的非零元的四元数 $(t,\mathbf{u})$，有 $(t,\mathbf{u})\cdot(t',\mathbf{u}')=(t',\mathbf{u}')\cdot(t,\mathbf{u})=(t,\mathbf{u})$；

(c) 存在逆元：对任意的非零元的四元数 $(t_1,\mathbf{u}_1)$，存在四元数 $(t_2,\mathbf{u}_2)$ 使得 $(t_1,\mathbf{u}_1)\cdot(t_2,\mathbf{u}_2)=(t_2,\mathbf{u}_2)\cdot(t_1,\mathbf{u}_1)=(t',\mathbf{u}')$，其中 $(t',\mathbf{u}')$ 是单位元。

（3）乘法与加法满足分配率，即：

(a) 满足左分配率：$((t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2))\cdot(t_3,\mathbf{u}_3)=(t_1,\mathbf{u}_1)\cdot(t_3,\mathbf{u}_3)+(t_2,\mathbf{u}_2)\cdot(t_3,\mathbf{u}_3)$；

(b) 满足右分配率：$(t_1,\mathbf{u}_1)\cdot((t_2,\mathbf{u}_2)+(t_3,\mathbf{u}_3))=(t_1,\mathbf{u}_1)\cdot(t_2,\mathbf{u}_2)+(t_1,\mathbf{u}_1)\cdot(t_3,\mathbf{u}_3)$。

解：（1）(a)$(t_1,\mathbf{u}_1)+((t_2,\mathbf{u}_2)+(t_3,\mathbf{u}_3))=(t_1+(t_2+t_3),\mathbf{u}_1+(\mathbf{u}_2+\mathbf{u}_3))=((t_1+t_2)+t_3,(\mathbf{u}_1+\mathbf{u}_2)+\mathbf{u}_3)=((t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2))+(t_3,\mathbf{u}_3)$。

(b)$(t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2)=(t_1+t_2,\mathbf{u}_1+\mathbf{u}_2)=(t_2+t_1,\mathbf{u}_2+\mathbf{u}_1)=(t_2,\mathbf{u}_2)+(t_1,\mathbf{u}_1)$。

(c) 令 $(t_0,\mathbf{u}_0)=(0,\mathbf{0})$，则 $(t,\mathbf{u})+(0,\mathbf{0})=(t+0,\mathbf{u}+\mathbf{0})=(t,\mathbf{u})$。

(d) 令 $(t_2,\mathbf{u}_2)=(-t_1,-\mathbf{u}_1)$,则 $(t_1,\mathbf{u}_1)+(-t_1,-\mathbf{u}_1)=(t_1-t_1,\mathbf{u}_1-\mathbf{u}_1)=(0,\mathbf{0})$。

（2）(a) 直接展开，有 $(t_1,\mathbf{u}_1)\cdot((t_2,\mathbf{u}_2)\cdot(t_3,\mathbf{u}_3))=(t_1t_2t_3-t_1\mathbf{u}_2\cdot\mathbf{u}_3-t_2\mathbf{u}_1\cdot\mathbf{u}_3-t_3\mathbf{u}_1\cdot\mathbf{u}_2-\mathbf{u}_1\cdot\mathbf{u}_2\times\mathbf{u}_3,t_1t_2\mathbf{u}_3+t_1t_3\mathbf{u}_2+t_1\mathbf{u}_2\times\mathbf{u}_3+t_2t_3\mathbf{u}_1-(\mathbf{u}_2\cdot\mathbf{u}_3)\mathbf{u}_1+t_2\mathbf{u}_1\times$

$\mathbf{u}_3+t_3\mathbf{u}_1\times\mathbf{u}_2+\mathbf{u}_1\times(\mathbf{u}_2\times\mathbf{u}_3))$，$((t_1,\mathbf{u}_1)\cdot(t_2,\mathbf{u}_2))\cdot(t_3,\mathbf{u}_3)=(t_1t_2t_3-t_3\mathbf{u}_1\cdot\mathbf{u}_2-t_1\mathbf{u}_2\cdot\mathbf{u}_3-t_2\mathbf{u}_1\cdot\mathbf{u}_3-\mathbf{u}_1\times\mathbf{u}_2\cdot\mathbf{u}_3, t_1t_2\mathbf{u}_3-(\mathbf{u}_1\cdot\mathbf{u}_2)\mathbf{u}_3+t_1t_3\mathbf{u}_2+t_2t_3\mathbf{u}_1+t_3\mathbf{u}_1\times\mathbf{u}_2+t_1\mathbf{u}_2\times\mathbf{u}_3+t_2\mathbf{u}_1\times\mathbf{u}_3+(\mathbf{u}_1\times\mathbf{u}_2)\times\mathbf{u}_3)$。由 $\mathbf{u}_1\cdot\mathbf{u}_2\times\mathbf{u}_3=\mathbf{u}_1\times\mathbf{u}_2\cdot\mathbf{u}_3$ 以及 $\mathbf{u}_1\times(\mathbf{u}_2\times\mathbf{u}_3)-(\mathbf{u}_2\cdot\mathbf{u}_3)\mathbf{u}_1=(\mathbf{u}_1\times\mathbf{u}_2)\times\mathbf{u}_3-(\mathbf{u}_1\cdot\mathbf{u}_2)\mathbf{u}_3$（见第 4 题），知两式相等。故 $(t_1,\mathbf{u}_1)\cdot((t_2,\mathbf{u}_2)\cdot(t_3,\mathbf{u}_3))=((t_1,\mathbf{u}_1)\cdot(t_2,\mathbf{u}_2))\cdot(t_3,\mathbf{u}_3)$。

(b) 令 $(t',\mathbf{u}')=(1,\mathbf{0})$，则 $(t,\mathbf{u})\cdot(1,\mathbf{0})=(1,\mathbf{0})\cdot(t,\mathbf{u})=(t,\mathbf{u})$。

(c) 先观察。$(t_1,\mathbf{u}_1)\cdot(t_2,\mathbf{u}_2)=(t_1t_2-\langle\mathbf{u}_1,\mathbf{u}_2\rangle, t_1\mathbf{u}_2+t_2\mathbf{u}_1+\mathbf{u}_1\times\mathbf{u}_2)=(1,\mathbf{0})$，而如果 $\mathbf{u}_2$ 不与 $\mathbf{u}_1$ 共线，那么 $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_1\times\mathbf{u}_2$ 不共面，因此三者的非零线性组合不可能为 $\mathbf{0}$。故只能存在 $t\in\mathbb{R}$，使得 $\mathbf{u}_2=t\mathbf{u}_1$。则 $(t_1,\mathbf{u}_1)\cdot(t_2,t\mathbf{u}_1)=(t_1t_2-t|\mathbf{u}_1|^2, tt_1\mathbf{u}_1+t_2\mathbf{u}_1)=(1,\mathbf{0})$，我们有方程组 $\begin{cases}t_1t_2-t|\mathbf{u}_1|^2=1\\(tt_1+t_2)\mathbf{u}_1=\mathbf{0}\end{cases}$。解，得 $\begin{cases}t=\frac{-1}{|\mathbf{u}_1|^2+t_1^2}\\t_2=\frac{t_1}{|\mathbf{u}_1|^2+t_1^2}\end{cases}$。因此逆元 $(t_2,\mathbf{u}_2)=(\dfrac{t_1}{|\mathbf{u}_1|^2+t_1^2},\dfrac{-\mathbf{u}_1}{|\mathbf{u}_1|^2+t_1^2})$。

（3）(a)$((t_1,\mathbf{u}_1)+(t_2,\mathbf{u}_2))\cdot(t_3,\mathbf{u}_3)=(t_1+t_2,\mathbf{u}_1+\mathbf{u}_2)\cdot(t_3,\mathbf{u}_3)=((t_1+t_2)t_3-(\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{u}_3,(t_1+t_2)\mathbf{u}_3+t_3(\mathbf{u}_1+\mathbf{u}_2)+(\mathbf{u}_1+\mathbf{u}_2)\times\mathbf{u}_3)=((t_1t_3-\mathbf{u}_1\cdot\mathbf{u}_3)+(t_2t_3-\mathbf{u}_2\cdot\mathbf{u}_3),(t_1\mathbf{u}_3+t_3\mathbf{u}_1+\mathbf{u}_1\times\mathbf{u}_3)+(t_2\mathbf{u}_3+t_3\mathbf{u}_2+\mathbf{u}_2\times\mathbf{u}_3))=(t_1,\mathbf{u}_1)\cdot(t_3,\mathbf{u}_3)+(t_2,\mathbf{u}_2)\cdot(t_3,\mathbf{u}_3)$。

(b) 证明方法与 (a) 完全类似。

7. 证明：E^3 中的四点 $P_1(x_1,y_1,z_1),P_2(x_2,y_2,z_2),P_3(x_3,y_3,z_3),P_4(x_4,y_4,z_4)$ 共面，当且仅当：

$$\begin{vmatrix}x_1-x_4 & x_2-x_4 & x_3-x_4\\ y_1-y_4 & y_2-y_4 & y_3-y_4\\ z_1-z_4 & z_2-z_4 & z_3-z_4\end{vmatrix}=0.$$

解：P_1,P_2,P_3,P_4 四点共面，当且仅当 $\overrightarrow{P_1P_2},\overrightarrow{P_1P_3},\overrightarrow{P_1P_4}$ 三个向量共面，当且仅当它们组成的平行六面体的体积为 0，当且仅当 $\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1P_3}\times\overrightarrow{P_1P_4}=\begin{vmatrix}x_1-x_4 & x_2-x_4 & x_3-x_4\\ y_1-y_4 & y_2-y_4 & y_3-y_4\\ z_1-z_4 & z_2-z_4 & z_3-z_4\end{vmatrix}=0$。

8. 判断下列各组的三个向量 $\mathbf{a},\mathbf{b},\mathbf{c}$ 是否共面？能否将 $\mathbf{c}$ 表示成 $\mathbf{a},\mathbf{b}$ 的线性组合（即：存在实数 λ,μ 使得 $\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}$）？若能表示则写出表示式。

（1）$\mathbf{a}=(5,2,1),\mathbf{b}=(-1,4,2),\mathbf{c}=(-1,-1,5)$；

（2）$\mathbf{a}=(6,4,2),\mathbf{b}=(-9,6,3),\mathbf{c}=(-3,6,3)$；

（3）$\mathbf{a}=(1,2,-3),\mathbf{b}=(-2,-4,6),\mathbf{c}=(1,0,5)$。

解：（1）由题，$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=120\neq 0$，故不共面，也不能将 $\mathbf{c}$ 写成 $\mathbf{a},\mathbf{b}$ 的线性组合。

（2）由题，$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$，故共面。验证可知 $\mathbf{c}=\frac{\mathbf{a}}{2}+\frac{2}{3}\mathbf{b}$。

（3）由题，$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$，故共面。但 $\mathbf{a},\mathbf{b}$ 共线而它们均不与 $\mathbf{c}$ 共线，因此不能将 $\mathbf{c}$ 写成 $\mathbf{a},\mathbf{b}$ 的线性组合。

9. 求顶点为 $P_1(1,2,3),P_2(2,4,1),P_3(1,-3,5),P_4(4,-2,3)$ 的四面体的体积。

解：这个四面体的体积是由 $\overrightarrow{P_1P_2},\overrightarrow{P_1P_3},\overrightarrow{P_1P_4}$ 组成的平行六面体的体积的 1/6。由 $\overrightarrow{P_1P_2}=(1,2,-2),\overrightarrow{P_1P_3}=(0,-5,2),\overrightarrow{P_1P_4}=(3,-4,0)$，知四面体体积 $V=\frac{1}{6}|\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1P_3}\times\overrightarrow{P_1P_4}|=|\frac{1}{6}\begin{vmatrix}1&2&-2\\0&-5&2\\3&-4&0\end{vmatrix}|=\frac{5}{3}$。

第五次作业答案

1. 证明：$\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{BC}\cdot\overrightarrow{AD}+\overrightarrow{CA}\cdot\overrightarrow{BD}=0$。

解：$\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{BC}\cdot\overrightarrow{AD}+\overrightarrow{CA}\cdot\overrightarrow{BD}=\overrightarrow{AB}\cdot(\overrightarrow{AD}-\overrightarrow{AC})+(\overrightarrow{AC}-\overrightarrow{AB})\cdot\overrightarrow{AD}-\overrightarrow{AC}\cdot(\overrightarrow{AD}-\overrightarrow{AB})=0$。

2. 求 $\mathbf{a}\times\mathbf{b}$ 和以 $\mathbf{a},\mathbf{b}$ 为边的平行四边形的面积：

(1)$\mathbf{a}=(2,3,1),\mathbf{b}=(5,6,4)$

(2)$\mathbf{a}=(5,-2,1),\mathbf{b}=(4,0,6)$

(3)$\mathbf{a}=(-2,6,4),\mathbf{b}=(3,-9,6)$

解：(1)$\mathbf{a}\times\mathbf{b}=(6,-3,-3)$，面积 $S=|\mathbf{a}\times\mathbf{b}|=3\sqrt{6}$。

(2)$\mathbf{a}\times\mathbf{b}=(-12,-26,8)$，面积 $S=|\mathbf{a}\times\mathbf{b}|=2\sqrt{221}$。

(3)$\mathbf{a}\times\mathbf{b}=(72,24,0)$，面积 $S=|\mathbf{a}\times\mathbf{b}|=24\sqrt{10}$。

3. 给定 $\mathbf{a}=(1,0,-1),\mathbf{b}=(1,-2,0),\mathbf{c}=(-1,2,1)$，求：

(1)$\mathbf{a}\times\mathbf{b},\mathbf{b}\times\mathbf{a}$

(2)$(3\mathbf{a}+\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{b}+\mathbf{c})$

(3)$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c},\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$

(4)$(\mathbf{a}\times\mathbf{b})\times\mathbf{c},\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

解：(1)$\mathbf{a}\times\mathbf{b}=(-2,-1,-2),\mathbf{b}\times\mathbf{a}=(2,1,2)$；

(2)$(3\mathbf{a}+\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{b}+\mathbf{c})=(5,-4,-4)\times(-1,4,0)=(16,4,16)$；

(3)$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=-2,\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-2$；

(4)$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(3,4,-5),\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(-1,2,-1)$。

4. 证明下列等式：

(1)$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\times\mathbf{d}=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$

(2)$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{b}\times\mathbf{c})\times\mathbf{a}+(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$

解： 以下设 $\mathbf{a}=(a_1,a_2,a_3),\mathbf{b}=(b_1,b_2,b_3),\mathbf{c}=(c_1,c_2,c_3),\mathbf{d}=(d_1,d_2,d_3)$。

(1)$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\times\mathbf{d}=(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(a_2b_3-a_3b_2)(c_2d_3-c_3d_2)+(a_3b_1-a_1b_3)(c_3d_1-c_1d_3)+(a_1b_2-a_2b_1)(c_1d_2-c_2d_1)$，$(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})=(a_1c_1+a_2c_2+a_3c_3)(b_1d_1+b_2d_2+b_3d_3)-(a_1d_1+a_2d_2+a_3d_3)(b_1c_1+b_2c_2+b_3c_3)$，易验证二者相等。

(2) 对于 x 分量，$((\mathbf{a}\times\mathbf{b})\times\mathbf{c})_x=a_3b_1c_3-a_1b_3c_3+a_1b_2c_2-a_2b_1c_2$，故 $((\mathbf{b}\times\mathbf{c})\times\mathbf{a})_x=b_3c_1a_3-b_1c_3a_3+b_1c_2a_2-b_2c_1a_2$，$((\mathbf{c}\times\mathbf{a})\times\mathbf{b})_x=c_3a_1b_3-c_1a_3b_3+c_1a_2b_2-c_2a_1b_2$，因此 $((\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{b}\times\mathbf{c})\times\mathbf{a}+(\mathbf{c}\times\mathbf{a})\times\mathbf{b})_x=0$。其它分量同理，故 $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{b}\times\mathbf{c})\times\mathbf{a}+(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$。

5. 求解以下三阶行列式：

(1)$\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$

(2)$\begin{vmatrix}1&1&1\\a&b&c\\a^2&b^2&c^2\end{vmatrix}$

解： (1)$\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}=1\cdot\begin{vmatrix}5&6\\8&9\end{vmatrix}-2\cdot\begin{vmatrix}4&6\\7&9\end{vmatrix}+3\cdot\begin{vmatrix}4&5\\7&8\end{vmatrix}=0$。

(2)$\begin{vmatrix}1&1&1\\a&b&c\\a^2&b^2&c^2\end{vmatrix}=1\cdot b\cdot c^2-1\cdot c\cdot b^2-1\cdot a\cdot c^2+1\cdot c\cdot a^2+1\cdot a\cdot b^2-1\cdot b\cdot a^2=(a-b)(b-c)(c-a)$。

6. 一个四面体的顶点为 $A(1,2,0),B(-1,3,4),C(-1,-2,-3),D(0,-1,3)$，求它的体积。

解： 体积 $V=\dfrac{1}{6}|\overrightarrow{AB}\times\overrightarrow{AC}\cdot\overrightarrow{AD}|=\dfrac{59}{6}$。

7. 如果 $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=\mathbf{0}$，那么 $\mathbf{a},\mathbf{b},\mathbf{c}$ 共面。

解：由题，在等式两边同时点乘 $\mathbf{c}$，得 $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}+(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}+(\mathbf{c}\times\mathbf{a})\cdot\mathbf{c}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$，故 $\mathbf{a},\mathbf{b},\mathbf{c}$ 共面。

8. 下列等式是否正确？

(1)$|\mathbf{a}|\mathbf{a}=\mathbf{a}^2$

(2)$\mathbf{a}(\mathbf{b}\cdot\mathbf{b})=\mathbf{a}\mathbf{b}^2$

(3)$\mathbf{a}(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}^2\mathbf{b}$

(4)$(\mathbf{a}\cdot\mathbf{b})^2=\mathbf{a}^2\mathbf{b}^2$

(5)$(\mathbf{a}\cdot\mathbf{b})\mathbf{c}=\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$

(6)$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

解：(2) 正确，其余均不正确。

9. 下列推论是否正确？

(1) 如果 $\mathbf{c}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$，且 $\mathbf{c}\neq\mathbf{0}$，那么 $\mathbf{a}=\mathbf{b}$

(2) 如果 $\mathbf{c}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$，且 $\mathbf{c}\neq\mathbf{0}$，那么 $\mathbf{a}=\mathbf{b}$

解：均不正确。

10. 讨论 $\mathbf{x},\mathbf{u}$ 的关系，已知：

(1)$\mathbf{x}$ 与 $\mathbf{x}\times\mathbf{y}$ 共线

(2)$\mathbf{x},\mathbf{y},\mathbf{x}\times\mathbf{y}$ 共面

解：关系均为 $\mathbf{x},\mathbf{y}$ 共线。

11. 写出下列直线方程的即约形式和相应的参数方程：

(1)$\begin{cases}x+y+z+3=0\\2x+3y-z+1=0\end{cases}$

(2)$\begin{cases}x-y+1=0\\z+1=0\end{cases}$

(3)$\begin{cases}3x-y+2=0\\4y+3z+1=0\end{cases}$

(4)$\begin{cases}y-1=0\\z+2=0\end{cases}$

解：(1) 即约形式 $\begin{cases}x+4z+8=0\\y-3z-5=0\end{cases}$，参数方程 $\begin{cases}x=-4t-8\\y=3t+5\\z=t\end{cases}$；

(2) 即约形式 $\begin{cases} x-y+1=0 \\ z+1=0 \end{cases}$，参数方程 $\begin{cases} x=t-1 \\ y=t \\ z=-1 \end{cases}$；

(3) 即约形式 $\begin{cases} x-\frac{1}{3}y+\frac{2}{3}=0 \\ z+\frac{4}{3}y+\frac{1}{3}=0 \end{cases}$，参数方程 $\begin{cases} x=\frac{1}{3}t-\frac{2}{3} \\ y=t \\ z=-\frac{4}{3}t-\frac{1}{3} \end{cases}$；

(4) 即约形式 $\begin{cases} y-1=0 \\ z+2=0 \end{cases}$，参数方程 $\begin{cases} x=t \\ y=1 \\ z=-2 \end{cases}$。

第六次作业答案

第一题 求下列平面的方程：

（1）过点 $(0,-1,4)$，法向量为 $(2,-1,0)$。

（2）过点 $(-1,-5,4)$，平行于平面 $3x-2y+5=0$。

（3）过点 $(1,3,5),(-1,-2,3),(2,0,-3)$。

（4）过点 $(3,-1,4),(1,0,-3)$，垂直于平面 $2x+5y+z+1=0$。

（5）过点 $(0,-1,3)$ 和 Y 轴。

（6）过点 $(-2,-1,3),(0,-1,2)$，平行于 Z 轴。

解：（1）由于法向量为 $(2,-1,0)$，可设平面方程为 $2x-y+D=0$。将点 $(0,-1,4)$ 代入，得 $D=-1$，故平面方程为 $2x-y-1=0$。

（2）由于平行于平面 $3x-2y+5=0$，可设平面方程为 $3x-2y+D=0$。将点 $(-1,-5,4)$ 代入，得 $D=-7$，故平面方程为 $3x-2y-7=0$。

（3）由题，平面的法向量垂直于 $(1,3,5)-(-1,-2,3)=(2,5,2)$ 和 $(2,0,-3)-(-1,-2,3)=(3,2,-6)$，故平面法向量为 $(2,5,2)\times(3,2,-6)=(-34,18,-11)$，故可设平面方程为 $34x-18y+11z+D=0$。将点 $(2,0,-3)$ 代入，得 $D=-35$，故平面方程为 $34x-18y+11z-35=0$。

（4）由题，平面的法向量垂直于 $(3,-1,4)-(1,0,-3)=(2,-1,7)$ 和 $(2,5,1)$，故平面法向量为 $(2,-1,7)\times(2,5,1)=(-36,12,12)$，故可设平面方程为 $3x-y-z+D=0$。将点 $(1,0,-3)$ 代入，得 $D=-6$，故平面方程为 $3x-y-z-6=0$。

（5）由题，平面的法向量垂直于 $(0,1,0)$ 和 $(0,-1,3)-(0,0,0)=(0,-1,3)$，故平面法向量为 $(0,1,0)\times(0,-1,3)=(3,0,0)$，故可设平面方程为 $x+D=0$，

将点 $(0,0,0)$ 代入，得 $D=0$，故平面方程为 $x=0$。
（6）由题，平面的法向量垂直于 $(0,-1,2)-(-2,-1,3)=(2,0,-1)$ 和 $(0,0,1)$，故平面法向量为 $(2,0,-1)\times(0,0,1)=(0,-2,0)$，故可设平面方程为 $y+D=0$。将点 $(0,-1,2)$ 代入，得 $D=1$，故平面方程为 $y+1=0$。

第二题 求下列直线的方程：
（1）过点 $(-2,3,5)$，方向向量为 $(-1,3,4)$。
（2）过点 $(0,3,1),(-1,2,7)$。
（3）过点 $(-1,2,9)$，垂直于平面 $3x+2y-z+5=0$。
（4）过点 $(2,4,-1)$，与三个坐标轴成等角。

解：（1）直接可得直线方程为 $\dfrac{x+2}{-1}=\dfrac{y-3}{3}=\dfrac{z-5}{4}$ 或 $\begin{cases}x=-t-2\\ y=3t+3\\ z=4t+5\end{cases}$；
（2）直线的方向向量为 $(-1,2,7)-(0,3,1)=(-1,-1,6)$，故直线方程为 $\dfrac{x}{-1}=\dfrac{y-3}{-1}=\dfrac{z-1}{6}$ 或 $\begin{cases}x=-t\\ y=-t+3\\ z=6t+1\end{cases}$；
（3）直线的方向向量为 $(3,2,-1)$，故直线方程为 $\dfrac{x+1}{3}=\dfrac{y-2}{2}=\dfrac{z-9}{-1}$。
（4）直线的方向向量为 $(1,\pm1,\pm1)$，故直线方程为 $\dfrac{x-2}{1}=\dfrac{y-4}{\pm1}=\dfrac{z+1}{\pm1}$。

第三题 给定直线 $l:\dfrac{x+1}{-2}=\dfrac{y-1}{1}=\dfrac{z+2}{-3}$，求：
（1）过 l 平行于 Z 轴的平面。
（2）l 在 XY 平面的投影。
解：(1) 由题,平面的法向量垂直于 $(-2,1,-3),(0,0,1)$,故法向量为 $(-2,1,-3)\times(0,0,1)=(1,2,0)$，故可设平面方程为 $x+2y+D=0$。将点 $(-1,1,-2)$ 代入，得 $D=-1$，故平面方程为 $x+2y-1=0$。
（2）由题，在 XY 平面的投影直线的方向向量为 $(-2,1)$，且过点 $(-1,1)$，故直线方程为 $\dfrac{x+2}{-2}=\dfrac{y-1}{1}$，即 $x+2y-1=0$。

第四题 求直线与平面的交点：
（1）$\dfrac{x-1}{2}=\dfrac{y+1}{3}=\dfrac{z-2}{-1}$ 与 $3x+2y+z=0$。
（2）$\begin{cases}2x+3y+z-1=0\\ x+2y-z+2=0\end{cases}$ 与 XZ 平面。

解：(1) 设交点为 $(1+2t,-1+3t,2-t)$，代入平面方程，得 $3(1+2t)+2(-1+3t)+(2-t)=0$，解得 $t=-\frac{3}{11}$，故交点坐标为 $(\frac{5}{11},-\frac{20}{11},\frac{25}{11})$。

（2）交点坐标必有 $y=0$，故得方程组 $\begin{cases}2x+z-1=0\\x-z+2=0\end{cases}$，解，得 $\begin{cases}x=-1/3\\z=5/3\end{cases}$，故交点坐标为 $(-\frac{1}{3},0,\frac{5}{3})$。

第五题 求直线 $l:\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}$ 与 Z 轴相交的条件。

解：要直线与 Z 轴相交，即在 $x=y=0$ 时方程组 $\begin{cases}C_1z+D_1=0\\C_2z+D_2=0\end{cases}$ 有解，故条件为 $C_1D_2-C_2D_1=0$，且 $C_1=C_2=0$ 时有 $D_1=D_2=0$。

第六题 证明：直线 $p:\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ 落在平面 $\pi:Ax+By+Cz+D=0$ 上必须且只需 $Al+Bm+Cn=0, Ax_0+By_0+Cz_0+D=0$。同时写出 p 平行于 π 但不在 π 上的条件。

解：直线 p 落在平面 π 上，当且仅当 p 的方向向量垂直于 π 的法向量，且 p 上有一个点在 π 上，即 $Al+Bm+Cn=0$ 且 $Ax_0+By_0+Cz_0+D=0$。p 平行于 π 但不在 π 上的条件是 $Al+Bm+Cn=0, Ax_0+By_0+Cz_0+D\neq 0$。

第七题 求经过直线 $\begin{cases}2x+3y+2z-9=0\\3x-2y+3z-1=0\end{cases}$ 和点 $(1,2,1)$ 的平面方程。

解：**方法一**：直线的方向向量为 $(2,3,2)\times(3,-2,3)=(13,0,-13)$，且直线上有一点 $(1,\frac{25}{13},\frac{8}{13})$，故平面的法向量垂直于 $(1,0,-1)$ 和 $(1,2,1)-(1,\frac{25}{13},\frac{8}{13})=(0,\frac{1}{13},\frac{5}{13})$，故法向量为 $(1,0,-1)\times(0,1,5)=(1,-5,1)$，可设平面方程为 $x-5y+z+D=0$。将 $(1,2,1)$ 代入，得 $D=8$，故平面方程为 $x-5y+z+8=0$。

方法二：用**第八题**的结论，设所求平面为 $\lambda(2x+3y+2z-9)+\mu(3x-2y+3z-1)=0$，将点 $(1,2,1)$ 代入，且不妨设 $\lambda=1$，则 $\mu=-1$，故平面方程为 $x-5y+z+8=0$。

第八题 设平面 π_1 与 π_2 不平行，它们的方程分别为：$A_1x+B_1y+C_1z+D_1=0$ 和 $A_2x+B_2y+C_2z+D_2=0$。证明：过 π_1 和 π_2 的交线的所有平面的方程都可以表示为 $\lambda(A_1x+B_1y+C_1z+D_1)+\mu(A_2x+B_2y+C_2z+D_2)=0$，其中 λ,μ 为不全为零的实数。

解：设过 π_1,π_2 交线的平面方程为 $Ax+By+Cz+D=0$。

首先，由于 π_1,π_2 交线的方向向量为 $(A_1,B_1,C_1)\times(A_2,B_2,C_2)$，故有 $(A,B,C)\cdot(A_1,B_1,C_2)\times(A_2,B_2,C_2)=0$，因此 $(A,B,C),(A_1,B_1,C_1),(A_2,B_2,C_2)$ 共面，又 $(A_1,B_1,C_1),(A_2,B_2,C_2)$ 不共线，因此存在不全为零的实数 λ,μ 使得 $A=\lambda A_1+\mu A_2, B=\lambda B_1+\mu B_2, C=\lambda C_1+\mu C_2$。
其次，取 π_1,π_2 交线上的一点 (x_0,y_0,z_0)，则有 $Ax_0+By_0+Cz_0+D=0$，得 $D=-Ax_0-By_0-Cz_0=-\lambda(A_1x_0+B_1y_0+C_1z_0)-\mu(A_2x_0+B_2y_0+C_2z_0)$。但我们还有 $A_1x_0+B_1y_0+C_1z_0+D_1=0, A_2x_0+B_2y_0+C_2z_0+D_2=0$，故 $A_1x_0+B_1y_0+C_1z_0=-D_1, A_2x_0+B_2x_0+C_2x_0=-D_2$。于是 $D=\lambda D_1+\mu D_2$。
综上，存在不全为零的实数 λ,μ 使得平面方程为 $Ax+By+Cz+D=\lambda(A_1x+B_1y+C_1z+D_1)+\mu(A_2x+B_2y+C_2z+D_2)=0$。

第七次作业答案

1. 设射线 $h=\{(x,0,0),x>0\}$ 与另一过 O 点的射线 k 构成 $\angle(h,k)$，另给定一平面 α' 上关于直线 l' 的一侧与 l' 上关于其上一点 O' 的一侧并记为射线 h'，则存在唯一一条过 O' 且落在 l' 给定一侧的射线 k'，使得 $\angle(h,k)\equiv\angle(h',k')$。
解： 先证存在性。设平面 α' 的法向量为 $\mathbf{n}$，取 h' 上异于 O' 的一点 Q，平面上的点 P 依 $\overrightarrow{O'Q}\times\overrightarrow{O'P}\cdot\mathbf{n}$ 的正负号分为两侧。不妨设现在取的是正号那一侧。设 $\angle(h,k)$ 的余弦为 $t\in(-1,1)$，$O'=(a,b,c), Q=(a+a_0,b+b_0,c+c_0), \mathbf{R}=(a_0,b_0,c_0)$，则取 $P=\begin{cases}(a,b,c)+\mathrm{sgn}(s)(\mathbf{R}+s\mathbf{n}\times\mathbf{R}), & t\neq 0\\ (a,b,c)+\mathbf{n}\times\mathbf{R}, & t=0\end{cases}$，其中 $s=\dfrac{\sqrt{1-t^2}}{t|\mathbf{n}|}, \mathrm{sgn}(s)=\begin{cases}1, & s\geq 0\\ -1, & s<0\end{cases}$。可验证射线 $O'P'$ 即满足条件的 k'。
下证唯一性。设存在另一条射线 k'' 使得 k'' 在 l' 给定的那一侧且 $\angle(h,k)\equiv\angle(h',k'')$。令 $|O'P'|=s$，则 $\dfrac{\overrightarrow{O'P'}\cdot\mathbf{R}}{|O'P'||\mathbf{R}|}=t$，故 $\overrightarrow{O'P'}=\dfrac{ts}{|\mathbf{R}|}\mathbf{R}+k\mathbf{n}\times\mathbf{R}$。于是 $|O'P'|=s=\sqrt{(ts)^2+(k|\mathbf{n}||\mathbf{R}|)^2}$，再由 $\overrightarrow{O'Q}\times\overrightarrow{O'P'}\cdot\mathbf{n}>0$ 得 $k=\dfrac{\sqrt{1-t^2}s}{|\mathbf{n}||\mathbf{R}|}$。因此 $\overrightarrow{O'P'}=s(\dfrac{t}{|\mathbf{R}|}\mathbf{R}+\dfrac{\sqrt{1-t^2}}{|\mathbf{n}||\mathbf{R}|}\mathbf{n}\times\mathbf{R})$。令 $s=\begin{cases}|\mathbf{R}|/|t| & t\neq 0\\ |\mathbf{n}||\mathbf{R}| & t=0\end{cases}$，得 $P'=P$，故 $k'\subset k''$，因此 $k'=k''$。故唯一性得证。

2. 设 l 为 E^3 中一平面 α 上一直线，P 为 α 上但不在 l 上的一点，令 $l_0=$

$\{(x,0)|x\in\mathbb{R}\}\subset E^2$，$P_0$ 为 l_0 外一点，则存在映射 $\phi: E^2 \xrightarrow{\sim} \alpha \subset E^3$，满足 $\phi(\text{直线}) = \text{直线}, \phi(P_0) = P, \phi(l_0) = l$。

解：设 $P_0 = (x_0, y_0)$。先构造 $\varphi: E^2 \xrightarrow{\sim} E^2, (x,y) \mapsto (x-x_0, y/y_0)$，则 φ 把直线映为直线，且 $\varphi(l_0) = l_0, \varphi(P_0) = (0,1)$。取直线 l 上两点 Q, R，则由讲义的**命题 23**，存在映射 $\xi: E^2 \to E^3$ 使得 $\xi((0,1)) = P, \xi((0,0)) = Q, \xi((1,0)) = R$，且该映射的像 $\mathrm{im}(\xi) = \alpha$，$\xi: E^2 \xrightarrow{\sim} \alpha$ 是双射，并把 E^2 上的直线映为直线，特别的，把过 $(0,0),(1,0)$ 的直线 l_0 映为过 Q, R 的直线 l。

令 $\phi = \xi\circ\varphi : E^2 \xrightarrow{\sim} \alpha$。则 ϕ 把直线映为直线，$\phi(l_0) = \xi(\varphi(l_0)) = \xi(l_0) = l, \phi(P_0) = \xi(\varphi(P_0)) = \xi((0,1)) = P$。

3. 求下列平面的方程：

(a) 过点 $(-1,0,3)$，垂直于向量 $(1,2,-5)$；

(b) 过点 $(2,4,-3)$，平行于向量 $(0,2,4)$ 和 $(-1,-2,1)$；

(c) 过点 $(1,0,3),(2,-1,2),(4,-3,7)$；

(d) 过直线 $\dfrac{x-1}{2} = \dfrac{y}{1} = \dfrac{z}{-1}$，平行于直线 $\dfrac{x}{2} = \dfrac{y}{1} = \dfrac{z-1}{-2}$；

(e) 过直线 $\begin{cases} 2x-y-2z+1=0 \\ x+y+4z-2=0 \end{cases}$ 且在 Y 轴与 Z 轴有相同的非零截距；

(f) 过点 $(2,-1,3)$ 和 $(4,1,5)$ 并且垂直于平面 $x+2y+3z+4=0$。

解：(a) 平面法向量为 $(1,2,-5)$，故可设平面方程为 $x+2y-5z+D=0$。将点 $(-1,0,3)$ 代入，得 $D=16$，故平面方程为 $x+2y-5z+16=0$。

(b) 平面的法向量为 $(0,2,4)\times(-1,-2,1) = (10,-4,2)$，故可设平面方程为 $5x-2y+z+D=0$。将点 $(2,4,-3)$ 代入，得 $D=1$，故平面方程为 $5x-2y+z+1=0$。

(c) 平面的法向量垂直于 $(2,-1,2)-(1,0,3) = (1,-1,-1)$ 和 $(4,-3,7)-(1,0,3) = (3,-3,4)$，于是法向量为 $(1,-1,-1)\times(3,-3,4) = (-7,-7,0)$，故可设平面方程为 $x+y+D=0$。将点 $(1,0,3)$ 代入，得 $D=-1$，故平面方程为 $x+y-1=0$。

(d) 平面的法向量为 $(2,1,-1)\times(2,1,-2) = (-1,2,0)$，故可设平面方程为 $x-2y+D=0$。将点 $(1,0,0)$ 代入，得 $D=-1$，故平面方程为 $x-2y-1=0$。

(e) 设平面方程为 $\lambda(2x-y-2x+1)+\mu(x+y+4z-2)=0$。令 $x=z=0$，得 $y=\dfrac{\lambda-2\mu}{\lambda-\mu}$；令 $x=y=0$，得 $z=1/2$。故有 $\dfrac{\lambda-2\mu}{\lambda-\mu}=\dfrac{1}{2}$，令 $\mu=1$，得 $\lambda=3$。故平面方程为 $7x-2y-2z+1=0$。

(f) 平面的法向量垂直于 $(4,1,5)-(2,-1,3) = (2,2,2)$ 与 $(1,2,3)$，于是平面法向量为 $(1,1,1)\times(1,2,3) = (1,-2,1)$。故可设平面方程为 $x-2y+z+D=0$，

将点 $(2,-1,3)$ 代入，得 $D=-7$，故平面方程为 $x-2y+z-7=0$。

4. 求下列直线的方程：
(a) 过点 $(1,0,-2)$，平行于向量 $(4,2,-3)$；
(b) 过点 $(0,2,3)$，垂直于平面 $2x+3y=0$；
(c) 过点 $(2,-1,3)$，与直线 $\dfrac{x-1}{-1}=\dfrac{y}{0}=\dfrac{z-2}{2}$ 相交且垂直；
(d) 过点 $(11,9,0)$，与直线 $\dfrac{x-1}{2}=\dfrac{y+3}{4}=\dfrac{z-5}{3}$ 和直线 $\dfrac{x}{5}=\dfrac{y-2}{-1}=\dfrac{z+1}{2}$ 相交；
(e) 直线 $\dfrac{x-1}{1}=\dfrac{y}{-3}=\dfrac{z}{2}$ 与直线 $\dfrac{x}{2}=\dfrac{y}{1}=\dfrac{z}{-2}$ 的公垂线。

解：(a) 直线方程为 $\dfrac{x-1}{4}=\dfrac{y}{2}=\dfrac{z+2}{-3}$。
(b) 直线方程为 $\dfrac{x}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{0}$。
(c) 点与直线所在平面的法向量为 $((2,-1,3)-(1,0,2))\times(-1,0,2)=(-2,-3,-1)$，因此直线的方向向量为 $(2,3,1)\times(-1,0,2)=(6,-5,3)$，故直线方程为 $\dfrac{x-2}{6}=\dfrac{y+1}{-5}=\dfrac{z-3}{3}$。
(d) 点与第一条直线所在平面的法向量为 $((11,9,0)-(1,-3,5))\times(2,4,3)=(56,-40,16)$，点与第二条直线所在平面的法向量为 $((11,9,0)-(0,2,-1))\times(5,-1,2)=(15,-17,-46)$，因此所求直线的方向向量为 $(7,-5,2)\times(15,-17,-46)=(264,352,-44)$，故直线方程为 $\dfrac{x-11}{6}=\dfrac{y-9}{8}=\dfrac{z}{-1}$。
(e) 直线的方向向量为 $(1,-3,2)\times(2,1,-2)=(4,6,7)$。设所求直线与第一条直线的交点为 $(1+t,-3t,2t)$，与第二条直线的交点为 $(1+t+4s,-3t+6s,2t+7s)$，则 $\dfrac{1+t+4s}{2}=\dfrac{-3t+6s}{1}=\dfrac{2t+7s}{-2}$，解得 $s=-\dfrac{4}{101},t=-\dfrac{19}{101}$，故直线方程为 $\dfrac{x-\frac{82}{101}}{4}=\dfrac{y-\frac{57}{101}}{6}=\dfrac{z+\frac{38}{101}}{7}$。

5.(a) 求点 $(-1,-3,5)$ 到直线 $\dfrac{x-1}{2}=\dfrac{y-1}{3}=\dfrac{z+1}{3}$ 的距离；
(b) 求两直线 $\dfrac{x}{2}=\dfrac{y+2}{-2}=\dfrac{z-1}{1}$ 与 $\dfrac{x-1}{4}=\dfrac{y-3}{2}=\dfrac{z+1}{-1}$ 之间的距离。

解：(a) 取直线上两个点 $A(1,1,-1),B(3,4,2)$，则这两个点与点 $C(-1,-3,5)$ 组成的三角形的面积 $S=\dfrac{d|AB|}{2}=\dfrac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{2}$，其中 d 为点 C 到直线的距离，故得 $d=\sqrt{\dfrac{614}{11}}$。
(b) 取两条直线上的点 $(0,-2,1),(1,3,-1)$，则向量 $(2,-2,1),(4,2,-1),(1,3,-1)-(0,-2,1)=(1,5,-2)$ 构成的平行六面体的体积为 $V=Sh=|(2,-2,1)\times(4,2,-1)|d=|((2,-2,1)\times(4,2,-1))\cdot(1,5,-2)|$，其中 d 为两直线的距离。故

得 $d=1/\sqrt{5}$。

6.(a) 给定点 $A(1,0,3)$ 与 $B(0,2,5)$ 和直线 $l:\dfrac{x-1}{2}=\dfrac{y+1}{1}=\dfrac{z}{3}$，设 A',B' 为 A,B 在 l 上的垂足，求 $|A'B'|$ 与 A',B' 的坐标；
(b) 给定点 $A(2,3,1)$ 与 $B(-1,0,4)$ 和平面 $\pi: x+2y-z+4=0$，设 A',B' 为 A,B 在 π 上的垂足，求 $|A'B'|$ 与通过 A',B' 的直线方程；
(c) 求点 $(0,2,1)$ 到平面 $2x-3y+5z-1=0$ 的距离。

解: (a) 设直线的方向向量 $\mathbf{t}=(2,1,3)$，则 $|A'B'|=\dfrac{|\overrightarrow{A'B'}\cdot\mathbf{t}|}{|\mathbf{t}|}=\dfrac{6}{\sqrt{14}}$。取直线上的点 $O=(1,-1,0)$，设 $A'=(1+2s,-1+s,3s)$，则 $|OA'|=\sqrt{14}s=\dfrac{|\overrightarrow{OA}\cdot\mathbf{t}|}{|\mathbf{t}|}$，解得 $s=5/7$，故 $A'=(\dfrac{17}{7},-\dfrac{2}{7},\dfrac{15}{7})$。同理得 $B'=(\dfrac{23}{7},\dfrac{1}{7},\dfrac{24}{7})$。
(b) 设平面的法向量为 $\mathbf{n}=(1,2,-1)$，则 $|A'B'|=\sqrt{|AB|^2-\dfrac{|\overrightarrow{AB}\cdot\mathbf{n}|^2}{|\mathbf{n}|^2}}=\sqrt{3}$。设 $A'=(2+t,3+2t,1-t)$，则代入平面方程，得 $t=-11/6$，故得 $A'=(\dfrac{1}{6},-\dfrac{2}{3},\dfrac{17}{6})$。同理得 $B'=(-\dfrac{5}{6},\dfrac{1}{3},\dfrac{23}{6})$，故直线 $A'B':\dfrac{x}{1}=\dfrac{y+1/2}{-1}=\dfrac{z-3}{-1}$。
(c) 取平面上的点 $(1,2,1)$，则距离为 $\dfrac{|((1,2,1)-(0,2,1))\cdot(2,-3,5)|}{|(2,-3,5)|}=\dfrac{2}{\sqrt{38}}$。

第八次作业答案

第一题 试证非空集合 X 上的等价关系与该集合的分拆存在一一对应。

解: 先证明：每一个等价关系都对应着一个分拆。设 X 上的等价关系 $\sim$，取 X 的子集族 $\mathcal{X}=\{[x]|x\in X\}$，其中 $[x]=\{y\in X|x\sim y\}$。我们证明 $\mathcal{X}$ 是 X 的一个分拆。首先，由于 $x\sim x$，对任意的 $x\in X$，都有 $x\in[x]$，因此 $\cup\mathcal{X}=X$（$\cup\mathcal{X}$ 指的是 $\mathcal{X}$ 中所有元素的并）；其次，如果 $[x_1]\cap[x_2]\neq 0$，则存在 $y\in X$ 使得 $x_1\sim y, x_2\sim y$，故 $x_1\sim x_2$，因此对任意的 $x\in[x_1]$，$x\sim x_2$，进而 $x\in[x_2]$，且对任意的 $x'\in[x_2]$，$x'\sim x_1$，进而 $x'\in[x_1]$，因此 $[x_1]=[x_2]$，故若 $[x_1]\neq[x_2]$，则 $[x_1]\cap[x_2]=0$。综上，X 是 $\mathcal{X}$ 中元素的不交并，因此 $\mathcal{X}$ 是 X 的一个分拆。

再证明：每一个分拆都对应着一个等价关系。设 $\mathcal{X}=\sqcup_i X_i$ 构成 X 的一个分拆。在 X 上构造二元关系 $\sim$ 满足：$a\sim b$ 当且仅当存在某个 i 使得 $a,b\in X_i$。则：显然 $a\sim a$；若 $a\sim b$，则存在 i 使得 $a,b\in X_i$，故 $b\sim a$；若 $a\sim b, b\sim c$，则存在 i,j 使得 $a,b\in X_i, b,c\in X_j$，由于对 $i\neq j$，$X_i\cap X_j=\emptyset$，故 $i=j$，因

此 $a, c \in X_i$，进而 $a \sim c$。综上，$\sim$ 是 X 上的等价关系。

第二题 定义 E^3 上的典范距离函数：对 $P_i = (x_i, y_i, z_i)(i = 1, 2)$，定义 $d_E = E^3 \times E^3 \longrightarrow \mathbb{R}$，$d_E(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$。证明 d_E 是一个距离函数，即为典范距离函数。

解：正定性与对称性是显然的。下面验证三角不等式。对三个点 P_1, P_2, P_3，要证明 $d_E(P_1, P_3) \leq d_E(P_1, P_2) + d_E(P_2, P_3)$。由于 $d_E(P, Q) = |\overrightarrow{PQ}|$，因此只要验证 $|\overrightarrow{P_1P_3}| \leq |\overrightarrow{P_1P_2}| + |\overrightarrow{P_2P_3}|$。根据余弦定理，$|\overrightarrow{P_1P_3}|^2 = |\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3}|^2 = ||\overrightarrow{P_1P_2}|^2 + |\overrightarrow{P_2P_3}|^2 + 2\overrightarrow{P_1P_2} \cdot \overrightarrow{P_2P_3}| = ||\overrightarrow{P_1P_2}|^2 + |\overrightarrow{P_2P_3}|^2 + 2|\overrightarrow{P_1P_2}||\overrightarrow{P_2P_3}|\cos\theta| \leq |\overrightarrow{P_1P_2}|^2 + |\overrightarrow{P_2P_3}|^2 + 2|\overrightarrow{P_1P_2}||\overrightarrow{P_2P_3}| = (|\overrightarrow{P_1P_2}| + |\overrightarrow{P_2P_3}|)^2$，其中 θ 为 $\overrightarrow{P_1P_2}, \overrightarrow{P_2P_3}$ 之间的夹角。因此 $|\overrightarrow{P_1P_3}| \leq |\overrightarrow{P_1P_2}| + |\overrightarrow{P_2P_3}|$。综上，$d_E$ 是距离函数。

第三题 刚体变换不满足交换律。规定记号：$T_{\mathbf{u}}$ 为沿着向量 $\mathbf{u}$ 的平移变换，R_θ 为绕 Z 轴从 X 向 Y 转动 θ 的旋转变换。请验证：当 $\mathbf{u} \nparallel \mathbf{e}_Z, \theta \neq 0$ 时，总有：$T_{\mathbf{u}} \circ R_\theta \neq R_\theta \circ T_{\mathbf{u}}$。

解：考察点 $O = (0, 0)$。一方面，$R_\theta(O) = O$，故 $T_{\mathbf{u}} \circ R_\theta = T_{\mathbf{u}}(O) = \mathbf{u}$。另一方面，$T_{\mathbf{u}}(O) = \mathbf{u}$ 不在 Z 轴上，故 $R_\theta \circ R_{\mathbf{u}}(O) = R_\theta(\mathbf{u}) \neq \mathbf{u}$。因此 $T_{\mathbf{u}} \circ R_\theta(O) \neq R_\theta \circ T_{\mathbf{u}}(O)$，进而 $T_{\mathbf{u}} \circ R_\theta \neq R_\theta \circ T_{\mathbf{u}}$。

第九次作业答案

1. 证明：三维欧氏空间中的刚体变换是旋转和平移的复合。即：对任何刚体变换 ϕ，存在平移 τ 和旋转 $\mathcal{R}$ 使得 $\phi = \mathcal{R} \circ \tau$。

解：**方法一**：由刚体变换是双射，故可逆，且其逆也是刚体变换。平移 $\tau_{\mathbf{u}}$ 的逆 $\tau_{-\mathbf{u}}$ 也是平移，而旋转 $\mathcal{R}$ 的逆固定原点，故也是旋转。对刚体变换 ϕ，有 $\phi^{-1} = \tau \circ \mathcal{R}$，故 $\phi = (\phi^{-1})^{-1} = \mathcal{R}^{-1} \circ \tau^{-1}$，因此刚体变换是旋转和平移的复合。

方法二：利用题 2（1），设 $\phi = \tau \circ \mathcal{R}$，设平移 $\tau' = \mathcal{R}^{-1} \circ \tau \circ \mathcal{R}$，则 $\mathcal{R} \circ \tau' = \tau \circ \mathcal{R} = \phi$，故刚体变换是旋转和平移的复合。

2.（1）证明：对任意的平移 τ 与旋转 $\mathcal{R}$，刚体变换 $\mathcal{R} \circ \tau \circ \mathcal{R}^{-1}$ 也是一个平移，即：所有平移构成的群是刚体变换群的正规子群。

（2）证明：所有旋转构成的群是刚体变换群的子群。这个群是否是刚体变换群的正规子群？请说明理由。

解：（1）设 $\tau = \tau_{\mathbf{u}}$，则 $\mathcal{R} \circ \tau \circ \mathcal{R}^{-1}(A) = \mathcal{R}(\mathcal{R}^{-1}(A) + \mathbf{u}) = A + \mathcal{R}(\mathbf{u})$ 是平移。

（2）设旋转 $\mathcal{R}_1, \mathcal{R}_2$，则 $\mathcal{R}_1, \mathcal{R}_2$ 固定原点，故其复合的刚体变换 $\mathcal{R}_1 \circ \mathcal{R}_2$ 固定原点，其逆 $\mathcal{R}_1^{-1}$ 固定原点，因此它们都是旋转。故所有旋转构成一个群。对并非恒等变换的 $\mathcal{R}$ 和平移 $\tau = \tau_{\mathbf{u}}$，$\mathcal{R}(\textbf{-u}) \neq \textbf{-u}$，故 $\tau \circ \mathcal{R} \circ \tau^{-1}(O) = \tau \circ \mathcal{R}(-\mathbf{u}) \neq \tau(-\mathbf{u}) = O$，因此 $\tau \circ \mathcal{R} \circ \tau^{-1}$ 这个刚体变换不是旋转，因此所有旋转并不构成刚体变换群的正规子群。

3. 用坐标写出从 x 轴正方向看绕 x 轴逆时针旋转 θ 角的旋转变换 $\mathcal{R}_x(\theta)$ 的表达式。并证明：$\mathcal{R}_z(\frac{\pi}{2}) \circ \mathcal{R}_x(\frac{\pi}{2}) \neq \mathcal{R}_x(\frac{\pi}{2}) \circ \mathcal{R}_z(\frac{\pi}{2})$。

解:$\mathcal{R}_x(\theta)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y\cos\theta - z\sin\theta \\ y\sin\theta + z\cos\theta \end{pmatrix}$。由于 $\mathcal{R}_z(\frac{\pi}{2}) \circ \mathcal{R}_x(\frac{\pi}{2})(1,0,0) = (0,-1,0)$，而 $\mathcal{R}_x(\frac{\pi}{2}) \circ \mathcal{R}_z(\frac{\pi}{2})(1,0,0) = (0,0,-1)$，故 $\mathcal{R}_z(\frac{\pi}{2}) \circ \mathcal{R}_x(\frac{\pi}{2}) \neq \mathcal{R}_x(\frac{\pi}{2}) \circ \mathcal{R}_z(\frac{\pi}{2})$。

4. 定义 E^2 上自然的距离函数 $d: E^2 \times E^2 \longrightarrow \mathbb{R}$，对 $P_i = (x_i, y_i)(i = 1, 2)$，有 $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$。我们称 $\phi: E^2 \longrightarrow E^2$ 是二维欧氏平面上的刚体变换，若 ϕ 满足如下两个条件：
(a)ϕ 保持距离 d，即：$\forall P, Q \in E^2$，$d(\phi(P), \phi(Q)) = d(P, Q)$；
(b)ϕ 保持定向，即 ϕ 保持平行四边形的有向面积不变。
请完成以下问题：
（1）任取 E^2 中的向量 $u_0 = (x_0, y_0)$，定义平移 $\tau_{u_0}: E^2 \longrightarrow E^2, u \mapsto u + u_0$。验证：平移是刚体变换。
（2）对任意的角度 θ，定义绕坐标原点 O 的旋转为 $\mathcal{R}_\theta: E^2 \longrightarrow E^2, (x, y) \mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$。验证：这样的旋转是刚体变换。
（3）设 E^2 上有直角坐标系 $\{O; \mathbf{e}_1, \mathbf{e}_2\}$，设刚体变换 $\phi: E^2 \longrightarrow E^2$，$\tilde{\phi} = \tau_{-\overrightarrow{O\phi(O)}} \circ \phi$。证明：$\{\phi(O); \tilde{\phi}(\mathbf{e}_1), \tilde{\phi}(\mathbf{e}_2)\}$ 也构成一个直角坐标系。
（4）设 $\phi: E^2 \longrightarrow E^2$ 为一个刚体变换，证明：存在 $O' = (x_0, y_0), \mathbf{e}_1' = (x_1, y_1), \mathbf{e}_2' = (x_2, y_2)$ 满足：$x_i^2 + y_i^2 = 1(i = 1, 2)$，$x_1x_2 + y_1y_2 = 0$，且 $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} > 0$，使得：$\phi(x, y) = x\mathbf{e}_1' + y\mathbf{e}_2' + O'$。
（5）证明：刚体变换是平移和旋转的复合。
（6）请对 E^2 上所有两个点的集合在刚体变换下进行分类。
（提示：对两个点的点对 (A, B) 进行分类，也就是找出一个点对可以经过某个刚体变换与第二个点对重合的条件，也就是找到两个点对合同的条件。在这一问里，即证明：两个点对合同当且仅当它们的距离相等。）
（7）试对 E^2 上所有三个点的集合在刚体变换下进行分类。

解：（1）$d(\tau_{u_0}(P),\tau_{u_0}(Q))=|\overrightarrow{\tau_{u_0}(P)\tau_{u_0}(Q)}|=|P+u_0-(Q+u_0)|=|P-Q|=|\overrightarrow{PQ}|=d(P,Q)$，而平移不改变向量，因此平移仍保持定向。
（2）用坐标可直接验证 $d(\mathcal{R}_\theta(P),\mathcal{R}_\theta(Q))=d(P,Q)$。要验证定向，只需证明 $\mathcal{R}_\theta$ 不改变 $\mathbf{e}_1,\mathbf{e}_2$ 张成的有限平行四边形的面积，而 $S(\mathcal{R}_\theta(\mathbf{e}_1),\mathcal{R}_\theta(\mathbf{e}_2))=\begin{vmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{vmatrix}=1$，故不改变定向。因此旋转也是刚体变换。
（3）由 $\tilde\phi$ 是刚体变换，知 $|\tilde\phi(\mathbf{e}_1)|=|\mathbf{e}_1|=1,|\tilde\phi(\mathbf{e}_2)|=|\mathbf{e}_2|=1$，且 $\langle\tilde\phi(\mathbf{e}_1),\tilde\phi(\mathbf{e}_2)\rangle=\dfrac{1}{2}(|\tilde\phi(\mathbf{e}_1)\tilde\phi(\mathbf{e}_2)|^2-|\tilde\phi(\mathbf{e}_1)|^2-|\tilde\phi(\mathbf{e}_2)|^2)=\dfrac{1}{2}(|\mathbf{e}_1\mathbf{e}_2|^2-|\mathbf{e}_1|^2-|\mathbf{e}_2|^2)=\langle\mathbf{e}_1,\mathbf{e}_2\rangle=0$，故 $\{\phi(O);\tilde\phi(\mathbf{e}_1),\tilde\phi(\mathbf{e}_2)\}$ 构成直角坐标系。
（4）设 $\phi(O)=O'$，令 $\tau=\tau_{\overrightarrow{OO'}}$，则刚体变换 $\tilde\phi=\tau^{-1}\circ\phi$ 满足 $\tilde\phi(O)=O$。令 $\mathbf{e}'_i=\tilde\phi(\mathbf{e}_i),i=1,2$，则与（3）的方法相同，可证 $\{O';\mathbf{e}'_1\mathbf{e}'_2\}$ 构成直角坐标系，即 $\mathbf{e}'_1=(x_1,y_1),\mathbf{e}'_2=(x_2,y_2)$ 满足 $x_i^2+y_i^2=1(i=1,2)$，$x_1x_2+y_1y_2=0$。又因为 ϕ 是刚体变换，不改变 $\mathbf{e}_1,\mathbf{e}_2$ 的有限平行四边形面积，因此 $\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}>0$。设 $\tilde\phi(x\mathbf{e}_1+y\mathbf{e}_2)=x'\mathbf{e}'_1+y'\mathbf{e}'_2$，则 $x=\langle x\mathbf{e}_1+y\mathbf{e}_2,\mathbf{e}_1\rangle=\langle x'\mathbf{e}'_1+y'\mathbf{e}'_2,\mathbf{e}'_1\rangle=x'$，同理 $y=y'$，故 $\phi(x,y)=x\mathbf{e}_1+y\mathbf{e}_2+O'$。
（5）设刚体变换 ϕ 满足 $\phi(O)=O'$，令 $\tau=\tau_{\overrightarrow{OO'}},\tilde\phi=\tau^{-1}\circ\tilde\phi$，则 $\tilde\phi(O)=O$。设 $\tilde\phi(\mathbf{e}_i)=\mathbf{e}'_i$，则由于保距，$\mathbf{e}'_1$ 必为 $\mathbf{e}_1$ 转过一个角度，进而由保距性、保向性和三角形的全等，知 $\mathbf{e}'_2$ 也为 $\mathbf{e}_2$ 转过同样的角度，因此 $\tilde\phi$ 是旋转，故刚体变换 $\phi=\tau\circ\tilde\phi$ 是平移与旋转的复合。
（6）若两个点对 $(A,B),(A',B')$ 合同，则存在刚体变换 ϕ 使 $\phi(A)=A',\phi(B)=B'$，故 $d(A',B')=d(A,B)$，因此它们的距离相等。若两个点对 $(A,B),(A',B')$ 的距离相等，令 $\overrightarrow{AB}$ 顺时针转到 $\overrightarrow{A'B'}$ 的角度为 θ，则令刚体变换 $\phi=\tau_{\overrightarrow{OA'}}\circ\mathcal{R}_\theta\circ\tau_{-\overrightarrow{OA}}$，知 $\phi(A)=A',\phi(B)=B'$，故 $(A,B),(A',B')$ 合同。
（7）三个点的点组 (A,B,C) 与 (A',B',C') 合同当且仅当 $\triangle ABC$ 与 $\triangle A'B'C'$ 全等且 $\overrightarrow{AB},\overrightarrow{AC}$ 的有向平行四边形面积等于 $\overrightarrow{A'B'},\overrightarrow{A'C'}$ 的有向平行四边形面积。证明类似（6）。

第十次作业答案

1. 设 $\phi:E^3\to E^3$ 为刚体变换，$\alpha\subset E^3$ 为平面。求证：$\phi(\alpha)\subset E^3$ 为平面。
解： 设平面 $\alpha:\mathbf{x}\cdot\mathbf{n}=b$，则 $\mathbf{x}\in\alpha$ 当且仅当 $(\phi(\mathbf{x})-\phi(O))\cdot(\phi(\mathbf{n})-\phi(O))=b$，即 $\phi(\mathbf{x})$ 满足 $\phi(\mathbf{x})\cdot(\phi(\mathbf{n})-\phi(O))=b+\phi(O)\cdot(\phi(\mathbf{n})-\phi(O))$，即 $\phi(\mathbf{x})\cdot\mathbf{n}'=b'$，

故 $\phi(\alpha)=\{\phi(\mathbf{x})|x\in\alpha\}$ 也是平面。

2. 用矩阵乘法将如下线性方程组写成矩阵形式：

$$\begin{cases}a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1\\a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2\\a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3\end{cases}$$

解： $\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ 或 $\begin{pmatrix}x_1&x_2&x_3\end{pmatrix}\begin{pmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{pmatrix}=\begin{pmatrix}b_1&b_2&b_3\end{pmatrix}$。注意两种形式的矩阵互为转置。

3.$A,B\in M_{n\times m}(\mathbb{R}),C\in M_{m\times k}(\mathbb{R})$，求证：$(A+B)\cdot C=A\cdot C+B\cdot C$。
解： 记 $A=(a_{ij})_{nm},B=(b_{ij})_{nm},C=(c_{ij})_{mk}$，则 $(A+B)\cdot C=(\sum_{l=1}^m(a_{il}+b_{il})c_{lj})_{nk}$，$A\cdot C+B\cdot C=(\sum_{l=1}^m a_{il}c_{lj}+\sum_{l=1}^m b_{il}c_{lj})_{nk}$，而 $\sum_{l=1}^m(a_{il}+b_{il})c_{lj}=\sum_{l=1}^m a_{il}c_{lj}+\sum_{l=1}^m b_{il}c_{lj}$，故 $(A+B)\cdot C=A\cdot C+B\cdot C$。

4.$A,B\in M_{n\times n}(\mathbb{R})$，求证：$(A\cdot B)^T=B^T\cdot A^T$。
解： 记 $A=(a_{ij})_{nn},B=(b_{ij})_{nn}$，则 $(A\cdot B)^T=(\sum_{l=1}^n a_{jl}b_{li})_{nn}$，$B^T\cdot A^T=(\sum_{l=1}^n b_{li}a_{jl})_{nn}$。而 $a_{jl}b_{li}=b_{li}a_{jl}$，故 $(A\cdot B)^T=B^T\cdot A^T$。

5. 给定 $\lambda\in\mathbb{R}$，以及 $A=(a_{ij})_{nm}\in M_{n\times m}(\mathbb{R})$，我们定义矩阵的数乘为 $\lambda\cdot A=(\lambda\cdot a_{ij})_{nm}$。验证如下性质：
（1）$\lambda\cdot(A+B)=\lambda\cdot A+\lambda\cdot B$。
（2）$(\lambda\mu)\cdot A=\lambda\cdot(\mu\cdot A)$。
解：（1）记 $A=(a_{ij})_{nm},B=(b_{ij})_{nm}$，则 $\lambda\cdot(A+B)=(\lambda\cdot(a_{ij}+b_{ij}))_{nm}=(\lambda\cdot a_{ij}+\lambda\cdot b_{ij})_{nm}=\lambda\cdot A+\lambda\cdot B$。
（2）$(\lambda\mu)\cdot A=((\lambda\mu)\cdot a_{ij})_{nm}=(\lambda\cdot(\mu\cdot a_{ij}))_{nm}=\lambda\cdot(\mu\cdot a_{ij})_{nm}=\lambda\cdot(\mu\cdot A)$。

6. 证明：方阵 $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ 可逆，当且仅当 $\begin{vmatrix}a&b\\c&d\end{vmatrix}\neq 0$。

（提示：计算 $\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$。）

解： $\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}d&-b\\-c&a\end{pmatrix}=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{vmatrix}a&b\\c&d\end{vmatrix}\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}$。故若

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$,则 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 当然可逆。另一方面,若方阵 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 可逆,则存在 x, y, z, w 使得 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,故 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,故 $cx + dy = 0$,因此存在 $k \in \mathbb{R}$ 使得 $x = kd, y = -kc$。又 $ax + by = 1$,知 $k(ad - bc) = 1$,因此 $ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$。

注:如果承认矩阵乘法的结合律(证明见下一题的注),那么“另一方面”开始有另一种方法:另一方面,若 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$,要证明 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 不可逆。假设它可逆,则存在方阵 $A \in M_{2\times 2}(\mathbb{R})$,使得 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,则 $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}(\begin{pmatrix} a & b \\ c & d \end{pmatrix} A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,而由结合律,$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}(\begin{pmatrix} a & b \\ c & d \end{pmatrix} A) = (\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix})A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,故 $a = b = c = d = 0$,而对任意方阵 B,$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,因此 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 不可逆,矛盾。

7. 设方阵 $A = (a_{ij})_{33} \in M_{3\times 3}(\mathbb{R})$。将元素 a_{ij} 所在的第 i 行和第 j 列元素划掉后,剩余的元素按照原来的排列顺序组成 2 阶方阵,我们称该方阵的行列式为元素 a_{ij} 的余子式,记为 M_{ij},称 $A_{ij} = (-1)^{i+j}M_{ij}$ 为元素 a_{ij} 的代数余子式。我们定义矩阵 A 的伴随矩阵 $A^* = (A_{ji})_{33}$。求证以下命题:

(1)验证:$A \cdot A^* = A^* \cdot A = |A| \cdot I$。

(2)验证:$|A \cdot B| = |A||B|$,$B \in M_{3\times 3}(\mathbb{R})$。

(3)求证:A 可逆当且仅当 $|A| \neq 0$,且 A 可逆时,$A^{-1} = |A|^{-1} \cdot A^*$。

(4)求证:若方阵 $A, B \in M_{3\times 3}(\mathbb{R})$ 均可逆,则 $A \cdot B$ 可逆,且 $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$。

解:(1)记 $A \cdot A^* = (e_{ij})_{33}$。则 $e_{11} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) = |A|$。类似地可验证 $e_{ij} = |A|\delta_{ij}$,故 $A \cdot A^* = |A| \cdot I$,$A^* \cdot A = |A| \cdot I$ 同理。

(2)慢慢算吧……加油。

(3)若 $|A| \neq 0$,令 $B = |A|^{-1} \cdot A^*$,则 $A \cdot B = B \cdot A = I$,故 A 可逆且 $A^{-1} = B = |A|^{-1} \cdot A^*$。另一方面,若 A 可逆,则存在 B 使得 $AB = BA = I$,

因此 $1=|I|=|AB|=|A||B|$，故 $|A|\neq 0$。
（4）由（3），A,B 均可逆，故 $|A|,|B|$ 均不为 0，故 $|A\cdot B|=|A||B|\neq 0$，因此 $A\cdot B$ 可逆。由 $A^{-1}=|A|^{-1}\cdot A^*, B^{-1}=|B|^{-1}\cdot B^*, (A\cdot B)^{-1}=|A\cdot B|^{-1}\cdot(A\cdot B)^*$，只需验证 $(A\cdot B)^*=B^*\cdot A^*$，就有 $(A\cdot B)^{-1}=B^{-1}\cdot A^{-1}$。慢慢算吧……
注：对（4），还有另一种思路：
先证明：矩阵乘法具有结合律。即证：$(A\cdot B)\cdot C=A\cdot(B\cdot C)$，其中 $A=(a_{ij})_{mn}, B=(b_{ij})_{np}, C=(c_{ij})_{pq}$。则 $(A\cdot B)\cdot C=(\sum_{l=1}^{p}(\sum_{k=1}^{n}a_{ik}b_{kl})c_{lj})_{mq}$，$A\cdot(B\cdot C)=(\sum_{k=1}^{n}a_{ik}(\sum_{l=1}^{p}b_{kl}c_{lj}))_{mq}$，显然二者相等。
再证明（4）。若 A,B 可逆，则由结合律，有 $(A\cdot B)\cdot(B^{-1}\cdot A^{-1})=A\cdot(B\cdot B^{-1})\cdot A^{-1}=A\cdot I\cdot A^{-1}=A\cdot A^{-1}=I$，同理 $(B^{-1}\cdot A^{-1})\cdot(A\cdot B)=I$，因此 $A\cdot B$ 也可逆且 $(A\cdot B)^{-1}=B^{-1}\cdot A^{-1}$。

第十一次作业答案

第一题　回忆在课堂上关于二次曲面标准方程的理论的一个关键命题是：
对任何对称方阵 $C\in M_3(\mathbb{R})$，总存在方阵 $A\in M_3(\mathbb{R})$ 使得：
（1）$ACA^T=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$；
（2）$AA^T=I$；
（3）$|A|>0$。
本习题将给出算法计算 $\lambda_i, i=1,2,3$ 和对应的方阵 A。
第一步：令 $C=\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{12}&c_{22}&c_{23}\\c_{13}&c_{23}&c_{33}\end{pmatrix}$，计算方程 $\begin{vmatrix}\lambda-c_{11}&-c_{12}&-c_{13}\\-c_{12}&\lambda-c_{22}&-c_{23}\\-c_{13}&-c_{23}&\lambda-c_{33}\end{vmatrix}=0$ 的三个解 $\lambda_1,\lambda_2,\lambda_3$。
第二步：
情形一：$\lambda_1=\lambda_2=\lambda_3=\lambda$。**试证明**：$C=\lambda I$，此时取 $A=I$ 即可。
情形二：$\lambda=\lambda_1=\lambda_2\neq\lambda_3=\mu$，此时线性方程组 $(C-\lambda I)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ 定义了一个过原点的平面，任取其上两个不同向的向量 u_1,u_2，作如下的正交化过程：$\tilde{u}_1=\dfrac{u_1}{\|u_1\|}, \tilde{u}_2=\dfrac{u_2-\langle u_2,\tilde{u}_1\rangle\tilde{u}_1}{\|u_2-\langle u_2,\tilde{u}_1\rangle\tilde{u}_1\|}$。另一方面，方程组 $(C-\mu I)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

定义了一个过原点的直线，任取直线上的单位向量 $\tilde{u}_3$，我们取 $A = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix}$ 或者 $A = \begin{pmatrix} \tilde{u}_2 \\ \tilde{u}_1 \\ \tilde{u}_3 \end{pmatrix}$ 使得 $|A| > 0$。

情形三：$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$，此时线性方程组 $(C - \lambda_i I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 均定义了过原点的两两不同的直线，分别取直线上的单位向量 $\tilde{u}_i$，并取 $A = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix}$ 或者 $A = \begin{pmatrix} \tilde{u}_2 \\ \tilde{u}_1 \\ \tilde{u}_3 \end{pmatrix}$ 使得 $|A| > 0$。

解： 由方程 $|xI - C| = (x-\lambda)^3$，可列出方程组（比较 x, x^2, x^3 各项的系数）$\begin{cases} c_{11} + c_{22} + c_{33} = 3\lambda \\ c_{11}c_{22} + c_{22}c_{33} + c_{33}c_{11} - c_{12}^2 - c_{13}^2 - c_{23}^2 = 3\lambda^2 \\ c_{11}c_{22}c_{33} + 2c_{12}c_{23}c_{13} - c_{11}c_{23}^2 - c_{22}c_{13}^2 - c_{33}c_{12}^2 = \lambda^3 \end{cases}$。设 $c_{ii} = \lambda + a_i, i = 1,2,3$，则 $a_1 + a_2 + a_3 = 0$，故必存在两个非异号的 a，不妨设为 a_1, a_2，也即 $a_1a_2 \geq 0$。由第二个式子，有 $a_1a_2 + a_2a_3 + a_1a_3 = c_{12}^2 + c_{13}^2 + c_{23}^2 \geq 0$，但 $a_1a_2 + a_2a_3 + a_1a_3 = -(a_1^2 + a_1a_2 + a_2^2) \leq 0$，于是只能 $c_{12}^2 + c_{13}^2 + c_{23}^2 = a_1^2 + a_1a_2 + a_2^2 = 0$，故 $c_{12} = c_{13} = c_{23} = a_1 = a_2 = 0$，进而 $a_3 = 0$，故 $c_{11} = c_{22} = c_{33} = \lambda$，因此 $C = \lambda I$。

第二题　按照课上的步骤，尝试将以下二次曲面化为标准形式：$xy - yz + 2 = 0$。

解： 该二次曲面可表示为 $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$。则 $C = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$，解方程 $|\lambda I - C| = 0$，得 $\lambda_1 = \frac{\sqrt{2}}{2}, \lambda_2 = -\frac{\sqrt{2}}{2}, \lambda_3 = 0$。

分别考察线性方程组 $(C-\lambda_i I)\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$，得 $u_1=(\frac{1}{2},\frac{\sqrt{2}}{2},-\frac{1}{2}), u_2=(-\frac{1}{2},\frac{\sqrt{2}}{2},\frac{1}{2}), u_3=(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2})$。令 $A=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}=\begin{pmatrix}1/2 & \sqrt{2}/2 & -1/2\\ -1/2 & \sqrt{2}/2 & 1/2\\ \sqrt{2}/2 & 0 & \sqrt{2}/2\end{pmatrix}$，则 $|A|=1$。故刚体变换 $\Phi(x,y,z)=(x,y,z)A$ 有 $F_2\circ\Phi=\frac{\sqrt{2}}{2}x^2-\frac{\sqrt{2}}{2}y^2$，$F_1\circ\Phi=0$，因此标准方程为 $\frac{x^2}{2\sqrt{2}}-\frac{y^2}{2\sqrt{2}}-1=0$。

第三题 本题中我们模仿 E^3 中二次曲面在刚体变换下的分类，来对 E^2 中二次曲线在（E^2 上）刚体变换下进行分类。关于 E^2 上的刚体变换大家可以回忆一下习题九中的第四题。

（1）定义 E^2 中二次曲线的全体。

（2）令 $F=F(x,y)$ 为二元齐次二次多项式，即 $F=ax^2+bxy+cy^2$，则存在唯一的对称方阵 $C\in M_2(\mathbb{R})$，使得 $F=\begin{pmatrix}x & y\end{pmatrix}C\begin{pmatrix}x\\y\end{pmatrix}$。

（3）对（2）中的方阵 C，我们总能找到 $A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},\theta\in\mathbb{R}$，使得 $ACA^T=\mathrm{diag}(\lambda_1,\lambda_2),\lambda_1,\lambda_2\in\mathbb{R}$。作为推论，我们对任意二次多项式 F，总存在旋转变换 Φ 使得 $(F\circ\Phi)(x,y)=\lambda_1x^2+\lambda_2y^2+a_1x+a_2y+b,\lambda_1^2+\lambda_2^2\neq 0$。

（4）按照 $\lambda_1\lambda_2=0,<0,>0$ 三种情形分别讨论，证明任意平面二次曲线在刚体变换下化归到下列标准方程定义的曲线：

(4.1)$\lambda_1\lambda_2=0$：

(4.1.1) 空集：$x^2+1=0$

(4.1.2) 两条平行直线：$x^2-a=0,a\geq 0$

(4.1.3) 抛物线：$x^2-ay=0,a>0$

(4.2)$\lambda_1\lambda_2>0$

(4.2.1) 点：$x^2+y^2=0$

(4.2.2) 椭圆：$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

(4.3)$\lambda_1\lambda_2<0$

(4.3.1) 两条相交直线：$x^2-ay^2=0,a>0$

(4.3.2) 双曲线：$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$。

解：（1）子集 $A\subset E^2$ 称为二次曲线，如果存在某一二元二次多项式 $F=$

$F(x,y)$，使得 $A=\{(x,y)\in E^2|F(x,y)=0\}$，即 A 是 F 的零点集。定义 E^2 中二次曲线的全体为集合 $S=\{A\in E^2|A\text{是二次曲线}\}$。

（2）设 $C=\begin{pmatrix}p & q\\ q & r\end{pmatrix}$，则 $\begin{pmatrix}x & y\end{pmatrix}C\begin{pmatrix}x\\ y\end{pmatrix}=px^2+2qxy+ry^2$，比较系数，得 $p=a,q=b/2,r=c$，故存在唯一的对称方阵 $C=\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}$ 满足条件。

（3）对称方阵 $C=\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}$，则

$$ACA^T=\begin{pmatrix}a\cos^2\theta+\frac{b}{2}\sin 2\theta+c\sin^2\theta & \frac{c-a}{2}\sin 2\theta+\frac{b}{2}\cos 2\theta\\ \frac{c-a}{2}\sin 2\theta+\frac{b}{2}\cos 2\theta & a\sin^2\theta-\frac{b}{2}\sin 2\theta+c\cos^2\theta\end{pmatrix}$$

故若 $a\neq c$，取 θ 使得 $\tan 2\theta=\dfrac{b}{a-c}$；若 $a=c$，取 $\theta=\dfrac{\pi}{4}$。

（4）(4.1) 若 $\lambda_1\lambda_2=0$，不妨设 $\lambda_2=0$。则：

(4.1.1) 如果 $a_2=0$ 且 $4\lambda_1 b-a_1^2>0$，则令平移 $x\mapsto x+\dfrac{a_1}{2\lambda_1}$，方程化为 $x^2+\dfrac{4\lambda_1 b-a_1^2}{4\lambda_1^2}=0$，它是空集，与 $x^2+1=0$ 等价。

(4.1.2) 如果 $a_2=0$ 且 $4\lambda_1 b-a_1^2\leq 0$，则令平移 $x\mapsto x+\dfrac{a_1}{2\lambda_1}$，方程化为 $x^2-\dfrac{a_1^2-4\lambda_1 b}{4\lambda_1^2}=0$，它是两条平行直线。

(4.1.3) 如果 $a_2\neq 0$，则令平移 $x\mapsto x+\dfrac{a_1}{2\lambda_1},y\mapsto y+\dfrac{4\lambda_1 b-a_1^2}{4\lambda_1 a_2}$，方程化为 $x^2+\dfrac{a_2}{\lambda_1}y=0$。再令旋转 $x\mapsto -\mathrm{sgn}(\dfrac{a_2}{\lambda_1})x,y\mapsto -\mathrm{sgn}(\dfrac{a_2}{\lambda_1})y$，则化为 $x^2-|\dfrac{a_2}{\lambda_1}|y=0$，它是抛物线。

(4.2) 若 $\lambda_1\lambda_2>0$，不妨设 $\lambda_1>0$，则：

(4.2.0) 如果 $b-\dfrac{a_1^2}{4\lambda_1^2}-\dfrac{a_2^2}{4\lambda_2^2}>0$，则令平移 $x\mapsto x+\dfrac{a_1}{2\lambda_1},y\mapsto y+\dfrac{a_2}{2\lambda_2}$，方程化为 $\lambda_1x^2+\lambda_2y^2+(b-\dfrac{a_1^2}{4\lambda_1^2}-\dfrac{a_2^2}{4\lambda_2^2})=0$，它是空集，与 (4.1.1) 等价。

(4.2.1) 如果 $b-\dfrac{a_1^2}{4\lambda_1^2}-\dfrac{a_2^2}{4\lambda_2^2}=0$，则令平移 $x\mapsto x+\dfrac{a_1}{2\lambda_1},y\mapsto y+\dfrac{a_2}{2\lambda_2}$，方程化为 $\lambda_1x^2+\lambda_2y^2=0$，它是点，与 $x^2+y^2=0$ 等价。

(4.2.2) 如果 $b-\dfrac{a_1^2}{4\lambda_1^2}-\dfrac{a_2^2}{4\lambda_2^2}<0$，则令平移 $x\mapsto x+\dfrac{a_1}{2\lambda_1},y\mapsto y+\dfrac{a_2}{2\lambda_2}$，方程化为 $\dfrac{x^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_1}+\dfrac{y^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_2}-1=0$，它是椭圆。

(4.3) 若 $\lambda_1\lambda_2<0$，不妨设 $\lambda_1>0$，则：
(4.3.1) 如果 $b-\frac{a_1^2}{4\lambda_1^2}-\frac{a_2^2}{4\lambda_2^2}=0$，则令平移 $x\mapsto x+\frac{a_1}{2\lambda_1},y\mapsto y+\frac{a_2}{2\lambda_2}$，方程化为 $x^2+\frac{\lambda_2}{\lambda_1}y^2=0$，它是两条相交直线。
(4.3.2.1) 如果 $b-\frac{a_1^2}{4\lambda_1^2}-\frac{a_2^2}{4\lambda_2^2}<0$，则令平移 $x\mapsto x+\frac{a_1}{2\lambda_1},y\mapsto y+\frac{a_2}{2\lambda_2}$，方程化为 $\frac{x^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_1}+\frac{y^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_2}-1=0$，它是双曲线。
(4.3.2.2) 如果 $b-\frac{a_1^2}{4\lambda_1^2}-\frac{a_2^2}{4\lambda_2^2}>0$，则令平移 $x\mapsto x+\frac{a_1}{2\lambda_1},y\mapsto y+\frac{a_2}{2\lambda_2}$，方程化为 $\frac{x^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_1}+\frac{y^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_2}-1=0$，再令 $x\mapsto y,y\mapsto -x$，方程化为 $\frac{x^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_2}+\frac{y^2}{(a_1^2\lambda_2^2+a_2^2\lambda_1^2-4b\lambda_1^2\lambda_2^2)/\lambda_1}$ $-1=0$，它也是双曲线。
综上，证明完成。

第十二次作业答案

1. 设 A 为 E^3 中的二次曲面，H 为 E^3 中一平面，令 $C=A\cap H$，证明：
(a) 若 A 为两相交平面，即标准型为 $[x^2-ay^2=0,a>0]$，则 C 有且仅有可能为一条直线，两条相交直线，两平行直线，一个平面；
(b) 若 A 为抛物柱面，即标准型为 $[x^2-ay=0,a>0]$，则 C 有且仅有可能为空集，一条直线，两条平行直线，抛物线；
(c) 若 A 为单叶双曲面，即标准型为 $[\frac{x^2}{a}+\frac{y^2}{b}-\frac{z^2}{c}-1=0,a\geq b>0,c>0]$，则 C 有且仅有可能为椭圆，双曲线，抛物线，两条相交直线，两条平行直线。
解：(a) 设两个平面为 α,β，其交线为 l。若 H 平行于 l，分三种情况：若 H 平行于 α 或 β，则 C 为一条直线；若 H 与 α 或 β 重合，则 C 为一个平面；否则，C 为两条平行直线。若 H 包含 l，则 C 为一条直线。若 H 不包含 l 也不平行于 l，则 C 为两条相交直线。综上，C 有且仅有可能为一条直线，两条相交直线，两平行直线，一个平面。
(b) 设平面面方程为 $Ax+By+Cz+D=0$。
若 $C=0$，则交线 $\begin{cases}x^2-ay=0\\ Ax+By+D=0\end{cases}$。设交线上的点满足 $x=at,y=at^2$，t

为参数，则 $Bat^2+Aat+D=0$。当 $B=0$ 时交线为 $\begin{cases}x=-D/A\\y=D^2/(A^2a)\\z=z\end{cases}$ 为一条直线；当 $B\neq 0$ 且 $(Aa)^2-4BDa<0$ 时为空集；当 $B\neq 0$ 且 $(Aa)^2-4BDa=0$ 时为一条直线；当 $(Aa)^2-4BDa>0$ 时为两条平行直线。

若 $C\neq 0$，则不妨 $C=1$。则交线 $\begin{cases}x^2-ay=0\\Ax+By+z+D=0\end{cases}$ 的参数方程为 $\begin{cases}x=at\\y=at^2\\z=D-aAt-aBt^2\end{cases}$。下面用刚体变换把平面变为平行于 xy 平面。令 $c_1=\sqrt{A^2+B^2}, c_2=\sqrt{A^2+B^2+1}$，则令 $T=\begin{pmatrix}-B/c_1 & A/(c_1c_2) & A/c_2\\A/c_1 & B/(c_1c_2) & B/c_2\\0 & -c_1/c_2 & 1/c_2\end{pmatrix}$，则刚体变换 $\Phi(x,y,z)=\begin{pmatrix}x & y & z\end{pmatrix}T$ 将平面 $Ax+By+z+D=0$ 变为 $z+D/c_2=0$。此时参数方程变为 $\begin{cases}x=\frac{aA}{c_1}t^2-\frac{aB}{c_1}t\\y=\frac{c_2aB}{c_1}t^2+\frac{c_2aA}{c_1}t-\frac{c_1D}{c_2}\\z=-\frac{D}{c_2}\end{cases}$，消去参数 t，得 $\dfrac{B^2}{ac_1^2}x^2+\dfrac{A^2}{c_1^2c_2^2a}y^2-\dfrac{2AB}{c_1^2c_2a}xy-(\dfrac{2ABD}{c_1c_2^2a}+\dfrac{A}{c_1})x-(\dfrac{2A^2D}{c_1c_2^3a}+\dfrac{B}{c_1c_2})y+(\dfrac{A^2D^2}{c_2^4a}-\dfrac{BD}{c_2^2})=0$。其二次项的矩阵 $I_2=\begin{pmatrix}B^2/(ac_1^2) & -AB/(c_1^2c_2a)\\-AB/(c_1^2c_2a) & A^2/(c_1^2c_2^2a)\end{pmatrix}$，$|\lambda I-I_2|=\lambda(\lambda-\dfrac{B^2+1}{a})$，得 $\lambda_1=0,\lambda_2=\dfrac{B^2+1}{a}$，$\lambda_1\lambda_2=0$，故它要么是空集，要么是两条平行直线，要么是抛物线。当 $A=B=0$ 时交线显然是抛物线，故抛物线是可能的。

综上，交线有且仅有可能是空集，一条直线，两条平行直线，抛物线。

(c) 设平面方程为 $Ax+By+Cz+D=0$。

若 $B=C=0$，则交线 $\begin{cases}\frac{x^2}{a}+\frac{y^2}{b}-\frac{z^2}{c}-1=0\\Ax+D=0\end{cases}$，即 $\begin{cases}\frac{y^2}{b}-\frac{z^2}{c}=1-\frac{D^2}{aA^2}\\x=-\frac{D}{A}\end{cases}$，它要么是一条双曲线，要么是两条相交直线。

若 $B\neq 0, C=0$，不妨设平面为 $Ax+y+D=0$。下面用刚体变换把平面变为

平行于 xz 平面。令 $c'=\sqrt{1+A^2}$，取 $T=\begin{pmatrix}1/c' & A/c' & 0\\ -A/c' & 1/c' & 0\\ 0 & 0 & 1\end{pmatrix}$，则刚体变换 $\Phi(x,y,z)=\begin{pmatrix}x & y & z\end{pmatrix}T$ 将平面变为 $y+D/c'=0$。此时二次曲面的二次项矩阵 $I_3=T\mathrm{diag}(\frac{1}{a},\frac{1}{b},-\frac{1}{c})T^T=\begin{pmatrix}\frac{1}{ac'^2}+\frac{A^2}{bc'^2} & -\frac{A}{ac'^2}+\frac{A}{bc'^2} & 0\\ -\frac{A}{ac'^2}+\frac{A}{bc'^2} & \frac{A^2}{ac'^2}+\frac{1}{bc'^2} & 0\\ 0 & 0 & -\frac{1}{c}\end{pmatrix}$，代入 $y=-D/c'$，得交线方程 $(\frac{1}{ac'^2}+\frac{A^2}{bc'^2})x^2-\frac{1}{c}z^2+(\frac{2AD}{ac'^3}-\frac{2AD}{bc'^3})x+(\frac{A^2D^2}{ac'^4}+\frac{D^2}{bc'^4})=0$。平移消去 x^1 项，知它是双曲线或两条相交直线。

若 $C\neq 0$，不妨设平面为 $Ax+By+z+D=0$。下面用刚体变换把平面变为平行于 xy 平面。结合 (b) 和 (c) 的方法即可。

综上，交线是椭圆，抛物线，双曲线，两条相交直线，两条平行直线。

2. 将二次曲面方程 F 记为 $F=F_2+F_1+F_0$，其中 F_i 为 i 次项部分，故存在刚体变换 Φ 使得 $F_2\circ\Phi=\lambda_1x^2+\lambda_2y^2+\lambda_3z^2$。在 $\lambda_1,\lambda_2,\lambda_3$ 中有且仅有一个为 0 时，给出 F 对应曲面的分类。

解： 见讲义。

3. 将下列二次曲面方程化为标准型，并判断是何种曲面：

(a)$2xy-2xz-2yz-1=0$；

(b)$x^2+5y^2+5z^2+2xy-4zx-1=0$。

解：(a)$C=\begin{pmatrix}0 & 1 & -1\\ 1 & 0 & -1\\ -1 & -1 & 0\end{pmatrix}$，$|\lambda I-C|=(\lambda-2)(\lambda+1)^2$，因此 $\lambda_1=2,\lambda_2=\lambda_3=-1$。分别求解 $(\lambda_iI-C)u=0$，得 $u_1=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}),u_2=(1,0,1),u_3=(0,1,1)$，将 u_2,u_3 标准正交化，得 $u_2=(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}),u_3=(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}})$。因此令 $A=\begin{pmatrix}u_2\\ u_1\\ u_3\end{pmatrix}=\begin{pmatrix}1/\sqrt{2} & 0 & 1\sqrt{2}\\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}\\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6}\end{pmatrix}$，则 $|A|=1$，刚体变换 $\Phi(x,y,z)=(x,y,z)A$ 将 C 对角化为 $ACA^T=\mathrm{diag}(-1,2,-1)$，再令 $\begin{cases}x\mapsto x\\ y\mapsto z\\ z\mapsto -y\end{cases}$，故标准型为 $x^2+y^2-2z^2+1=0$，它是双叶双曲面。

(b)$C=\begin{pmatrix}1&1&-2\\1&5&0\\-2&0&5\end{pmatrix}$,$|\lambda I-C|=\lambda(\lambda-5)(\lambda-6)$,因此 $\lambda_1=6,\lambda_2=5,\lambda_3=0$。分别求解 $(\lambda_i I-C)u=0$，得 $u_1=(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}),u_2=(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}),u_3=(\frac{5}{\sqrt{30}},-\frac{1}{\sqrt{30}},\frac{2}{\sqrt{30}})$。令 $A=\begin{pmatrix}u_2\\u_1\\u_3\end{pmatrix}=\begin{pmatrix}0&2/\sqrt{5}&1/\sqrt{5}\\1/\sqrt{6}&1/\sqrt{6}&-2/\sqrt{6}\\5/\sqrt{30}&-1/\sqrt{30}&2/\sqrt{30}\end{pmatrix}$，则 $|A|=1$，刚体变换 $\Phi(x,y,z)=(x,y,z)A$ 将 C 对角化为 $ACA^T=\mathrm{diag}(5,6,0)$，故标准型为 $5x^2+6y^2-1=0$，它是椭圆柱面。

4. 证明如下命题：

(a)E^2 上的正交变换 Φ 在标准直角坐标系下有如下表达：$\Phi(x,y)=\begin{pmatrix}x&y\end{pmatrix}\begin{pmatrix}\cos\theta&\pm\sin\theta\\-\sin\theta&\pm\cos\theta\end{pmatrix}+\begin{pmatrix}x_0&y_0\end{pmatrix},\theta\in\mathbb{R}$。

(b) 定义 $\triangle ABC$ 与 $\triangle A'B'C'$ 相似若存在两个角相等，证明：若 $\triangle ABC$ 与 $\triangle A'B'C'$ 相似，则存在相似变换 Φ 使得 $\Phi(\triangle ABC)=\triangle A'B'C'$，并由此证明对应边长比例为定值。

(c) 设 $\Phi:E^2\to E^2$ 为仿射变换,P,Q,R 为 E^3 中互异三点,令 $P'=\Phi(P),Q'=\Phi(Q),R'=\Phi(R)$，证明：

(i) 若 P,Q,R 共线，则 P',Q',R' 共线且有等式 $\frac{|PQ|}{|PR|}=\frac{|P'Q'|}{|P'R'|}$ 成立；

(ii) 若 P,Q,R 不共线，则 P',Q',R' 不共线，且比值 $\frac{S_{\triangle P'Q'R'}}{S_{\triangle PQR}}$ 为常数，即不依赖于 P,Q,R 的选取。

解:(a) 设 $\Phi(O)=O'$,令平移 $\tau=\tau_{-\overline{OO'}}$,则正交变换 $\tilde{\Phi}=\tau\circ\Phi$ 满足 $\tilde{\Phi}(O)=O$。与刚体变换的分析相同，可以证明 $\tilde{\Phi}$ 保距离保内积，进而 $\{O;\tilde{\Phi}(\mathbf{e}_1),\tilde{\Phi}(\mathbf{e}_2)\}$ 也是平面直角坐标系。如果 $\mathbf{e}_1,\mathbf{e}_2$ 的有向平行四边形面积 S 与 $\tilde{\Phi}(\mathbf{e}_1),\tilde{\Phi}(\mathbf{e}_2)$ 的有向平行四边形面积 $\tilde{S}$ 相等，则 $\tilde{\Phi}$ 保距保定向，知 $\tilde{\Phi}$，进而 Φ 是刚体变换，因此 $\Phi(x,y)=\begin{pmatrix}x&y\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}+\begin{pmatrix}x_0&y_0\end{pmatrix},\theta\in\mathbb{R}$；如果 S 与 $\tilde{S}$ 反号，则取反射 $\mathcal{R}(x,y)=\begin{pmatrix}x&y\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$，令 $\tilde{\Phi}'=\mathcal{R}\circ\tilde{\Phi}$，则 $\tilde{\Phi}'(\mathbf{e}_1),\tilde{\Phi}'(\mathbf{e}_2)$ 的有向平行四边形面积 $S'=S$，进而 $\tilde{\Phi}'$ 是（保持原点的）刚体变换，因此 $\Phi=\tau\circ\tilde{\Phi}=\tau\circ\mathcal{R}\circ\tilde{\Phi}=\begin{pmatrix}x&y\end{pmatrix}\begin{pmatrix}\cos\theta&-\sin\theta\\-\sin\theta&-\cos\theta\end{pmatrix}+\begin{pmatrix}x_0&y_0\end{pmatrix},\theta\in\mathbb{R}$。

(b) 在射线 $A'B'$ 上取点 B'' 使得 $|A'B''|=|AB|$，令 $\lambda=|A'B'|/|A''B''|$。则

存在正交变换 Φ' 使得 $\Phi'(A)=A',\Phi'(B)=B''$ 且 $\triangle\Phi(A)\Phi(B)\Phi(C)$ 的有向面积与 $\triangle A'B'C'$ 的有向面积同号。令相似变换 $\Phi=\Phi'\circ\tau_{\overrightarrow{OA}}\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\circ\tau_{-\overrightarrow{OA}}$，则 $\Phi(B)=B'$，又 Φ 保角，故 $\angle\Phi(B)=\angle B'$，于是 $\Phi(C)=C'$，故相似变换 $\Phi(\triangle ABC)=\triangle A'B'C'$。而由于 $\overrightarrow{\Phi(P)\Phi(Q)}=\lambda\overrightarrow{\Phi'(P)\Phi'(Q)}$ 且 $|\Phi'(P)\Phi'(Q)|=|PQ|$，知 $|\Phi'(P)\Phi'(Q)|=|\lambda|\cdot|PQ|,\forall P,Q\in E^2$。因此对应边长比例为定值。

(c) 由于平移不影响结论，不妨设仿射变换 $\Phi(x,y)=\begin{pmatrix}x & y\end{pmatrix}A,|A|\neq 0$。

(i) 由于 P,Q,R 共线，知 $\overrightarrow{PQ},\overrightarrow{PR}$ 共线。设 $\overrightarrow{PQ}=(x_1,y_1),\overrightarrow{PR}=(x_2,y_2)$，$(x_2,y_2)=\lambda(x_1,y_1)$，则 $\overrightarrow{P'R'}=\begin{pmatrix}x_2 & y_2\end{pmatrix}A=\lambda\begin{pmatrix}x_1 & y_1\end{pmatrix}A=\lambda\overrightarrow{P'Q'}$，因此 P',Q',R' 共线且 $\dfrac{|PQ|}{|PR|}=\dfrac{|P'Q'|}{|P'R'|}=\dfrac{1}{|\lambda|}$。

(ii) 设 $\overrightarrow{PQ}=(x_1,y_1),\overrightarrow{PR}=(x_2,y_2)$，则 $S_{\triangle PQR}=\dfrac{1}{2}|\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}|$，由于 P,Q,R 不共线，故 $S_{\triangle PQR}\neq 0$。而 $\overrightarrow{P'Q'}=\begin{pmatrix}x_1 & y_1\end{pmatrix}A,\overrightarrow{P'R'}=\begin{pmatrix}x_2 & y_2\end{pmatrix}A$，因此 $S_{\triangle P'Q'R'}=\dfrac{1}{2}|\det(\begin{pmatrix}x_1 & y_1\\ x_2 & y_2\end{pmatrix}A)|=\dfrac{1}{2}|\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}|A||=S_{\triangle PQR}|\det A|\neq 0$，因此 P',Q',R' 不共线，且 $\dfrac{S_{\triangle P'Q'R'}}{S_{\triangle PQR}}=|\det A|$ 是常数。

第十三次作业答案

令 $(X:Y:Z)$ 为 $\mathbb{RP}^2$ 的射影坐标环。回忆在课堂上我们有 $\mathbb{RP}^2=\mathbb{R}^2\sqcup l_\infty$。注意到其中 $\mathbb{R}^2=\{(X:Y:Z)\in\mathbb{RP}^2|Z\neq 0\},l_\infty=\{(X:Y:0)\in\mathbb{RP}^2\}$。对 $Z\neq 0$，我们有：$(X:Y:Z)=(\dfrac{X}{Z}:\dfrac{Y}{Z}:1)$，且对于 $Z,Z'\neq 0$，$(X:Y:Z)=(X':Y':Z')$ 当且仅当 $\dfrac{X}{Z}=\dfrac{X'}{Z'},\dfrac{Y}{Z}=\dfrac{Y'}{Z'}$。

1. 设 (x,y) 为 $\mathbb{R}^2$ 坐标，且直线 $l^0\subset\mathbb{R}^2$ 由如下方程定义：$ax+by+c=0,a^2+b^2\neq 0$。我们称由 $aX+bY+cZ=0$ 定义的射影直线 $l\subset\mathbb{RP}^2$ 为 l^0 的射影化。令 $l_i^0\subset\mathbb{R}^2$，$i=1,2$ 为 $\mathbb{R}^2$ 中两条不同的直线。

（1）若 l_1^0 与 l_2^0 不平行，则 $l_1\cap l_2=l_1^0\cap l_2^0\in\mathbb{R}^2$

（2）若 l_1^0 与 l_2^0 平行，则 $l_1\cap l_2=l_1\cap l_\infty=l_2\cap l_\infty\in l_\infty$

（3）求 $\mathbb{R}^2$ 直线 $5x+6y+7=0$ 的射影化交于无穷远直线 l_∞ 的射影坐标。

解： 设 $l_1^0:ax+by+c=0,l_2^0:a'x+b'y+c'=0$。

（1）由于 l_1^0, l_2^0 不平行，故 $ab'-ba'\neq 0$。求解 $\begin{cases} aX+bY+cZ=0 \\ a'X+b'Y+c'Z=0 \end{cases}$，得

$\begin{cases} X=(bc'-b'c)t \\ Y=(a'c-ac')t \\ Z=(ab'-a'b)t \end{cases}$ $,t\neq 0$。于是 $Z\neq 0$，故 $(X:Y:Z)=(\frac{X}{Z}:\frac{Y}{Z}:1)\in\mathbb{R}^2$。

验证知 $(\frac{X}{Z},\frac{Y}{Z})=(\frac{bc'-b'c}{ab'-a'b},\frac{a'c-ac'}{ab'-a'b})\in l_i^0, i=1,2$，于是 $l_1\cap l_2=l_1^0\cap l_2^0\in\mathbb{R}^2$。

（2）若 l_1^0, l_2^0 平行，不妨设 $a'=a, b'=b$，则 $c\neq c'$。求解 $\begin{cases} aX+bY+cZ=0 \\ aX+bY+c'Z=0 \end{cases}$，

得 $\begin{cases} X=bt \\ Y=-at \\ Z=0 \end{cases}$ $,t\neq 0$。由于 $Z=0$，知 $(X:Y:Z)=(b:-a:0)\in l_\infty$。因此 $l_1\cap l_2=l_1\cap l_\infty=l_2\cap l_\infty\in l_\infty$。

（3）求解 $\begin{cases} 5X+6Y+7Z=0 \\ Z=0 \end{cases}$，得交点的射影坐标为 $(6:-5:0)$。

2. 设 $F(X,Y,Z)=\sum\limits_{i+j+k=d} a_{ijk}X^iY^jZ^k$ 为次数为 d 的三元齐次多项式，试证：

（1）$F(\lambda X,\lambda Y,\lambda Z)=\lambda^d F(X,Y,Z), \lambda\in\mathbb{R}$

（2）$F(X,Y,Z)$ 的零点集在 $\mathbb{RP}^2$ 中是良好定义的，从而它定义了一条 d 次射影曲线。

解：（1）$F(\lambda X,\lambda Y,\lambda Z)=\sum\limits_{i+j+k=d} a_{ijk}(\lambda X)^i(\lambda Y)^j(\lambda Z)^k=\sum\limits_{i+j+k=d} a_{ijk}\lambda^{i+j+k}X^iY^jZ^k=\lambda^d\sum\limits_{i+j+k=d} a_{ijk}X^iY^jZ^j=\lambda^d F(X,Y,Z)$。

（2）任取 F 的零点 (X,Y,Z)，即 $F(X,Y,Z)=0$，则对任意的 $\lambda\neq 0$，$F(\lambda X,\lambda Y,\lambda Z)=\lambda^d F(X,Y,Z)=0$，因此 $(\lambda X,\lambda Y,\lambda Z)$ 也是 F 的零点。故 $(X:Y:Z)\in\mathbb{RP}^2$ 作为 F 的零点是良好定义的。

3. 类似于 $\mathbb{R}^2$ 中直线在 $\mathbb{RP}^2$ 中的射影化，我们对 $\mathbb{R}^2$ 中的 d 次多项式 $f(x,y)$ 零点集定义的 d 次曲线的射影化定义为 $\{(X:Y:Z)\in\mathbb{RP}^2|F(X,Y,Z)=Z^d f(\frac{X}{Z},\frac{Y}{Z})=0\}$

（1）求 $y^2=x^3+5x-3$ 在 $\mathbb{RP}^2$ 中的射影化。记为 E。

（2）计算 $E\cap l_\infty$

（3）试证：E 相交于 $\mathbb{RP}^2$ 中任意射影直线最多三个点。

解：（1）$f(x,y)=x^3-y^2+5x-3$ 是一个三次多项式，$F(X,Y,Z)=Z^3f(X/Z,$

$Y/Z) = X^3 - Y^2Z + 5XZ^2 - 3Z^3$，$E = \{(X:Y:Z) \in \mathbb{RP}^2 | X^3 - Y^2Z + 5XZ^2 - 3Z^3 = 0\}$。

（2）$E \cap l_\infty = \{(X:Y:Z) \in E | Z = 0\} = \{(0:1:0)\}$。

（3）对射影直线 l，$E \cap l$ 可能有两类点：$\mathbb{R}^2$ 上的点和 l_∞ 上的点。由于 $E \cap l_\infty = (0,1,0)$，因此分以下三种情况讨论：

若 $l \cap l_\infty \neq (0:1:0)$，则 E 与 l 的交点全在 $\mathbb{R}^2$ 上，此时可设 $l: ax+y+c=0$，故 E 与 l 的交点满足方程 $\begin{cases} y^2 = x^3 + 5x - 3 \\ ax + y + c = 0 \end{cases}$，利用第二个式子将 $y = -ax - c$ 代入第一个式子，得到一个关于 x 的三次方程，它至多有三个根，而 y 由 x 唯一确定，因此交点至多有 3 个。

若 $l \cap l_\infty = (0:1:0)$，则设 $l: x + c = 0$，此时 E 与 l 在 l_∞ 上有一个交点。下面考虑它们在 $\mathbb{R}^2$ 上的交点。这些交点满足方程 $\begin{cases} y^2 = x^3 + 5x - 3 \\ x + c = 0 \end{cases}$，利用第二个式子将 $x = -c$ 代入第一个式子，得到一个关于 y 的二次方程，它至多有两个根，而 x 已经确定，因此 $\mathbb{R}^2$ 上交点至多有 2 个。再加上 l_∞ 上的一个交点，E 与 l 的交点至多有 3 个。

若 $l = l_\infty$，则由（2），E 与 l 只有 1 个交点。

综上，E 相交于 $\mathbb{RP}^2$ 中任意射影直线最多三个点。

4.$\mathbb{RP}^2$ 中有三个自然的 $\mathbb{R}^2$，它们分别是：

$$\begin{cases} [X:Y:Z] = \{X:Y:Z | Z \neq 0\} = \mathbb{R}_1^2 \\ [X:Y:Z] = \{X:Y:Z | Y \neq 0\} = \mathbb{R}_2^2 \\ [X:Y:Z] = \{X:Y:Z | X \neq 0\} = \mathbb{R}_3^2 \end{cases}$$

试证：

（1）$\mathbb{RP}^2 = \mathbb{R}_1^2 \cup \mathbb{R}_2^2 \cup \mathbb{R}_3^2$，所以 $\mathbb{RP}^2$ 是由三片 $\mathbb{R}^2$ 粘合而成的，这三片 $\mathbb{R}^2$ 可以拼成 $\mathbb{RP}^2$ 的一个“地图册”。

（2）我们定义自然映射：

$$\begin{cases} \Phi_1 : \mathbb{R}^2 \to \mathbb{RP}^2, (x,y) \to (x:y:1) \\ \Phi_2 : \mathbb{R}^2 \to \mathbb{RP}^2, (u,v) \to (u:1:v) \\ \Phi_3 : \mathbb{R}^2 \to \mathbb{RP}^2, (\xi:\eta) \to (1:\xi:\eta) \end{cases}$$

验证 Φ_i 给出 $\mathbb{R}^2$ 到 $\mathbb{R}_i^2$ 的双射，并称之为 $\mathbb{R}_i^2$ 上的坐标映射。

（3）考察在 $\mathbb{R}^2$ 中坐标为 $(20,21)$ 的点。通过 Φ_1 我们得到射影坐标为 $(20:21:1)\in\mathbb{RP}^2$ 的点 P。注意到 $P\in\mathbb{R}^2_1\cap\mathbb{R}^2_2\cap\mathbb{R}^2_3$，问 P 在坐标映射 Φ_2,Φ_3 下原像的坐标分别是多少？

（4）你能给出一般的坐标变换的公式吗？

（5）写出 $\mathbb{R}_2$ 曲线 $f_1(x,y)=x^3+y^3-1=0$ 在 $\mathbb{RP}^2$ 中的射影化 E。试用坐标 (u,v) 写出平面曲线 $E\cap\mathbb{R}^2_2\subset\mathbb{R}^2_2$ 的方程 $f_2(u,v)=0$。试用坐标 (ξ,η) 写出平面曲线 $E\cap\mathbb{R}^2_3\subset\mathbb{R}^2_3$ 的方程 $f_3(\xi,\eta)=0$。

解:（1）显然。

（2）这里只验证 Φ_1 给出 $\mathbb{R}^2$ 到 $\mathbb{R}^2_1$ 的双射（其余的同理）。

先证明 Φ_1 是满射。对任意的 $(X:Y:Z)\in\mathbb{R}^2_1, Z\neq 0$，有 $\Phi_1(\dfrac{X}{Z},\dfrac{Y}{Z})=(\dfrac{X}{Z}:\dfrac{Y}{Z}:1)=(X:Y:Z)$，故 Φ_1 为满射。

再证明 Φ_1 是单射。若 $\Phi_1(x,y)=\Phi_1(x',y')$，则 $(x:y:1)=(x':y':1)$，因此 $\dfrac{x}{1}=\dfrac{x'}{1},\dfrac{y}{1}=\dfrac{y'}{1}$，故 $x=x',y=y'$，因此 Φ_1 是单射。

（3）$P=(20,21,1)=(\dfrac{20}{21},1,\dfrac{1}{21})$，故 P 在 Φ_2 下的原像坐标是 $(\dfrac{20}{21},\dfrac{1}{21})$。同理，$P$ 在 Φ_3 下的原像坐标是 $(\dfrac{21}{20},\dfrac{1}{20})$。

（4）点 $(x,y),xy\neq 0$ 在 Φ_2 下的原像是 $(\dfrac{x}{y},\dfrac{1}{y})$，在 Φ_3 下的原像是 $(\dfrac{y}{x},\dfrac{1}{x})$。

（5）$E:X^3+Y^3-Z^3=0$，$f_2(u,v)=u^3-v^3+1, f_3(\xi,\eta)=\xi^3-\eta^3+1$。

第十四次作业答案

1. 若 $\mathbb{RP}^2$ 上 $A(2:1:-1),B(1:-1:1),C(1:0:0),D(1:5:-5)$ 为共线四点，求 $R(A,B;C,D)$。

解: 通过 $\mathbb{R}^2$ 上 x 轴方向的无穷远点将它们投影到 y 轴上，得 $A'(2:-1),B'(1:1),C'(1:0),D'(1:-5)$，它们在直线上可以对应为实数 $-2,1,\infty,-1/5$。它们的交比为 $R(A,B;C,D)=R(A',B';C',D')=\dfrac{\infty-(-2)}{\infty-1}\dfrac{-1/5-1}{-1/5-(-2)}=-\dfrac{2}{3}$。

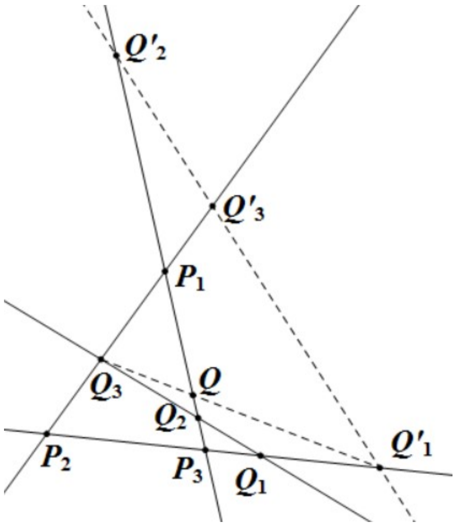
另一种方法: 观察可知这四个点在直线 $Y+Z=0$ 上。故 $\phi(u:v)=(u:v:-v)$ 为该直线的一个自然射影坐标，即 $A(2:1),B(1:-1),C(1:0),D(1:5)$，它们对应为实数 $2,-1,\infty,1/5$，它们的交比为 $R(A,B;C,D)=-2/3$。

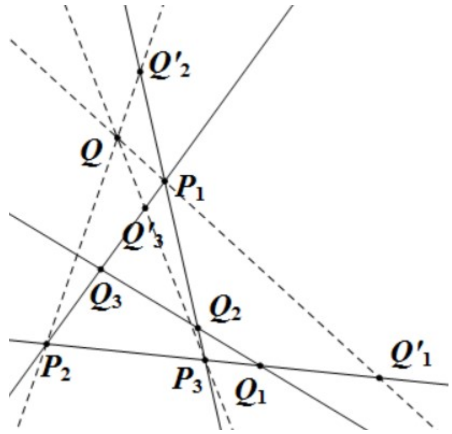
2. 在 $\mathbb{RP}^2$ 上三点 $A(1:4:1),B(0:1:1),C(2:3:-3)$，证明 A,B,C 三点共线，并且求此直线上的一点 D 使 $R(A,B;C,D)=-4$。

解： 由 A, B 所在的直线为 $\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} x - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} y + \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} z = 0$，即 $3x - y + z = 0$。代入 $C(2:3:-3)$ 确实满足方程，故 A, B, C 共线。该直线上有射影坐标 $\phi(u:v) = (u:v:v-3u)$，则有射影坐标 $A(1:4), B(0:1), C(2:3)$，它们对应的实数为 $1/4, 0, 2/3$。设 D 对应的实数为 x，则交比 $R(A,B;C,D) = \dfrac{2/3-1/4}{2/3-0}\dfrac{x-0}{x-1/4} = -4$，解得 $x = \dfrac{8}{37}$，于是直线上的射影坐标 $D(8:37)$，进而平面上的射影坐标 $D(8:37:13)$。

3. 设 $P_i, 1 \le i \le 6$ 是 $\mathbb{RP}^2$ 上六个不同的共线点，求证：
（1）$R(P_1,P_2;P_3,P_4)R(P_1,P_2;P_5,P_6) = R(P_1,P_2;P_3,P_6)R(P_1,P_2;P_5,P_4)$；
（2）如果 $R(P_1,P_2;P_3,P_4) = R(P_2,P_3;P_4,P_1)$，则 $R(P_1,P_3;P_2,P_4) = -1$。
解： 设这六个点对应的实数分别为 $p_i, 1 \le i \le 6$。
(1) 等式即 $\dfrac{p_3-p_1}{p_3-p_2}\dfrac{p_4-p_2}{p_4-p_1}\cdot\dfrac{p_5-p_1}{p_5-p_2}\dfrac{p_6-p_2}{p_6-p_1} = \dfrac{p_3-p_1}{p_3-p_2}\dfrac{p_6-p_2}{p_6-p_1}\cdot\dfrac{p_5-p_1}{p_5-p_2}\dfrac{p_4-p_2}{p_4-p_1}$，这显然成立。
(2) 若 $R(P_1,P_2;P_3,P_4) = R(P_2,P_3;P_4,P_1)$，也就是 $\dfrac{p_3-p_1}{p_3-p_2}\dfrac{p_4-p_2}{p_4-p_1} = \dfrac{p_4-p_2}{p_4-p_3}\dfrac{p_1-p_3}{p_1-p_2}$，故 $-\dfrac{p_2-p_1}{p_2-p_3}\dfrac{p_4-p_3}{p_4-p_1} = 1$，也即 $R(P_1,P_3;P_2,P_4) = -1$。

4. 一直线顺序交三角形 $P_1P_2P_3$ 之三边 P_2P_3, P_3P_1, P_1P_2 于点 Q_1, Q_2, Q_3，在此三边上顺序取 Q_1', Q_2', Q_3'，使 $R(P_2,P_3;Q_1',Q_1) = k_1, R(P_3,P_1;Q_2',Q_2) = k_2, R(P_1,P_2;Q_3',Q_3) = k_3$，求证：
（1）Q_1', Q_2', Q_3' 共线的充要条件是 $k_1k_2k_3 = 1$；
（2）$P_1Q_1', P_2Q_2', P_3Q_3'$ 共点的充要条件是 $k_1k_2k_3 = -1$。
解：（1）如下左图。设 $Q_1'Q_3$ 交 P_1P_3 于点 Q。则 Q_1', Q_2', Q_3' 共线当且仅当 $(P_1,P_3,Q_2',Q) = P_1P_2 \overset{Q_1'}{\wedge} P_1P_3(P_1,P_2,Q_3',Q_3)$，即 $R(P_1,P_2;Q_3',Q_3) = R(P_1,P_3;Q_2',Q)$。而通过 3（1），我们有 $R(P_3,P_1;Q_2',Q_2)R(P_3,P_1;Q,Q_2') = R(P_2,P_1;Q_2',Q_2')R(P_3,P_1,Q,Q_2) = R(P_3,P_1;Q,Q_2)$，故 $R(P_1,P_3;Q_2',Q) = R(P_3,P_1;Q,Q_2') = \dfrac{R(P_3,P_1;Q,Q_2)}{R(P_3,P_1;Q_2',Q_2)}$，因此 $R(P_1,P_2;Q_3',Q_3)R(P_3,P_1;Q_2',Q_2)$ $= R(P_3,P_1;Q,Q_2)$。又由于 $(P_3,P_1,Q,Q_2) = P_1P_3 \overset{Q_3}{\wedge} P_2P_3(P_3,P_2,Q_1',Q_1)$，故 $R(P_3,P_1;Q,Q_2) = R(P_3,P_2;Q_1',Q_1) = \dfrac{1}{R(P_2,P_3;Q_1',Q_1)}$，因此 $R(P_1,P_2;Q_3',Q_3)R(P_3,P_1;Q_2',Q_2) = \dfrac{1}{R(P_2,P_3;Q_1',Q_1)}$，即 $k_1k_2k_3 = 1$。





（2）如上右图。由习题四 1，知 $P_1Q_1', P_2Q_2', P_3Q_3'$ 共点，当且仅当 $\frac{P_1Q_2'}{Q_2'P_3}\frac{P_3Q_1'}{Q_1'P_2}\frac{P_2Q_3'}{Q_3'P_1}=1$。而由习题三 9，因为 Q_1, Q_2, Q_3 共线，有 $\frac{P_1Q_2}{Q_2P_3}\frac{P_3Q_1}{Q_1P_2}\frac{P_2Q_3}{Q_3P_1}=-1$。两式相除，得 $\frac{Q_1'P_2}{Q_1'P_3}\frac{Q_1P_3}{Q_1P_2}\cdot\frac{Q_2'P_3}{Q_2'P_1}\frac{Q_2P_1}{Q_2P_3}\cdot\frac{Q_3'P_1}{Q_3'P_2}\frac{Q_3P_2}{Q_3P_1}=-1$，即 $k_1k_2k_3=-1$。

注：（1），（2）各自的方法也可以证明另一问。

5. 在 $\mathbb{RP}^2$ 上给了 5 个点 $A(1:-1:0), B(2:0:-1), C(0:2:-1), D(1:4:-2), E(2:3:-2)$，求由它们所确定的二次曲线。它是什么类型的？

解：由题，过 AB 的直线方程为 $x+y+2z=0$，过 CD 的直线方程为 $y+2z=0$，故二次方程 $(x+y+2z)(y+2z)=0$ 确定的二次曲线通过点 A, B, C, D。另一方面，过 AC 的直线方程为 $x+y+2z=0$，过 BD 的直线方程为 $4x+3y+8z=0$，故二次方程 $(x+y+2z)(4x+3y+8z)=0$ 确定的二次曲线也通过点 A, B, C, D。故对任意实数 λ，$(x+y+2z)(y+2z)+\lambda(x+y+2z)(4x+3y+8z)=0$ 均为通过 A, B, C, D 的二次曲线。将点 $E(2:3:-2)$ 代入方程，得 $\lambda=1$，因此这五个点确定的二次曲线方程为 $(x+y+2z)(y+2z)+(x+y+2z)(4x+3y+8z)=0$，即 $2x^2+2y^2+10z^2+4xy+9yz+9xz=0$。

这个二次曲线的二次项矩阵为 $C=\begin{pmatrix} 2 & 2 & 9/2 \\ 2 & 2 & 9/2 \\ 9/2 & 9/2 & 10 \end{pmatrix}$，求解 $|\lambda I-C|=0$，得 $\lambda=0, \frac{14\pm 3\sqrt{22}}{2}$，标准方程 $x^2-y^2=0$，故是两条相交直线。

6. 试证明：$\mathbb{RP}^2$ 上任给一般位置的五个点（即其中任意三点不共线），有且只有一条二次曲线通过它们。

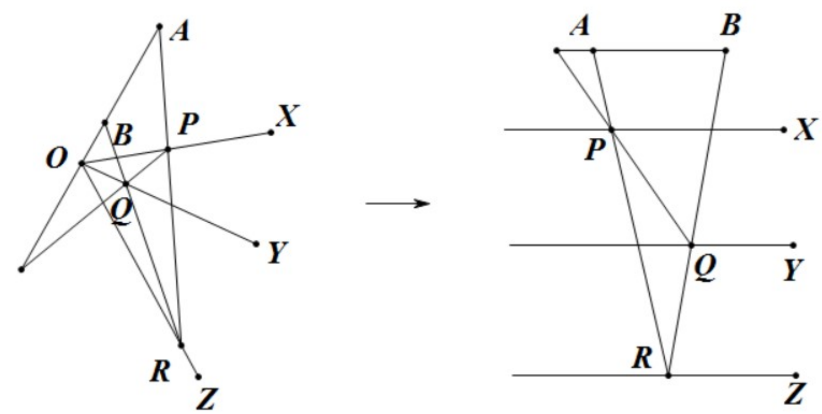
解：设通过 AB 的直线方程为 $a_1x+b_1y+c_1z=0$，通过 CD 的直线方程为

$a_2x+b_2y+c_2z=0$，则二次曲线 $(a_1x+b_1y+c_1z)(a_2x+b_2y+c_2z)=0$ 通过 $ABCD$ 四点。设通过 AC 的直线方程为 $a_2x+b_3y+c_3z=0$，通过 BD 的直线方程为 $a_4x+b_4y+c_4z=0$，则二次曲线 $(a_3x+b_3y+c_3z)(a_4x+b_4y+c_4z)=0$ 也通过 $ABCD$ 四点。由于三三不共线，因此两条曲线不相同，且 E 不在这两条曲线上。故 $(a_1x+b_1y+c_1z)(a_2x+b_2y+c_2z)+\lambda(a_3x+b_3y+c_3z)(a_4x+b_4y+c_4z)=0$ 给出了过 $ABCD$ 的一族二次曲线。将 E 的坐标代入，得到关于 λ 的一次方程，解出 λ，即得到过点 $ABCDE$ 的二次曲线。故过一般位置的五个点必有一条二次曲线通过它们。

下证明唯一性。由于唯一性成立与否在射影变换下不变，因此不妨通过射影变换将点 A,B,C,D 映到点 $(1:0:0),(0:1:0),(0:0:1),(1:1:1)$，并设此时 $E=(x_0:y_0:z_0)$。设有两条二次曲线 $a_ix^2+b_iy^2+c_iz^2+d_ixy+e_ixz+f_iyz=0, i=1,2$ 均通过这五个点。则将 A,B,C 三点代入，得 $a_i=b_i=c_i=0, i=1,2$。将点 D,E 代入，得 $\begin{cases} d_i+e_i+f_i=0 \\ d_ix_0y_0+e_ix_0z_0+f_iy_0z_0=0 \end{cases}$。因此 $(d_i,e_i,f_i), i=1,2$ 在 $\mathbb{R}^3$ 的两个平面 $\begin{cases} x+y+z=0 \\ x_0y_0x+x_0z_0y+y_0z_0z=0 \end{cases}$ 的交集上。由五点三三不共线，知这两个平面不重合，因此它们交于一条过原点的直线，因此这两个向量平行，故这两条二次曲线相同，唯一性得证。

注：事实上，唯一性的证明过程中也顺带又证明了一次存在性。

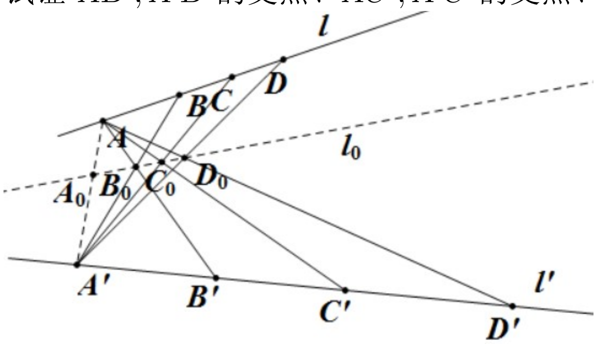
7. 利用射影变换试证：OX,OY,OZ 为三条直线，A,B 为两定点，其连线过 O。设 R 为 OZ 上的动点，且 RA,RB 分别交 OX,OY 于点 P,Q，则 PQ 必经过 AB 上一定点。



解：如图。由于这是一个射影命题，在射影变换下保持命题的真假性不变，故可通过一个射影变换将点 O 变为无穷远点。则此时 OX,OY,OZ,AB 相互平

行。不妨设 AB, PQ 的交点为 T，OZ, PQ 的交点为 S，则 $\dfrac{TA}{TB} = \dfrac{TA}{RS}\dfrac{RS}{TB} = \dfrac{d_{AB,OX}}{d_{OX,OZ}}\dfrac{d_{OY,OZ}}{d_{AB,OY}}$ 为定值，因此 T 为定点。

8. 设共面两直线 l, l' 上的点 $A, B, C, D, \cdots$ 与 $A', B', C', D', \cdots$ 成射影对应，试证 $AB', A'B$ 的交点、$AC', A'C$ 的交点、$AD', A'D$ 的交点……共线。

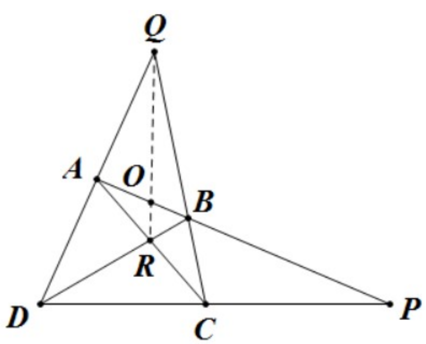


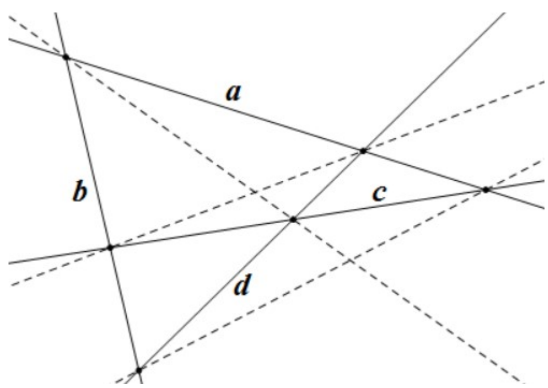
解： 只需证明 $AB', A'B$ 的交点（设为 B_0），$AC', A'C$ 的交点（设为 C_0），$AD', A'D$ 的交点（设为 D_0）共线。如图，设 A_0, B_0 所在的直线为 l_0，设 l_0 交 AA' 于点 A_0，交 AD' 于点 D_1，交 $A'D$ 于点 D_2。则由于 $(A_0, B_0, C_0, D_1) = l' \overset{A}{\wedge} l_0(A', B', C', D')$，故 $R(A_0, B_0; C_0, D_1) = R(A', B'; C', D')$；由于 $(A_0, B_0, C_0, D_2) = l \overset{A'}{\wedge} l_0(A, B, C, D)$，故 $R(A_0, B_0; C_0, D_2) = R(A, B; C, D)$。又由于 A, B, C, D 与 A', B', C', D' 构成射影对应，故 $R(A, B; C, D) = R(A', B'; C', D')$，因此 $R(A_0, B_0; C_0, D_1) = R(A_0, B_0; C_0, D_2)$，于是 $D_1 = D_2 = D_0$。

9. 在 $\mathbb{RP}^2$ 上一般位置的四个点 A, B, C, D（称它们为顶点）和由它们两两相连的六条直线（称它们为边）构成的图形称为完全四边形。不经过同一顶点的两条边称为对边，对边的交点称为对角点。

（1）证明：完全四边形的三个对角点不共线。

（2）试叙述（1）的对偶命题。





解：（1）如左图，完全四边形 $ABCD$，P, Q, R 为三个对角点。由于 AQ, BR, CP

交于一点 D，因此根据 4（2），$R(A,B;P,O)R(B,C;Q,Q)R(C,A;R,R) = R(A,B;P,O) = -1$。假设 P,Q,R 三点共线，则由 4（1），$R(A,B;P,O)R(B,C;Q,Q)R(C,A;R,R) = R(A,B;P,O) = 1$，矛盾。因此 P,Q,R 三点不共线。
（2）如右图。$\mathbb{RP}^2$ 上一般位置的四条直线 a,b,c,d（称为边）和由它们两两相交的六个交点（称为顶点）构成一个图。不在同一条边上的两个顶点称为对顶点，则三组对顶点的连线不共点。

10. 设一射影直线 $\mathbb{RP}^1$ 上的点 $0,1,2$ 经过射影变换变为 $-1,0,-2$，求该射影变换的分式线性函数，化为射影齐次坐标形式，并求出 $\mathbb{RP}^1$ 上 ∞ 的像点和原像点。

解： 设分式线性函数为 $f(x) = \dfrac{ax+b}{cx+d}$，则 $\begin{cases} \frac{b}{d} = -1 \\ \frac{a+b}{c+d} = 0 \\ \frac{2a+b}{2c+d} = -2 \end{cases}$，解得 $\begin{cases} a = 4t \\ b = -4t \\ c = -3t \\ d = 4t \end{cases}$，

故分式线性函数为 $f(x) = \dfrac{4x-4}{-3x+4}$。齐次坐标形式 $f(u:v) = \dfrac{4u-4v}{-3u+4v}$。$\infty$ 的像点与原像点分别为 $f(\infty) = -4/3, f^{-1}(\infty) = 4/3$。

11. 求 $\mathbb{RP}^2$ 上的一射影变换，使点 $(1:0:1),(0:1:1),(1:1:1),(0:0:1)$ 顺次映到点 $(1:0:0),(0:1:0),(0:0:1),(1:1:1)$。
解： 考虑其逆变换，即将点 $(1:0:0),(0:1:0),(0:0:1),(1:1:1)$ 顺次映到点 $(1:0:1),(0:1:1),(1:1:1),(0:0:1)$ 的射影变换。设 $\Phi(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} A$，则由前三组点知 $A = \begin{pmatrix} a & 0 & a \\ 0 & b & b \\ c & c & c \end{pmatrix}, a,b,c \in \mathbb{R}$，最后由第四组点解 a,b,c 得 $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$。因此所求的射影变换为该变换的逆，即 $\Psi(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} A^{-1} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$。

注： 求三阶矩阵的逆有如下方法：设 $A = \begin{pmatrix} \alpha^T \\ \beta^T \\ \gamma^T \end{pmatrix}$，其中 α,β,γ 为列向量，则 $A^* = \begin{pmatrix} \beta\times\gamma & \gamma\times\alpha & \alpha\times\beta \end{pmatrix}$，$A^{-1} = \dfrac{A^*}{|A|} = \dfrac{A^*}{\alpha\cdot\beta\times\gamma}$。

12. 求射影变换 $\rho:(x:y:z)\mapsto(4x-y:6x-3y:x-y-z)$ 的不动点。

解： 设不动点为 $(x_0:y_0:z_0)$，则有 $\begin{cases}ax_0=4x_0-y_0\\ ay_0=6x_0-3y_0\\ az_0=x_0-y_0-z_0\end{cases}$，$a\in\mathbb{R}/\{0\}$，解得 $a=-2,3$。$a=3$ 时，$\begin{cases}x_0=t\\ y_0=t\\ z_0=0\end{cases}$；$a=-2$ 时，$\begin{cases}x_0=t\\ y_0=6t\\ z_0=5t\end{cases}$。故不动点为 $(1:1:0)$ 和 $(1:6:5)$。

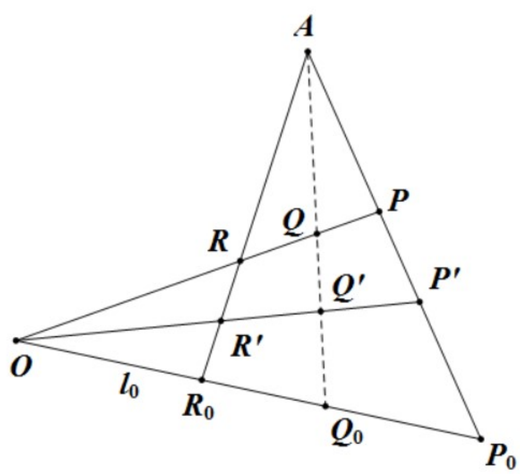
注： 事实上，该映射有不动直线 $X-Y+Z=0$。

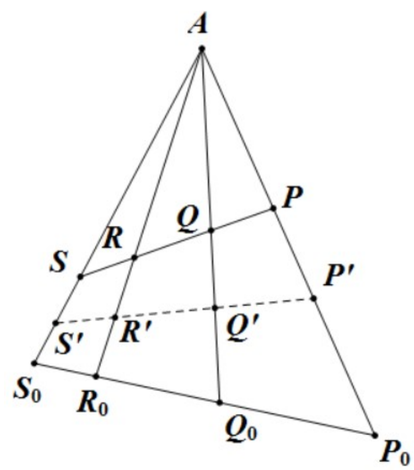
13. 设射影平面上的一个变换 σ 有不动点 A 和不动直线 l_0（l_0 上的点均不动），A 不在 l_0 上。并且对于每个其它点 P，σ 把 P 变到 AP 上一点 P'，使得 $R(A,P_0;P,P')=e$，其中 P_0 是 AP 和 l_0 的交点，$e\neq 0,1$ 是常数。证明：σ 是射影变换。

解： 分两步，先证明它保直线，再证明它保交比。

如下左图，设 $\sigma(P)=P',\sigma(R)=R'$，取直线 PR 上一点 Q，$\sigma(Q)=Q'$，设 $PR\cap l_0=O$，要证明 P',Q',O 共线（则同理有 P',R',O 共线）。它当且仅当 $Q'=AP\overset{O}{\wedge}AQ(P')$，当且仅当 $R(A,Q_0;Q,Q')=R(A,P_0;P,P')$（这是因为 $(A,Q_0,Q)=AP\overset{O}{\wedge}AQ(A,P_0,P)$），而这就是题目条件。

如下右图，即证 $R(P',Q';R',S')=R(P,Q;R,S)$。由于 $(P',Q',R',S')=PS\overset{A}{\wedge}P'S'(P,Q,R,S)$，故这是自然成立的。





注： 实际上根据 17 题，保交比的证明是不需要的。

14. 试证平面上所有形如 $\rho_\lambda:(x,y)\mapsto(\lambda x+a,\lambda y+b),\lambda>0$ 的变换的集合构

成群。

解：恒等映射 $(x,y)\mapsto(x,y)$ 属于该集，因为此时 $\lambda=1>0$；两该集中的映射 $(x,y)\mapsto(\lambda x+a_1,\lambda y+b_1),(x,y)\mapsto(\mu x+a_2,\mu y+b_2)$（$\lambda,\mu>0$）的复合 $(x,y)\mapsto(\lambda\mu x+(\mu a_1+a_2),\lambda\mu y+(\mu b_1+b_2))$ 也属于该集，这是因为 $\lambda\mu>0$；该集的映射 $(x,y)\mapsto(\lambda x+a,\lambda y+b)$（$\lambda>0$）有逆映射 $(x,y)\mapsto(\frac{1}{\lambda}x-\frac{a}{\lambda},\frac{1}{\lambda}y-\frac{b}{\lambda})$，它也属于该集，因为 $1/\lambda>0$。综上，这个集合在映射的复合下构成一个群。

15. 设 X 为一集合，我们令 $\mathrm{Aut}(X)=\{f:X\mapsto X,f\text{为双射}\}$。

（1）证明：$\mathrm{Aut}(X)$ 关于映射的复合构成一个群。

（2）给定一个子集族 $\mathcal{Y}=\{Y_\alpha|Y_\alpha\subset X\}_{\alpha\in\Lambda}$，证明：$\mathrm{Aut}_{\mathcal{Y}}(X)=\{f\in\mathrm{Aut}(X)|f(Y_\alpha)=Y_\alpha,\forall\alpha\in\Lambda\}$ 构成 $\mathrm{Aut}(X)$ 的一个子群。

解：（1）由恒等映射是双射，双射的逆和复合还是双射，它当然构成群。

（2）恒等映射当然属于 $\mathrm{Aut}_{\mathcal{Y}}(X)$，而对任意的 $f,g\in\mathrm{Aut}_{\mathcal{Y}}(X)$：$f(Y_\alpha)=Y_\alpha$，两边取 f^{-1}，得 $f^{-1}(Y_\alpha)=Y_\alpha$，因此 $f^{-1}\in\mathrm{Aut}_{\mathcal{Y}}(X)$；$g(Y_\alpha)=Y_\alpha$，两边取 f，得 $f\circ g(Y_\alpha)=f(Y_\alpha)=Y_\alpha$，故 $f\circ g\in\mathrm{Aut}_{\mathcal{Y}}(X)$。综上，$\mathrm{Aut}_{\mathcal{Y}}(X)$ 构成 $\mathrm{Aut}(X)$ 的一个子群。

注：在 14 题和 15 题（1）中，结合律都是显然的，因此略过了。

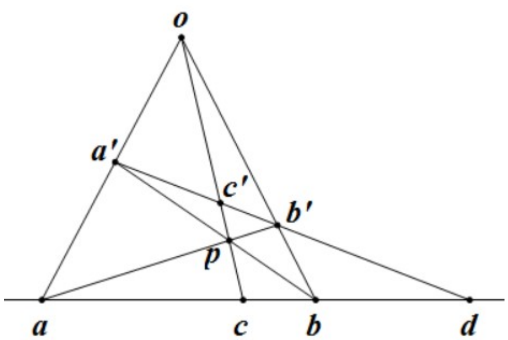
16. 在讲义 175-180 页，我们证明了 $\mathbb{RP}^2$ 上保无穷远直线 l_∞ 的射影变换的全体构成了射影变换群的子群，且它们自然同构于 $\mathbb{R}^2$ 上的仿射变换群。试证明 $\mathbb{RP}^2$ 上保无穷远直线点点不动的射影变换全体也构成了群，并从几何上理解这个群。

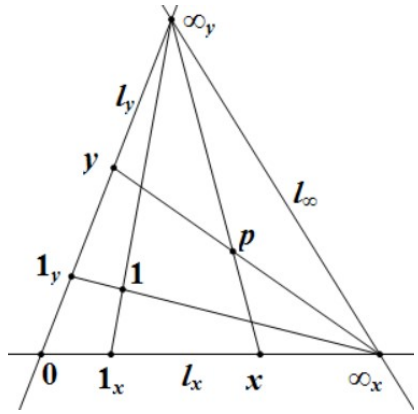
解：由 15 题（2），取 $X=\mathbb{RP}^2,\mathcal{Y}=\{\{t\}|t\in l_\infty\}$，知保无穷远直线点点不动的射影变换构成了群。设一个保无穷远直线点点不动的变换为 $\phi(x,y,z)=\begin{pmatrix}x & y & z\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$。则 $\phi(1:0:0)=(a_{11}:a_{12}:a_{13})=(1:0:0)$，故 $a_{12}=a_{13}=0,a_{11}\neq0$；$\phi(0:1:0)=(a_{21}:a_{22}:a_{23})=(0:1:0)$，故 $a_{21}=a_{23}=0,a_{22}\neq0$；$\phi(1:1:0)=(a_{11}:a_{22}:0)=(1:1:0)$，故 $a_{11}=a_{22}$。进而 $a_{33}\neq0$，因此 $A=\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ a & b & 1\end{pmatrix}=\begin{pmatrix}\lambda I_2 & \mathbf{0}\\ \mathbf{u}^T & 1\end{pmatrix},\lambda\neq0$，此时 ϕ 是平移与位似的复合。这就是这类变换的几何表示。

17. 回忆在课堂上刻画了 $\mathbb{RP}^2$ 上的射影变换：它们是保直线和保交比的双射。试证：若 $\mathbb{RP}^2$ 上的一双射保直线，则它必为射影变换。

解：我们分如下几步来证明这个定理。

第一步：我们称 $\mathbb{RP}^1$ 上四个点 $(a,b;c,d)$ 是调和点列，如果 $R(a,b;c,d)=-1$；我们称双射 $\tau:\mathbb{RP}^1\to\mathbb{RP}^1$ 是调和映射如果它把调和点列映成调和点列。我们把 $\mathbb{RP}^1$ 上的点 x 等同于 $R(0,\infty;x,1)$，则满足 $\tau(\infty)=\infty$ 的调和映射 τ 诱导了一个 $\mathbb{R}$ 上的双射 f_τ，它满足：$\tau(p)=q$ 当且仅当 $f_\tau(R(0,\infty;p,1))=R(0,\infty;q,1)$。则容易验证 $f_\tau(x+y)=f_\tau(x)+f_\tau(y), f_\tau(xy)=f_\tau(x)f_\tau(y)$，因此（由于 $\mathbb{R}$ 上的自同构只有恒等映射本身）有 $f_\tau(x)=x$。因此 τ 是恒等映射。





第二步：如左图，直线上三点 a,b,c，在直线外任取一点 o，在 oc 上任取一点 p，设 $bp\cap oa=a', ap\cap ob=b', a'b'\cap ab=d, a'b'\cap oc=c'$，则我们要证：$R(a,b;c,d)=-1$。由于 $(a',b',c',d)=ab\overset{o}{\wedge}a'b'(a,b,c,d)$，因此 $R(a,b;c,d)=R(a',b';c',d)$；又由于 $(a',b',c',d)=ab\overset{p}{\wedge}a'b'(b,a,c,d)$，因此 $R(b,a;c,d)=R(a',b';c',d)$。故 $R(a,b;c,d)=R(b,a;c,d)$，因此 $R(a,b;c,d)=-1$。

第三步：我们称 $\mathbb{RP}^2$ 上保直线的双射是调和映射，如果它把调和点列映成调和点列。这一步里我们要证明：$\mathbb{RP}^2$ 上保直线的双射是调和映射。根据第二步，调和点列中第四个点可以仅通过关联关系由其它三个点得到，而保直线的双射也保持关联关系，所以它自然保持调和点列，因此它是调和映射。

第四步：如右图。给定 $\mathbb{RP}^2$ 上任意三三不共线的四个点 $0,1,\infty_x,\infty_y$，令 $1\infty_y\cap 0\infty_x=1_x, 1\infty_x\cap 0\infty_y=1_y$。则任意不在 $l_\infty=\infty_x\vee\infty_y$ 上的点 p 都唯一对应着 $l_x=0\vee\infty_x, l_y=0\vee\infty_y$ 上的一对点 (x,y)，而它们又唯一地对应着一对数 $(R(0,\infty_x;x,1_x),R(0,\infty_y;y,1_y))$。这样的对应仅通过关联关系即可保证。

第五步：在 $\mathbb{RP}^2$ 上任取不共线的四个点，把它们记为 $0,1,\infty_x,\infty_y$，设保直线的双射 ψ 把这四个点映到 $0',1',\infty_x',\infty_y'$。我们知道，存在射影变换 ϕ 使得 $\phi(0')=0,\phi(1')=1,\phi(\infty_x')=\infty_x,\phi(\infty_y')=\infty_y$，那么 $\tau=\phi\circ\psi$ 是一个保直线的，以 $0,1,\infty_x,\infty_y$ 为不动点的双射。因此，根据第一步，在直线 $l_x=0\vee\infty_x, l_y=0\vee\infty_y, l_\infty=\infty_x\vee\infty_y$ 上，τ 是恒等映射。再根据第四步，知 τ 在 $\mathbb{RP}^2$ 上是恒等映射，因此 $\psi=\phi^{-1}$ 为射影变换。

附：交比的一些基本结论

1.A,B,C,D 是四个不同点，则 $R(A,B;A,D)=R(A,B;C,B)=0, R(A,B;B,D)=R(A,B;C,A)=\infty, R(A,B;C,C)=R(A,A;C,D)=1$。

2. 设 $R(a,b;c,d)=x$，则：

（1）$R(a,b;c,d)=R(b,a;d,c)=R(d,c;b,a)=R(c,d;a,b)=x$；

（2）$R(a,b;d,c)=R(b,a;c,d)=R(c,d;b,a)=R(d,c;a,b)=\dfrac{1}{x}$；

（3）$R(a,c;b,d)=R(c,a;d,b)=R(d,b;c,a)=R(b,d;a,c)=1-x$；

（4）$R(a,c;d,b)=R(c,a;b,d)=R(b,d;c,a)=R(d,b;a,c)=\dfrac{1}{1-x}$；

（5）$R(a,d;c,b)=R(d,a;b,c)=R(b,c;d,a)=R(c,b;a,d)=\dfrac{x}{x-1}$；

（6）$R(a,d;b,c)=R(d,a;c,b)=R(c,b;d,a)=R(b,c;a,d)=\dfrac{x-1}{x}$。

3.$R(0,\infty;x,1)=x$。

第十五次作业答案

1. 求证：一个连通图可以一笔画，当且仅当具有奇数条边经过的点的个数为 0 或 2。

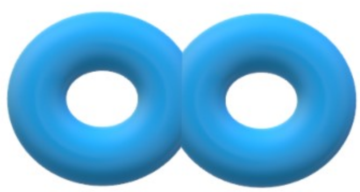
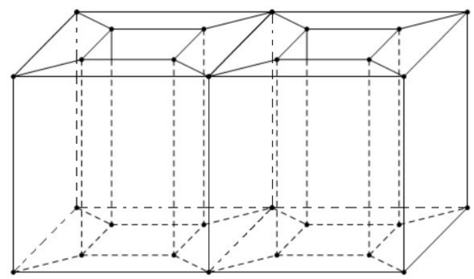
解： 一方面，设一个连通图可以一笔画。因为一笔画过程中每个并非起点和终点的点都恰好被经过一次，所以这些点都具有偶数条边经过。如果一笔画的起点和终点是同一个点，那么这个点也显然被偶数条边经过，此时具有奇数条边经过的点的个数为 0；如果一笔画的起点和和终点不是同一个点，则这两个点均经过奇数条边，此时具有奇数条边经过的点的个数为 2。综上，如果一个连通图可以一笔画，则具有奇数条边经过的点的个数为 0 或 2。

另一方面，设一个连通图具有奇数条边经过的点的个数为 0 或 2。我们称图中每个点经过的边的个数为这个点的度，从一个点沿边到达另一个点叫做一条路径，一个图去掉有限条不共边的路径（指去掉对应的边，如果去掉这些边后某个点的度为 0，那么也去掉这个点）后剩下的部分称为该连通图的子图。记一条路径为 $A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} A_n$，其中 a_i 为边，A_i 为点。若有另一条路径 $A_i \xrightarrow{b_1} B_1 \xrightarrow{b_2} B_2 \xrightarrow{b_3} \cdots \xrightarrow{b_{m-1}} B_{m-1} \xrightarrow{b_m} A_i$，则称路径 $A_0 \xrightarrow{a_1} \cdots \xrightarrow{a_i} A_i \xrightarrow{b_1} B_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} A_i \xrightarrow{a_{i+1}} \cdots \xrightarrow{a_n} A_n$ 为以上两条路径的和。

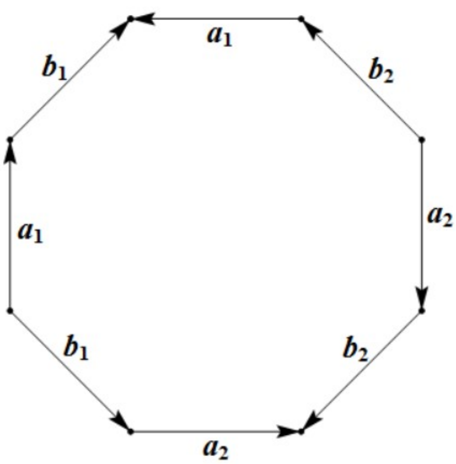
若该连通图度为奇数的点的个数为 2，记这两个点为 A,B，则从 A 点出发必存在一条路径到达点 B（这是因为图是连通的），记这条路径为 t_0。如果 t_0 不是该连通图的一笔画，则在去掉 t_0 后的子图中必存在 t_0 中的一个点（记为

T）和一条通过 T 的闭合路径，记这两条路径的和为 t_1。类比以上步骤，以此类推，可以每次将路径 t_{i-1} 与一条经过 t_{i-1} 中的某个点的闭合路径求和得到路径 t_i。由于该连通图的点的个数和每个点的度都是有限的，因此该过程总能在有限步结束，得到路径 t_n。又因为该图是连通图，故 t_n 就是所求的一笔画。对于连通图度为奇数的点的个数为 0 的情形，与上面的情形类似，只是 t_0 取为从图中任意一点 A 出发的一条闭合路径。

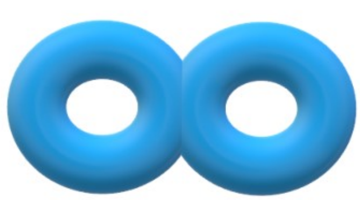
2. 将双“洞”球面，拓扑变换为多面体，画出该图形并计算该多面体的欧拉数。
解：如图。欧拉数为 -2。

3. 考虑将如下图形黏合后得到的曲面，并计算欧拉数。



解：黏出来是双重环面，如下图，欧拉数由第 2 题知是 -2。



A COMPARATIVE REVIEW OF RECENT RESEARCHES IN GEOMETRY.[1]

(*PROGRAMME ON ENTERING THE PHILOSOPHICAL FACULTY AND THE SENATE OF THE UNIVERSITY OF ERLANGEN IN 1872.)*

BY PROF. FELIX KLEIN.

Prefatory Note by the Author. - My 1872 Programme, appearing as a separate publication (Erlangen, A. Deichert), had but a limited circulation at first. With this I could be satisfied more easily, as the views developed in the Programme could not be expected at first to receive much attention. But now that the general development of mathematics has taken, in the meanwhile, the direction corresponding precisely to these views, and particularly since *Lie* has begun the publication in extended form of his *Theorie der Transformationsgruppen* (Liepzig, Teubner, vol. I. 1888, vol. II. 1890), it seems proper to give a wider circulation to the expositions in my Programme. An Italian translation by M. Gina Fano was recently published in the *Annali di Matematica*, ser. 2, vol. 17. A kind reception for the English translation, for which I am much indebted to Mr. Haskell, is likewise desired.

The translation is an absolutely literal one; in the two or three places where a few words are changed, the new phrases are enclosed in square brackets [ ]. In the same way are indicated a number of additional footnotes which it seemed desirable to append, most of them having already appeared in the Italian translation. - F. KLEIN.

1. Translated by Dr. M. W. HASKELL, Assistant Professor of Mathematics in the University of California. Published in Bull. New York Math. Soc. 2, (1892-1893), 215-249.

Among the advances of the last fifty years in the field of geometry, the development of *projective geometry*[2] occupies the first place. Although it seemed at first as if the so-called metrical relations were not accessible to this treatment, as they do not remain unchanged by projection, we have nevertheless learned recently to regard them also from the projective point of view, so that the projective method now embraces the whole of geometry. But metrical properties are then to be regarded no longer as characteristics of the geometrical figures *per se*, but as their relations to a fundamental configuration, the imaginary circle at infinity common to all spheres.

When we compare the conception of geometrical figures gradually obtained in this way with the notions of ordinary (elementary) geometry, we are led to look for a general principle in accordance with which the development of both methods has been possible. This question seems the more important as, beside the elementary and the projective geometry, are arrayed a series of other methods, which albeit they are less developed, must be allowed the same right to an individual existence. Such are the geometry of reciprocal radii vectores, the geometry of rational transformations, etc., which will be mentioned and described further on.

In undertaking in the following pages to establish such a principle, we shall hardly develop an essentially new idea, but rather formulate clearly what has already been more or less definitely conceived by many others. But it has seemed the more justifiable to publish connective observations of this kind, because geometry, which is after all one in substance, has been only too much broken up in the course of its recent rapid development into a series of almost distinct theories[3], which are advancing in comparative independence of each other. At the same time I was influenced especially by a wish to present certain methods and views that have been developed in recent investigation by *Lie* and myself. Our respective investigations, different as has been the nature of the subjects treated, have led to the same generalized conception here presented; so that it has become a sort of necessity to thoroughly discuss this view and on this basis to characterize the contents and general scope of those investigations.

Though we have spoken so far only of geometrical investigations, we will include investigations on manifoldnesses of any number of dimensions[4], which have been developed from geometry by making abstraction from the geometric image, which is not essential for purely mathematical investigations[5]. In the investigation of manifoldnesses the same different types occur as in geometry; and, as in geometry, the problem is to bring out what is common and what is distinctive in investigations undertaken independently of each other. Abstractly speaking, it would in what follows be sufficient to speak throughout of manifoldnesses of n dimensions simply; but it will render the exposition simpler and more intelligible to make use of the more familiar space-perceptions. In proceeding from the consideration of geometric objects and developing the general ideas by using these as an example, we follow the path which our science has taken in its development and which it is generally best to pursue in its presentation.

A preliminary exposition of the contents of the following pages is here scarcely possible, as it can hardly be presented in a more concise form[6]; the headings of the sections will indicate the general course of thought.

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2. See Note I of the appendix.
3. See Note II.
4. See Note IV.
5. See Note III.
6. This very conciseness is a defect in the following presentation which I fear will render the understanding of it essentially more difficult. But the difficulty could hardly be removed except by a very much fuller exposition, in which the separate theories, here only touched upon, would have been developed at length.

At the end I have added a series of notes, in which I have either developed further single points, wherever the general exposition of the text would seem to demand it, or have tried to define with reference to related points of view the abstract mathematical one predominant in the observations of the text.

1 GROUPS OF SPACE-TRANSFORMATIONS. PRINCIPAL GROUP. FORMULATION OF A GENERAL PROBLEM.

The most essential idea required in the following discussion is that of a *group* of space-transformations.

The combination of any number of tranformations of space[7] is always equivalent to a single transformation. If now a given system of transformations has the property that any transformation obtained by combining any tranformations of the system belongs to that system, it shall be called a *group of transformations*[8].

An example of a group of transformations is afforded by the totality of motions, every motion being regarded as an operation performed on the whole of space. A group contained in this group is formed, say, by the rotations about one point[9]. On the other hand, a group containing the group of motions is presented by the totality of the collineations. But the totality of the dualistic transformations does not form a group; for the combination of two dualistic transformations is equivalent to a collineation. A group is, however, formed by adding the totality of the dualistic to that of the collinear transformations[10].

Now there are space-transformations by which the geometric properties of configurations in space remain entirely unchanged. For geometric properties are, from their very idea, independent of the position occupied in space by the configuration in question, of its absolute magnitude, and finally of the sense[11] in which its parts are arranged. The properties of a configuration remain therefore unchanged by any motions of space, by transformation into similar configurations, by transformation into symmetrical configurations with regard to a plane (reflection), as well as by any combination of these transformations. The totality of all these transformations we designate as the *principal group*[12] of space-transformations; *geometric properties are not changed by the transformations of the principal group*. And, conversely, *geometric properties are characterized by*

7. We always regard the totality of configurations in space as simultaneously affected by the transformations, and speak therefore of *transformations of space*. The transformations may introduce other elements in place of points, like dualistic transformations, for instance; there is no distinction in the text in this regard.

8. [This definition is not quite complete, for it has been tacitly assumed that the groups mentioned always include the inverse of every operation they contain; but, when the number of operations is infinite, this is by no means a necessary consequence of the group idea, and this assumption of ours should therefore be explicitly added to the definition of this idea given in the text.]

The ideas, as well as the notation, are taken from the *theory of substitutions*, with the difference merely that there instead of the transformations of a continuous region the permutations of a finite number of discrete quantities are considered.

9. *Camille Jordan* has formed all the groups contained in the general group of motions: *Sur les groupes de mouvements*, Annali di Matematica, vol. 2.

10. It is not at all necessary for the transformations of a group to form a continuous succession, although the groups to be mentioned in the text will indeed always have that property. For example, a group is formed by the finite series of motions which superpose a regular body upon itself, or by the infinite but discrete series which superpose a sine-curve upon itself.

11. By "sense" is to be understood that peculiarity of the arrangement of the parts of a figure which distinguishes it from the symmetrical figure (the reflected image). Thus, for example, a right-handed and a left-handed helix are of opposite "sense".

12. The fact that these tranformations form a group results from their very idea.

their remaining invariant under the transformations of the principal group. For, if we regard space for the moment as immovable, etc., as a rigid manifoldness, then every figure has an individual character; of all the properties possessed by it as an individual, only the properly geometric ones are preserved in the transformations of the principal group. The idea, here formulated somewhat indefinitely, will be brought out more clearly in the course of the exposition.

Let us now dispose with the concrete conception of space, which for the mathematician is not essential, and regard it only as a manifoldness of n dimensions, that is to say, of three dimensions, if we hold to the usual idea of the point as space element. By analogy with the transformations of space we speak of transformations of the manifoldness; they also form groups. But there is no longer, as there is in space, one group distinguished above the rest by its signification; each group is of equal importance with every other. As a generalization of geometry arises then the following comprehensive problem:

Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group.

To make use of a modern form of expression, which to be sure is ordinarily used only with reference to a particular group, the group of all the linear transformation, the problem might be stated as follows:

Given a manifoldness and a group of transformations of the same; to develop the theory of invariants relating to that group.

This is the general problem, and it comprehends not alone ordinary geometry, but also and in particular the more recent geometrical theories which we propose to discuss, and the different methods of treating manifoldnesses of n dimensions. Particular stress is to laid upon the fact that the choice of the group of transformations to be adjoined is quite arbitrary, and that consequently all the methods of treatment satisfying our general condition are in this sense of equal value.

2 GROUPS OF TRANSFORMATIONS, ONE OF WHICH INCLUDES THE OTHER, ARE SUCCESIVELY ADJOINED. THE DIFFERENT TYPES OF GEOMETRICAL INVESTIGATION AND THEIR RELATION TO EACH OTHER.

As the geometrical properties of configurations in space remain unaltered under *all* the transformations of the principal group, it is by the nature of the question absurd to inquire for such properties as would remain unaltered under only a part of those transformations. This inquiry becomes justified, however, as soon as we investigate the configurations of space in their relation to elements regarded as fixed. Let us, for instance, consider the configurations of space with reference to one particular point, as in spherical trigonometry. The problem then is to develop the properties remaining invariant under the transformations of the principal group, not for the configurations taken independently, but for the system consisting of these configurations together with the given point. But we can state this problem in this other form: to examine configurations in space with regard to such properties as remain unchanged by those transformations of the principal group which can still take place when the point is kept fixed. In other words, it is exactly the same thing whether we investigate the configurations of space taken in connection with the given point from the point of view of the principal group or whether, without any such connection, we replace the principal group by that partial group whose transformations leave the point in question unchanged.

This is a principle which we shall frequently apply; we will therefore at once formulate it generally, as follows:

Given a manifoldness and a group of transformations applying to it. Let it be proposed to examine the configurations contained in the manifoldness with reference to a given configuration. *We may, then, either add the given configuration to the system, and then we have to investigate the properties of the extended system from the point of view of the given group, or we may leave the system unextended, limiting the transformations to be employed to such transformations of the given group as leave the given configuration unchanged. (These transformations necessarily form a group by themselves.)*

Let us now consider the converse of the problem proposed at the beginning of this section. This is intelligible from the outset. We inquire what properties of the configurations of space remain unaltered by a group of transformations which contains the principal group as a part of itself. Every property found by an investigation of this kind is a geometric property of the configuration itself; but the converse is not true. In the converse problem we must apply the principle just enunciated, the principal group being now the smaller. We have then:

If the principal group be replaced by a more comprehensive group, a part only of the geometric properties remain unchanged. The remainder no longer appear as properties of the configurations of space by themselves, but as properties of the system formed by adding to them some particular configuration. This latter is defined, in so far as it is a definite[13] configuration at all, by the following condition: *The assumption that it is fixed must restrict us to those transformations of the given group which belong to the principal group.*

In this theorem is to be found the peculiarity of the recent geometrical methods to be discussed here, and their relation to the elementary method. What characterizes them is just this, that they base their investigations upon an extended group of space-transformations instead of upon the principal group. Their relation to each other is defined, when one of the groups includes the other, by a corresponding theorem. The same is true of the various methods of treating manifoldnesses of n dimensions which we shall take up. We shall now consider the separate methods from this point of view, and this will afford an opportunity to explain on concrete examples the theorems enunciated in a general form in this and the preceding sections.

3 PROJECTIVE GEOMETRY.

Every space-transformation not belonging to the principal group can be used to transfer the properties of known configurations to new ones. Thus we apply the results of plane geometry to the geometry of surfaces that can be represented (*abgebildet*) upon a plane; in this way long before the origin of a true projective geometry the properties of figures derived by projection from a given figure were inferred from those of the given figure. But projective geometry only arose as it became customary to regard the original figure as essentially identical with all those deducible from it by projection, and to enunciate the properties transferred in the process of projection in such a way as to put in evidence their independence of the change due to the projection. By this process *the group of all the projective transformations* was made the basis of the theory in the sense of §1, and that is just what created the antithesis between projective and ordinary geometry.

A course of development similar to the one here described can be regarded as possible in the case of every kind of space-transformation; we shall often refer to it again. It has gone on still further in two directions within the domain of projective geometry itself. On the one hand, the concep-

13. Such a configuration can be generated, for instance, by applying the transformations of the principal group to any arbitrary element which cannot be converted into itself by any transformation of the given group.

tion was broadened by admitting the *dualistic* transformations into the group of the fundamental transformation. From the modern point of view two reciprocal figures are not to be regarded as two distinct figures, but as essentially one and the same. A further advance consisted in extending the fundamental group of collinear and dualistic transformations by the admission in each case of the *imaginary* transformations. This step requires that the field of true space-elements has previously been extended so as to include imaginary elements, - just exactly as the admission of dualistic transformations into the fundamental group requires the simultaneous introduction of point and line as space-elements. This is not the place to point out the utility of introducing imaginary elements, by means of which alone we can attain an exact correspondence of the theory of space with the established system of algebraic operations. But, on the other hand, it must be remembered that the reason for introducing the imaginary elements is to be found in the consideration of algebraic operations and not in the group of projective and dualistic transformations. For, just as we can in the latter case limit ourselves to real transformations, since the real collineations and dualistic transformations form a group by themselves, so we can equally well introduce imaginary space-elements even when we are not employing the projective point of view, and indeed must do so in strictly algebraic investigations.

How metric properties are to be regarded from the projective point of view is determined by the general theorem of the preceding section. Metrical properties are to be considered as projective relations to a fundamental configuration, the circle at infinity [14], a configuration having the property that it is transformed into itself only by those transformations of the projective group which belong at the same time to the principal group. The proposition thus broadly stated needs a material modification owing to the limitation of the ordinary view taken of geometry as treating only of *real* space-elements (and allowing only *real* transformations). In order to conform to this point of view, it is necessary expressly to adjoin to the circle at infinity the system of real space-elements (points); properties in the sense of elementary geometry are projectively either properties of the configurations by themselves, or relations to this system of the real elements, or to the circle at infinity, or finally to both.

We might here make mention further of the way in which *von Staudt* in his “Geometrie der Lage” (Nürnberg, 1847) develops projective geometry, - i.e., that projective geometry which is based on the group containing all the real projective and dualistic transformations [15].

We know how, in his system, he selects from the ordinary matter of geometry only such features as are preserved in projective transformations. Should we desire to proceed to the consideration of metrical properties also, what we should have to do would be precisely to introduce these latter as relations to the circle at infinity. The course of thought thus brought to completion is in so far of great importance for the present considerations, as a corresponding development of geometry is possible for every one of the methods we shall take up.

4 TRANSFER OF PROPERTIES BY REPRESENTATIOINS (ABBILDUNG).

Before going further in the discussion of the geometrical methods which present themselves beside the elementary and the projective geometry, let us develop in a general form certain conside-

14. This view is to be regarded as one of the most brilliant achievements of [the French school]; for it is precisely what provides a sound foundation for that distinction between properties of position and metrical properties, which furnishes a most desirable starting-point for projective geometry.

15. The extended horizon, which includes *imaginary* transformations, was first used by *von Staudt* as the basis of his investigation in his later work, “Beiträge zur Geometrie der Lage” (Nürnberg, 1856-60).

rations which will continually recur in the course of the work, and for which a sufficient number of examples are already furnished by the subjects touched upon up to this point. The present section and the following one will be devoted to these discussions.

Suppose a manifoldness A has been investigated with reference to a group B. If, by any transformation whatever, A be then converted into a second manifoldness A', the group B of transformations, which transformed A into itself, will become a group B', whose transformations are performed upon A'. It is then a self-evident principle that *the method of treating A with reference to B at once furnishes the method of treating A' with reference to B'*, i.e., every property of a configuration contained in A obtained by means of the group B furnishes a property of the corresponding configuration in A' to be obtained by the group B'.

For example, let A be a straight line and B the ∞^3 linear transformations which transform A into itself. The method of treating A is then just what modern algebra designates as the theory of binary forms. Now, we can establish a correspondence between the straight line and a conic section A' in the same plane by projection from a point of the latter. The linear transformations B of the straight line into itself will then become, as can easily be shown, linear transformations B' of the conic into itself, i.e., the changes of the conic resulting from those linear transformations of the plane which transform the conic into itself.

Now, by the principle stated in §2[16], the study of the geometry of the conic section is the same, whether the conic be regarded as fixed and only those linear transformations of the plane which transform the conic into itself be taken into account, or whether all the linear transformations of the plane be considered and the conic be allowed to vary too. The properties which we recognized in systems of points on the conic are accordingly projective properties in the ordinary sense. Combining this consideration with the result just deduced, we have, then:

The theory of binary forms and the projective geometry of systems of points on a conic are one and the same, i.e., to every proposition concerning binary forms corresponds a proposition concerning such systems of points, and vice versa.[17]

Another suitable example to illustrate these considerations is the following. If a quadric surface be brought into correspondence with a plane by stereographic projection, the surface will have one fundamental point, - the centre of projection. In the plane there are two, - the projections of the generators passing through the centre of projection. It then follows directly: the linear transformations of the plane which leave the two fundamental points unaltered are converted by the representation (*Abbildung*) into linear transformations of the quadric itself, but only into those which leave the centre of projection unaltered. By linear transformations of the surface into itself are here meant the changes undergone by the surface when linear space-transformations are performed which transform the surface into itself. According to this, the projective investigation of a plane with reference to two of its points is identical with the projective investigation of a quadric surface with reference to one of its points. Now, if imaginary elements are also taken into account, the former is nothing else but the investigation of the plane from the point of view of elementary geometry. For the principal group of plane transformations comprises precisely those linear transformations which leave two points (the circular points at infinity) unchanged. We obtain then finally:

Elementary plane geometry and the projective investigation of a quadric surface with reference to one of its points are one and the same.

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16. The principle might be said to be applied here in a somewhat extended form.

17. Instead of the plane conic we may equally well introduce a twisted cubic, or indeed a corresponding configuration in an n-dimensional manifoldness.

These examples may be multiplied at pleasure[18]; the two here developed were chosen because we shall have occasion to refer to them again.

5 ON THE ARBITRARINESS IN THE CHOICE OF THE SPACE-ELEMENT. HESSE'S PRINCIPLE OF TRANSFERENCE. LINE GEOMETRY.

As element of the straight line, of the plane, of space, or of any manifoldness to be investigated, we may use instead of the point any configuration contained in the manifoldness, - a group of points, a curve or surface[19], etc. As there is nothing at all determined at the outset about the number of arbitrary parameters upon which these configurations shall depend, the number of dimensioins of our line, plane, space, etc., may be anything we like, according to our choice of the element. *But as long as we base our geometrical investigation upon the same group of transformations, the substance of the geometry remains unchanged.* That is to say, every proposition resulting from *one* choice of the space-element will be a true proposition under any other assumption; but the arrangement and correlation of the propositions will be changed.

The essential thing is, then, the group of transformations; the number of dimensions to be assigned to a manifoldness appears of secondary importance.

The combination of this remark with the principle of the last section furnishes many interesting applications, some of which we will now develop, as these examples seem better fitted to explain the meaning of the general theory than any lengthy exposition.

Projective geometry on the straight line (the theory of binary forms) is, by the last section, equivalent to projective geometry on the conic. Let us now regard as element on the conic the point-pair instead of the point. Now, the totality of the point-pairs of the conic may be brought into correspondence with the totality of the straight lines in the plane, by letting every line correspond to that point-pair in which it intersects the conic. By this representation (*Abbildung*) the linear transformations of the conic into itself are converted into those linear transformations of the plane (regarded as made up of straight lines) which leave the conic unaltered. But whether we consider the group of the latter, or whether we base our investigation on the totality of the linear transformations of the plane, always adjoining the conic to the plane configurations under investigation, is by §2 one and the same thing. Uniting all these considerations, we have:

The theory of binary forms and projective geometry of the plane with reference to a conic are identical.

Finally, as projective geometry of the plane with reference to a conic, by reason of the equality of its group, coincides with that projective metrical geometry which in the plane can be based upon a conic[20], we can also say:

The theory of binary forms and general projective metrical geometry in the plane are one and the same.

In the preceding consideration the conic in the plane might be replaced by the twisted cubic, etc., but we will not carry this out further. The correlation here explained between the geometry of the plane, of space, or of a manifoldness of any number of dimensions is essentially identical with the principle of transference proposed by *Hesse* (Borchardt's Journal, vol. 66).

18. For other examples, and particularly for the extension to higher dimensions of which those here presented are capable, let me refer to an article of mine: *Ueber Liniengeometrie und metrische Geometrie* (Mathematische Annalen, vol. 5), and further to *Lie*'s investigations cited later.

19. See Note III.

20. See Note V.

An example of much the same kind is furnished by the projective geometry of space; or, in other words, the theory of quaternary forms. If the straight line be taken as space-element and be determined, as in line geometry, by six homogeneous co-ordinates connected by a quadratic equation of condition, the linear and dualistic transformations of space are seen to be those linear transformations of the six variables (regarded as independent) which transform the equation of condition into itself. By a combination of considerations similar to those just developed, we obtain the following theorem:

The theory of quaternary forms is equivalent to projective measurement in a manifoldness generated by six homogeneous variables.

For a detailed exposition of this view I will refer to an article in the Math. Annalen (vol. 6): "Ueber die sogenannte Nicht-Euklidische Geometrie" [Zweiter Aufsatz], and to a note at the close of this paper[21].

To the foregoing expositions I will append two remarks, the first of which is to be sure implicitly contained in what has already been said, but needs to be brought out at length, because the subject to which it applies is only too likely to be misunderstood.

Through the introduction of arbitrary configurations as space-elements, space becomes of any number of dimensions we like. But if we then keep to the (elementary or projective) space-perception with which we are familiar, the fundamental group for the manifoldness of n dimensions is given at the outset; in the one case it is the principal group, in the other the group of projective transformations. If we wished to take a different group as a basis, we should have to depart from the ordinary (or from the projective) space-perception. Thus, while it is correct to say that, with a proper choice of space-elements, space represents manifoldnesses of any number of dimensions, it is equally important to add that *in this representation either a definite group must form the basis of the investigation of the manifoldness, or else, if we wish to choose the group, we must broaden our geometrical perception accordingly.* If this were overlooked, an interpretation of line geometry, for instance, might be sought in the following way. In line geometry the straight line has six co-ordinates: the conic in the plane has the same number of coefficients. The interpretation of line geometry would then be the geometry in a system of conics separated from the aggregation of all conics by a quadratic equation between the coefficients. This is correct, provided we take as fundamental group for the plane geometry the totality of the transformations represented by the linear transformations of the coefficients of the conic which transform the quadratic equation into itself. But if we retain the elementary or the projective view of plane geometry, we have no interpretation at all.

The second remark has reference to the following line of reasoning: Suppose in space some group or other, the principal group for instance, be given. Let us then select a single configuration, say a point, or a straight line, or even an ellipsoid, etc., and apply to it all the transformations of the principal group. We thus obtain an infinite manifoldness with a number of dimensions in general equal to the number of arbitrary parameters contained in the group, but reducing in special cases, namely, when the configuration originally selected has the property of being transformed into itself by an infinite number of the transformations of the group. Every manifoldness generated in this way may be called, with reference to the generating group, a *body*[22].

21. See Note VI.

22. In choosing this name I follow the precedent established by *Dedekind* in the theory of numbers, where he applies the name *body* to a system of numbers formed from given elements by given operations (*Dirichlet*'s Vorlesungen über Zahlentheorie, 2. Aufl.)

If now we desire to base our investigations upon the group, selecting at the same time certain definite configurations as space-elements, and if we wish to represent uniformly things which are of like characteristics, *we must evidently choose our space-elements in such a way that their manifoldness either is itself a body or can be decomposed into bodies.* This remark, whose correctness is evident, will find application later (§9). This idea of a body will come under discussion once more in the closing section, in connection with certain related ideas[23].

6 THE GEOMETRY OF RECIPROCAL RADII. INTERPRETATION OF $x + iy$.

With this section we return to the discussion of the various lines of geometric research, which was begun in §§2 and 3.

As a parallel in many respects to the processes of projective geometry, we may consider a class of geometric investigations in which the transformation by reciprocal radii vectores (geometric inversion) is continually employed. To these belong investigations on the so-called eyelides and other anallagmatic surfaces, on the general theory of orthogonal systems, likewise on potential, etc. It is true that the processes here involved have not yet, like projective geometry, been united into a special geometry, *whose fundamental group would be the totality of the transformations resulting from a combination of the principal group with geometric inversion*; but this may be ascribed to the fact that the theories named have never happened to receive a connected treatment. To the individual investigators in this line of work some such systematic conception can hardly have been foreign.

The parallel between this geometry of reciprocal radii and projective geometry is apparent as soon as the question is raised; it will therefore be sufficient to call attention in a general way to the following points:

In projective geometry the elementary ideas are the point, line, and plane. The circle and the sphere are but special cases of the conic section and the quadric surface. The region at infinity of elementary geometry appears as a plane; the fundamental configuration to which elementary geometry is referred is an imaginary conic at infinity.

In the geometry of reciprocal radii the elementary ideas are the point, circle, and sphere. The line and the plane are special cases of the latter, characterized by the property that they contain a point which, however, has no further special significance in the theory, namely, the point at infinity. If we regard this point as fixed, elementary geometry is the result.

The geometry of reciprocal radii admits of being stated in a form which places it alongside of the theory of binary forms and of line geometry, provided the latter be treated in the way indicated in the last section. To this end we will for the present restrict our observations to plane geometry and therefore to the geometry of reciprocal radii in the plane[24].

We have already referred to the connection between elementary plane geometry and the projective geometry of the quadric surface with one distinctive point (§4). If we disregard the distinctive

23. [In the text sufficient attention is not paid to the fact that the proposed group may contain so-called self-conjugate subgroups. If a geometrical configuration remain unchanged by the operations of a self-conjugate subgroup, the same is true for all configurations into which it is transformed by the operations of the whole group; i.e., for all configurations of the body arising from it. But a body so formed would be absolutely unsuited to represent the operations of the group. In the text, therefore, are to be admitted only bodies formed of space-elements which remain unchanged by no self-conjugate subgroup of the given group whatever.]

24. The geometry of reciprocal radii on the straight line is equivalent to the projective investigation of the line, as the transformations in question are the same. Thus in the geometry of reciprocal radii, also, we can speak of the anharmonic ratio of four points on a line and of four points on a circle.

point, that is to say, if we consider the projective geometry on the surface by itself, we have a representation of the geometry of reciprocal radii in the plane. For it is easy to see[25] that to the group of geometric inversion in the plane corresponds by virtue of the representation (*Abbildung*) of the quadric surface the totality of the linear transformations of the latter into itself. We have, therefore,

The geometry of reciprocal radii in the place and the projective geometry on a quadric surface are one and the same; and, similarly:

The geometry of reciprocal radii in space is equivalent to the projective treatment of a manifoldness represented by a quadratic equation between five homogeneous variables.

By means of the geometry of reciprocal radii space geometry is thus brought into exactly the same connection with a manifoldness of four dimensions as by means of [projective] geometry with a manifoldness of five dimensions.

The geometry of reciprocal radii in the plane, if we limit ourselves to *real* transformations, admits of an interesting interpretation, or application, in still another direction. For, representing the complex variable $x + iy$ in the plane in the usual way, to its linear transformations corresponds the group of geometric inversion, with the above-mentioned restriction to real operations[26]. But the investigation of functions of a complex variable, regarded as subject to any linear transformations whatever, is merely what, under a somewhat different mode of representation, is called the theory of binary forms. In other words:

The theory of binary forms finds interpretation in the geometry of reciprocal radii in the real plane, and precisely in the way in which complex values of the variables are represented.

From the plane we will ascend to the quadric surface, to return to the more familiar circle of ideas of the projective transformations. As we have taken into consideration only real elements of the plane, it is not a matter of indifference how the surface is chosen; it can evidently not be a ruled surface. In particular, we may regard it as a spherical surface, - as is customary for the interpretation of a complex variable, - and obtain in this way the theorem:

The theory of the binary forms of a complex variable finds representation in the projective geometry of the real spherical surface.

I could not refrain from setting forth in a note[27] how admirably this interpretation illustrates the theory of binary cubics and quartics.

7 EXTENSION OF THE PRECEDING CONSIDERATIONS. LIE'S SPHERE GEOMETRY.

With the theory of binary forms, the geometry of reciprocal radii, and line geometry, which in the foregoing pages appear co-ordinated and only distinguished by the number of variables, may be connected certain further developments, which shall now be explained. In the first place, these developments are intended to illustrate with new exmaples the idea that the group determining the treatment of given subjects can be extended indefinitely; but, in the second place, the intention was

25. See the article already cited; *Ueber Liniengeometrie und metrische Geometrie*, Mathematische Annalen, vol. 5.

26. [The language of the text is inexact. To the linear transformations $z' = \frac{\alpha z+\beta}{\gamma z+\delta}$ (where $z' = x' + iy'$, $z = x + iy$) correspond only those operations of the group of geometric inversion by which no reversion of the angles takes place (in which the two circular points of the plane are not interchanged). If we wish to include the whole group of geometric inversion, we must, in addition to the transformations mentioned, take account of the other (not less important) ones given by the formula $z' = \frac{\alpha \bar{z}+\beta}{\gamma \bar{z}+\delta}$ (where again $z' = x' + iy'$, but $\bar{z} = x - iy$).]

27. See Note VII.

particularly to explain the relation to the views here set forth of certain considerations presented by *Lie* in a recent article[28]. The way by which we here arrive at *Lie*'s sphere geometry differs in this respect from the one pursued by *Lie*, that he proceeds from the conceptions of line geometry, while we assume a smaller number of variables in our exposition. This will enable us to be in agreement with the usual geometric perception and to preserve the connection with what precedes. The investigation is independent of the number of variables, as *Lie* himself has already pointed out (Göttinger Nachrichten, 1871, Nos. 7, 22). It belongs to that great class of investigations concerned with the projective discussion of quadratic equations between any number of variables, - investigations upon which we have already touched several times, and which will repeatedly meet us again (see §10, for instance).

I proceed from the connection established between the real plane and the sphere by stereographic projection. In §5 we connected plane geometry with the geometry on a conic section by making the straight line in the plane correspond to the point-pair in which it meets the conic. Similarly we can establish a connection between space geometry and the geometry on the sphere, by letting every plane of space correspond to the circle in which it cuts the sphere. If then by stereographic projection we transfer the geometry on the sphere from the latter to the plane (every circle being thereby transformed into a circle), we have the following correspondence:

the space geometry whose element is the plane and whose group is formed of the linear transformations converting a sphere into itself, and

the plane geometry whose element is the circle and whose group is the group of geometric inversion.

The former geometry we will now generalize in two directions by substituting for its group a more comprehensive group. The resulting extension may then be immediately transferred to plane geometry by representation (*Abbildung*).

Instead of those linear transformations of space (regarded as made up of planes) which convert the sphere into itself, it readily suggests itself to select either the totality of the *linear* transformations of space, or the totality of those plane-transformations which leave the sphere unchanged [in a sense yet to be examined]; in the former case we dispense with the sphere, in the latter with the linear character of the transformations. The former generalization is intelligible without further explanation; we will therefore consider it first and follow out its importance for plane geometry. To the second case we shall return later, and shall then in the first place have to determine the most general transformation of that kind.

Linear space-transformations have the common property of converting pencils and sheafs of planes into like pencils and sheafs. Now, transferred to the sphere, the pencil of planes gives a pencil of circles, i.e., a system of ∞^1 circles with common intersections; the sheaf of planes gives a sheaf of circles, i.e., a system of ∞^2 circles perpendicular to a fixed circle (the circle whose plane is the polar plane of the point common to the planes of the given sheaf). Hence to linear space-transformations there correspond on the sphere, and furthermore in the plane, circle-transformations characterized by the property that they convert pencils and sheafs of circles into the same[29] *The plane geometry which employs the group of transformations thus obtained is the representation of ordinary projective space geometry.* In this geometry the point cannot be used as element of the plane, for the points do not form a *body* (§5) for the chosen group of transformations; but circles shall be chosen as elements.

28. *Partielle Differentialgleichungen und Complexe*, Mathematische Annalen, vol. 5.
29. Such transformations are considered in Grassmann's *Ausdehnungslehre* (edition of 1862, p. 278).

In the case of the second extension named, the first question to be settled is with regard to the nature of the group of transformations in question. The problem is, to find plane-transformations converting every [pencil] of planes whose [axis touches] the sphere into a like [pencil]. For brevity of expression, we will first consider the reciprocal problem and, moreover, go down a step in the number of dimensions; we will therefore look for point-transformations of the plane which convert every tangent to a given conic into a like tangent. To this end we regard the plane with its conic as the representation of a quadric surface projected on the plane from a point of space not in the surface in such a way that the conic in question represents the boundary curve. To the tangents to the conic correspond the generators of the surface, and the problem is reduced to that of finding the totality of the point-transformations of the surface into itself by which generators remain generators.

Now, the number of these transformations is, to be sure, ∞^n, where n may have any value. For we only need to regard the point on the surface as intersection of the generators of the two systems, and to transform each system of lines into itself in any way whatever. But among these are in particular the linear transformations, and to these alone will we attend. For, if we had to do, not with a surface, but with an n-dimensional manifoldness represented by a quadratic equation, the linear transformations alone would remain, the rest would disappear[30].

These linear transformations of the surface into itself, transferred to the plane by projection (other than stereographic), give two-valued point-transformations, by which from every tangent to the boundary conic is produced, it is true, a tangent, but from every other straight line in general a conic having double contact with the boundary curve. This group of transformations will be conveniently characterized by basing a projective measurement upon the boundary conic. The transformations will then have the property of converting points whose distance apart is zero by this measurement, and also points whose distance from a given point is constant, into points having the same properties.

All these considerations may be extended to any number of variables, and can in particular be applied to the original inquiry, which had reference to the sphere and plane as elements. We can then give the result an especially perspicuous form, because the angle formed by two planes according to the projective measurement referred to a sphere is equal to the angle in the ordinary sense formed by the circles in which they intersect the sphere.

We thus obtain upon the sphere, and furthermore in the plane, a group of circle-transformations having the property that *they convert circles which are tangent to each other (include a zero angle), and also circles making equal angles with another circle, into like circles.* The group of these transformations contains on the sphere the linear transformations, in the plane the transformations of the group of geometric inversion[31].

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30. If the manifoldness be stereographically projected, we obtain the well-known theorem: in regions of n dimensions (even in space) there are no isogonal point-transformations except the transformations of the group of geometric inversion. In the plane, on the other hand, there are any number besides. See the articles by *Lie* already cited.

31. [Perhaps the addition of some few analytic formulae will materially help to explain the remarks in the text. Let the equation of the sphere, which we project stereographically on the plane, be in ordinary tetrahedral co-ordinates:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

The x's satisfying this equation of condition we then interpret as tetracyclic co-ordinates in the plane.

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0$$

will be the general circular equation of the plane. If we compute the radius of the circle represented in this way, we come upon the square root $\sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$, which we may denote by iu_3. We can now regard the circles as elements of the plane. The group of geometric inversion is then represented by the totality of those homogeneous

The circle geometry based on this group is analogous to the sphere geometry which *Lie* has devised for space and which appears of particular importance for investigations on the curvature of surfaces. It includes the geometry of reciprocal radii in the same sense as the latter includes elementary geometry.

The circle- (sphere-) transformations thus obtained have, in particular, the property of converting circles (spheres) which touch each other into circles (spheres) having the same property. If we regard all curves (surfaces) as envelopes of circles (spheres), then it results from this fact that curves (surfaces) which touch each other will always be transformed into curves (surfaces) having the same property. The transformations in question belong, therefore, to the class of *contact-transformations* to be considered from a general standpoint further on, i.e., transformations under which the contact of point-configurations is an invariant relation. The first circle-transformations mentioned in the present section, which find their parallel in corresponding sphere-transformations, are not contact transformations.

While these two kinds of generalization have here been applied only to the geometry of reciprocal redii, they nevertheless hold in a similar way for line geometry and in general for the projective investigation of a manifoldness defined by a quadratic equation, as we have already indicated, but shall not develop further in this connection.

8 ENUMERATION OF OTHER METHODS BASED ON A GROUP OF POINT-TRANSFORMATIONS.

Elementary geometry, the geometry of reciprocal radii, and likewise projective geometry, if we disregard the dualistic transformations connected with the interchange of the space-element, are included as special cases among the large number of conceivable methods based on groups of point-transformations. We will here mention especially only the three following methods, which agree in this respect with those named. Though these methods are far from having been developed into independent theories in the same degree as projective geometry, yet they can clearly be traced in the more recent investigations[32].

8.1 The Group of Rational Transformations.

In the case of rational transformations we must carefully distinguish whether they are rational for *all* points of the region under consideration, viz., of space, or of the plane, etc., or only for the points of a manifoldness contained in the region, viz., a surface or curve. The former alone are to be employed when the problem is to develop a geometry of space or of the plane in the meaning hitherto understood; the latter obtain a meaning, from our point of view, only when we wish to study the geometry on a given surface or curve. The same distinction is to be drawn in the case of the *analysis situs* to be discussed presently.

The investigations in both subjects up to this time have been occupied mainly with transformations of the second kind. Since in these investigations the question has not been with regard to the geometry on the surface or curve, but rather to find the criteria for the transformability of two

linear transformations of u_1,u_2,u_3,u_4, by which $u_1^2+u_2^2+u_3^2+u_4^2$ is converted into a multiple of itself. But the extended group which corresponds to *Lie*'s sphere geometry consists of those homogeneous linear transformations of the five variables u_1,u_2,u_3,u_4, which convert $u_1^2+u_2^2+u_3^2+u_4^2+u_5^2$ into a multiple of itself.]

32. [Groups with a finite number of parameters having been treated in the examples hitherto taken up, the so-called infinite groups will now be the subject of consideration in the text.]

surfaces or curves into each other, they are to be excluded from the sphere of the investigations here to be considered[33]. For the general synopsis here outlined does not embrace the entire field of mathematical research, but only brings certain lines of thought under a common point of view.

Of such a geometry of rational transformations as must result on the basis of the transformations of the first kind, only a beginning has so far been made. In the region of the first grade, viz., on the straight line, the rational transformations are identical with the linear transformations and therefore furnish nothing new. In the plane we know the totality of rational transformations (the Cremona transformations); we know that they can be produced by a combination of quadratic transformations. We know further certain invariant properties of plane curves [with reference to the totality of rational transformations], viz., their deficiency, the existence of moduli; but these considerations have not yet been developed into a geometry of the plane, properly speaking, in the meaning here intended. In space the whole theory is still in its infancy. We know at present but few of the rational transformations, and use them to establish correspondences between known and unknown surfaces.

8.2 Analysis situs.

In the so-called analysis situs we try to find what remains unchanged under transformations resulting from a combination of infinitesimal distortions. Here, again, we must distinguish whether the whole region, all space, for instance, is to be subjected to the transformations, or only a manifoldness contained in the same, a surface. It is the transformations of the first kind on which we could found a space geometry. Their group would be entirely different in constitution from the groups heretofore considered. Embracing as it does all transformations compounded from (real) infinitesimal point-transformations, it necessarily involves the limitation to real space-elements, and belongs to the domain of arbitrary functions. This group of transformations can be extended to advantage by combining it with those real collineations which at the same time affect the region at infinity.

8.3 The Group of all Point-transformations.

While with reference to this group no surface possesses any individual characteristics, as any surface can be converted into any other by transformations of the group, the group can be employed to advantage in the investigation of higher configurations. Under the view of geometry upon which we have taken our stand, it is a matter of no importance that these configurations have hitherto not been regarded as geometric, but only as analytic, configurations, admitting occasionally of geometric application, and, furthermore, that in their investigation methods have been employed (these very point-transformations, for instance) which we have only recently begun to consciously regard

33. [From another point of view they are brought back again, which I did not yet know in 1872, very nicely into connection with the considerations in the text. Given any algebraic configuration (curve, or surface, etc.), let it be transferred into a higher space by introducing the ratios

$$\phi_1 : \phi_2 : \ldots : \phi_P$$

of the intergrands of the first species belonging to it as homogeneous co-ordinates. In this space we have then simply to take the group of homogeneous linear transformations as a basis for our further considerations. See various articles by *Brill*, *Nöther*, and *Weber*, and (to mention a single recent article) my own paper: *Zur Theorie der Abelschen Functionen* in vol. 36 of the Math. Annalen.]

as geometric transformations. To these analytic configurations belong, above all, homogeneous differential expressions, and also partial differential equations. For the general discussion of the latter, however, as will be explained in detail in the next section, the more comprehensive group of all contact-transformations seems to be more advantageous.

The principal theorem in force in the geometry founded on the group of all point-transformations is this: *that for an infinitesimal portion of space a point-transformation always has the value of a linear transformation.* Thus the developments of projective geometry will have their meaning for infinitesimals; and, whatever be the choice of the group for the treatment of the manifoldness, *in this fact lies a distinguishing characteristic of the projective view.*

Not having spoken for some time of the relation of methods of treatment founded on groups, one of which includes the other, let us now give one more example of the general theory of §2. We will consider the question how projective properties are to be understood from the point of view of "all point-transformations," disregarding here the dualistic transformations which, properly speaking, form part of the group of projective geometry. This question is identical with the other question, What condition differentiates the group of linear point-transformations from the totality of point-transformations? What characterizes the linear group is this, that to every plane it makes correspond a plane; it contains those transformations under which the manifoldness of planes (or, what amounts to the same thing, of straight lines) remains unchanged. *Projective geometry is to be obtained from the geometry of all point-transformations by adjoining the manifoldness of planes, just as elementary is obtained from projective geometry by adjoining the imaginary circle at infinity.* Thus, for instance, from the point of view of all point-transformations the designation of a surface as an algebraic surface of a certain order must be regarded as an invariant relation to the manifoldness of planes. This becomes very clear if we connect, as *Grassmann* (Crelle's Journal, vol. 44) does, the generation of algebraic configurations with their construction by lines.

9 ON THE GROUP OF ALL CONTACT-TRANSFORMATIONS.

Particular cases of contact-transformations have been long known; *Jacobi* has even made use of the most general contact-transformations in analytical investigations, but an effective geometrical interpretation has only been given them by recent researches of *Lie*'s[34]. It will therefore not be superfluous to explain here in detail what a contact-transformation is. In this we restrict ourselves, as hitherto, to point-space with its three dimensions.

By a contact-transformation is to be understood, analytically speaking, any substitution which expresses the values of the variables x,y,z and their partial derivatives $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$ in terms of new variables x',y',z',p',q'. It is evident that such substitutions, in general, convert surfaces that are in contact into surfaces in contact, and this accounts for the name. Contact-transformations are divided into three classes (the point being taken as space-element), viz., those in which *points* correspond to the ∞^3 points (the point-transformations just considered); those converting the points into curves; lastly, those converting them into surfaces. This classification is not to be regarded as essential, inasmuch as for other ∞^3 space-elements, say for planes, while a division into three classes again occurs, it does not coincide with the division occurring under the assumption of points as elements.

34. See, in particular, the article already cited: *Ueber partielle Differentialgleichungen und Complexe*, Mathematische Annalen, vol. 5. For the details given in the text in regard to partial differential equations I am indebted mainly to oral communications of *Lie*'s; see his note, *Zur Theorie partieller Differentialgleichungen*, Göttinger Nachrichten, October 1872.

If a point be subjected to all contact-transformations it is converted into the totality of points, curves, and surfaces. Only in their entirety, then, do points, curves, and surfaces form a *body* of our group. From this may be deduced the general rule that the formal treatment of a problem from the point of view of all contact-transformations (e.g., the theory of partial differential equations considered below) must be incomplete if we operate only with point- (or plane-) co-ordinates, for the very reason that the chosen space-elements do not form a body.

If, however, we wish to preserve the connection with the ordinary methods, it will not do to introduce as space-elements all the individual configurations contained in the body, as their number is ∞^∞. This makes it necessary to introduce in these considerations as space-element not the point, curve, or surface, but the "surface-element," i.e., the system of values x,y,z,p,q. Each contact-transformation converts every surface-element into another; the ∞^5 surface-elements accordingly form a body.

From this point of view, point, curve, and surface must be uniformly regarded as aggregates of surface-elements, and indeed of ∞^2 elements. For the surface is covered by ∞^2 elements, the curve is tangent to the same number, through the point pass the same number. But these aggregates of ∞^2 elements have another characteristic property in common. Let us designate as the *united position* of two consecutive surface-elements x,y,z,p,q and $x + dx$,$y + dy$,$z + dz$,$p + dp$,$q + dq$ the relation defined by the equation

$$dz - pdx - qdy = 0.$$

Thus point, curve, and surface agree in being *manifoldnesses of* ∞^2 *elements, each of which is united in position with the* ∞^1 *adjoining elements*. This is the common characteristic of point, curve, and surface; and this must serve as the basis of the analytical investigation, if the group of contact-transformations is to be used.

The united position of consecutive elements is an invariant relation under any contact-transformation whatever. And, conversely, contact-transformations may be defined as *those substitutions of the five variables* x,y,z,p,q, *by which the relation*

$$dz - pdx - qdy = 0$$

is converted into itself. In these investigations space is therefore to be regarded as a manifoldness of five dimensions; and this manifoldness is to be treated by taking as fundamental group the totality of the transformations of the variables which leave a certain relation between the differentials unaltered.

First of all present themselves as subjects of investigation the manifoldnesses defined by one or more equations between the variables, i.e., *by partial differential equations of the first order, and systems of such equations*. It will be one of the principal problems to select out of the manifoldnesses of elements satisfying given equation systems of ∞^1, or of ∞^2, elements which are all united in position with a neighboring element. A question of this kind forms the sum and substance of the problem of the solution of a partial differential equation of the first order. It can be formulated in the following way: to select from among the ∞^4 elements satisfying the equation all the twofold manifoldnesses of the given kind. The problem of the complete solution thus assumes the definite form: to classify in some way the ∞^4 elements satisfying the equation into ∞^2 manifoldnesses of the given kind.

It cannot be my intention to pursue this consideration of partial differential equations further; on this point I refer to *Lie*'s articles already cited. I will only point out one thing further, that from the point of view of the contact-transformations a partial differential equation of the first order has no invariant, that every such equation can be converted into any other, and that therefore linear equations in particular have no distinctive properties. Distinctions appear only when we return to the point of view of the point-transformations.

The groups of contact-transformations, of point-transformations, finally of projective transformations, may be defined in a uniform manner which should here not be passed over[35]. Contact-transformations have already been defined as those transformations under which the united position of consecutive surface-elements is preserved. But, on the other hand, point-transformations have the characteristic property of converting consecutive line-elements which are united in position into line-elements similarly situated; and, finally, linear and dualistic transformations maintain the united position of consecutive connex-elements. By a connex-element is meant the combination of a surface-element with a line-element contained in it; consecutive connex-elements are said to be united in position when not only the point but also the line-element of one is contained in the surface-element of the other. The term connex-element (though only preliminary) has reference to the configurations recently introduced into geometry by *Clebsch*[36] and represented by an equation containing simultaneously a series of point-coordinates as well as a series of plane- and a series of line-coordinates whose analogues in the plane *Clebsch* denotes as connexes.

10 ON MANIFOLDNESSES OF ANY NUMBER OF DIMENSIONS.

We have already repeatedly laid stress on the fact that in connecting the expositions thus far with space-perception we have only been influenced by the desire to be able to develop abstract ideas more easily through dependence on graphic examples. But the considerations are in their nature independent of the concrete image, and belong to that general field of mathematical research which is designated as the theory of manifoldnesses of any dimensions, - called by *Grassmann* briefly "theory of extension" (Ausdehnungslehre). How the transference of the preceding development from space to the simple idea of a manifoldness is to be accomplished is obvious. It may be mentioned once more in this connection that in the abstract investigation we have the advantage over geometry of being able to choose arbitrarily the fundamental group of transformations, while in geometry a minimum group - the principal group - was given at the outset.

We will here touch, and that very briefly, only on the following three methods:

10.1 The Projective Method or Modern Algebra (Theory of Invariants).

Its group consists of the totality of linear and dualistic transformations of the variables employed to represent individual configurations in the manifoldness; it is the generalization of projective geometry. We have already noticed the application of this method in the discussion of infinitesimals in a manifoldness of one more dimension. It includes the two other methods to be mentioned, in so far as its group includes the groups upon which those methods are based.

10.2 The Manifoldness of Constant Curvature.

The notion of such a manifoldness arose in *Riemann*'s theory from the more general idea of a manifoldness in which a differential expression in the variables is given. In his theory the group

35. I am indebted to a remark of *Lie*'s for these definitions.

36. Göttinger Abhandlungen, 1872 (vol. 17): *Ueber eine Fundamentalaufgabe der Invariantentheorie*, and especially Göttinger Nachrichten, 1872, No. 22: *Ueber ein neues Grundgebilde der analytischen Geometrie der Ebene.*

consists of the totality of those transformations of the variables which leave the given expression unchanged. On the other hand, the idea of a manifoldness of constant curvature presents itself when a projective measurement is based upon a given quadratic equation between the variables. From this point of view as compared with *Riemann*'s the extension arises that the variables are regarded as complex; the variability can be limited to the real domain afterwards. Under this head belong the long series of investigations touched on in §§5, 6, 7.

10.3 The Plane Manifoldness.

Riemann designates as a plane manifoldness one of constant zero curvature. Its theory is the immediate generalization of elementary geometry. Its group can, like the principal group of geometry, be separated from out the group of the projective method by supposing a configuration to remain fixed which is defined by two equations, a linear and a quadratic equation. We have then to distinguish between real and imaginary if we wish to adhere to the form in which the theory is usually presented. Under this head are to be counted, in the first place, elementary geometry itself, then for instance the recent generalizations of the ordinary theory of curvature, etc.

CONCLUDING REMARKS.

In conclusion we will introduce two further remarks closely related to what has thus far been presented, - one with reference to the analytic form in which the ideas developed in the preceding pages are to be represented, the other marking certain problems whose investigation would appear important and fruitful in the light of the expositions here given.

Analytic geometry has often been reproached with giving preference to arbitrary elements by the introduction of the system of co-ordinates, and this objection applied equally well to every method of treating manifoldnesses in which individual configurations are characterized by the values of variables. But while this objection has been too often justified owing to the defective way in which, particularly in former times, the method of co-ordinates was manipulated, yet it disappears when the method is rationally treated. The analytical expressions arising in the investigation of a manifoldness with reference to its group must, from their meaning, be independent of the choice of the co-ordinate system; and the problem is then to clearly set forth this independence analytically. That this can be done, and how it is to be done, is shown by modern algebra, in which the abstract idea of an invariant that we have here in view has reached its clearest expression. It possesses a general and exhaustive law for constructing invariant expressions, and operates only with such expressions. This object should be kept in view in any formal (analytical) treatment, even when other groups than the projective group form the basis of the treatment[37]. For the analytical formulation should, after all, be congruent with the conceptions whether it be our purpose to use it only as a precise and perpicuous expression of the conceptions, or to penetrate by its aid into still unexplored regions.

The further problems which we wished to mention arise on comparing the views here set forth with the so-called *Galois* theory of equations.

In the *Galois* theory, as in ours, the interest centres on groups of transformations. The objects to which the transformations are applied are indeed different; there we have to do with a finite number of discrete elements, here with the infinite number of elements in a continuous manifoldness. But

37. [For instance, in the case of the groups of rotations of three-dimensional space about a fixed point, such a formalism is furnished by quaternions.]

still the comparison may be pursued further owing to the identity of the group-idea[38], an I am the more ready to point it out here, as it will enable us to characterize the position to be awarded to certain investigations begun by *Lie* and myself[39] in accordance with the views here developed.

In the Galois theory, as it is presented for instance in *Serret*'s "Cours d'Algèbre supérieure" or in *C. Jordan*'s "Traité des Substitutions," the real subject of investigation is the group theory of substitution theory itself, from which the theory of equations results as an application. Similarly we require a *theory of transformations*, a theory of the groups producible by transformations of any given characteristics. The ideas of commutativity, of similarity, etc., will find application just as in the theory of substitutions. As an application of the theory of transformations appears that treatment of a manifoldness which results from taking as a basis the groups of transformations.

In the theory of equations the first subjects to engage the attention are the symmetric functions of the coefficients, and in the next place those expressions which remain unaltered, if not under all, yet under a considerable number of permutations of the roots. In treating a manifoldness on the basis of a group our first inquiry is similarly with regard to the bodies (§5), viz., the configurations which remain unaltered under all the transformations of the group. But there are configurations admitting not all but some of the transformations of the group, and they are next of particular interest from the point of view of the treatment based on the group; they have distinctive characteristics. It amounts, then, to distinguishing in the sense of ordinary geometry symmetric and regular bodies, surfaces of revolution and helicoidal surfaces. If the subject be regarded from the point of view of projective geometry, and if it be further required that the transformations converting the configurations into themselves shall be commutative, we arrive at the configurations considered by *Lie* and myself in the article cited, and the general problem proposed in §6 of that article. The determination (given in §§1, 3 of that article) of all groups of an infinite number of commutative linear transformations in the plane forms a part of the general theory of transformations named above[40].

38. I should like here to call to mind *Grassmann*'s comparison of combinatory analysis and extensive algebra in the introduction to the first edition of his "Ausdehnungslehre" (1844).

39. See our article: *Ueber diejenigen Curven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen*, Mathematische Annalen, Bd. IV.

40. I must refrain from referring in the text to the fruitfulness of the consideration of infinitesimal transformations in the theory of differential equations. In §7 of the article cited, *Lie* and I have shown that ordinary differential equations which admit the same infinitesimal transformations present like difficulties of integration. How the considerations are to be employed for partial differential equations, *Lie* has illustrated by various examples in several places; for instance, in the article named above (Math. Annalen, vol. 5). See in particular the proceedings of the Christiania Academy, May 1872.

[At this time I may be allowed to refer to the fact that it is exactly the two problems mentioned in the text which have influenced a large part of the further investigations of *Lie* and myself. I have already called attention to the appearance of the two first volumes of *Lie*'s "Theorie der Transformationsgruppen." Of my own work might be mentioned the later researches on regular bodies, on elliptic modular functions, and on single-valued functions with linear transformations into themselves, in general. An account of the first of these was given in a special work: "Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen fünften Grades" (Leipzig, 1884); an exposition of the theory of the elliptic modular functions, elaborated by *Dr. Fricke* is in course of publication.]

NOTES.

I. On the Antithesis between the Synthetic and the Analytic Method in Modern Geometry.

The distinction between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form in both. We choose therefore in the text as common designation of them both the term *projective geometry*. Although the synthetic method has more to do with space-perception and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulae of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations. On the other hand, the advantage to original research of a well formulated analysis should not be underestimated, - an advantage due to its moving, so to speak, in advance of the thought. But it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aid of analysis is only a first, though a very important, step.

II. Division of Modern Geometry into Theories.

When we consider, for instance, how persistently the mathematical physicist disregards the advantages afforded him in many cases by only a moderate cultivation of the projective view, and how, on the other hand, the student of projective geometry leaves untouched the rich mine of mathematical truths brought to light by the theory of the curvature of surfaces, we must regard the present state of mathematical knowledge as exceedingly incomplete and, it is to be hoped, as transitory.

III. On the Value of Space-perception.

When in the text we designated space-perception as something incidental, we meant this with regard to the purely mathematical contents of the ideas to be formulated. Space-perception has then only the value of illustration, which is to be estimated very highly from the pedagogical standpoint, it is true. A geometric model, for instance, is from this point of view very instructive and interesting.

But the question of the value of space-perception in itself is quite another matter. I regard it as an independent question. There is a true geometry which is not, like the investigations discussed in the text, intended to be merely an illustrative form of more abstract investigations. Its problem is to grasp the full reality of the figures of space, and to interpret - and this is the mathematical side of the question - the relations holding for them as evident results of the axioms of space-perception. A model, whether constructed and observed or only vividly imagined, is for this geometry not a means to an end, but the subject itself.

This presentation of geometry as an independent subject, apart from and independent of pure mathematics, is nothing new, of course. But it is desirable to lay stress explicitly upon this point of view once more, as modern research passes it over almost entirely. This is connected with the fact that, *vice versa*, modern research has seldom been employed in investigations on the form-relations of space-configurations, while it appears well adapted to this purpose.

IV. On Manifoldnesses of any Number of Dimensions.

That space, regarded as the locus of points, has only three dimensions, does not need to be discussed from the mathematical point of view; but just as little can anybody be prevented from that point of view from claiming that space really has four, or any unlimited number of dimensions, and that we are only able to perceive three. The theory of manifoldnesses, advancing as it does with the course of time more and more into the foreground of modern mathematical research, is by its nature fully independent of any suh claim. But a nomenclature has become established in this theory which has indeed been derived from this idea. Instead of the elements of a manifoldness we speak of the points of a higher space, etc. The nomenclature itself has certain advantages, in that it facilitates the interpretation by calling to mind the perceptions of geometry. but it has had the unfortunate result of causing the whide-spread opinion that investigations on manifoldnesses of any number of dimensions are inseparably connected with the above-mentioned idea of the nature of space. Nothing is more unsound than this opinion. The mathematical investigations in question would, it is true, find an immediate application to geometry, if the idea were correct; but their value and purport is absolutely independent of this idea, and depends only on their own mathematical contents.

It is quite another matter when *Plücker* shows how to regard actual space as a manifoldness of any number of dimensions by introducing as space-element a configuration depending on any number of parameters, a curve, surface, etc. (see §5 of the text).

The conception in which the element of a manifoldness (of any number of dimensions) is regarded as analogous to the point in space was first developed, I suppose, by *Grassmann* in his "Ausdehnungslehre" (1844). With him the thought is absolutely free of the above-mentioned idea of the nature of space; this idea goes back to occasional remarks by *Gauss*, and became more widely known through *Riemann*'s investigations on manifoldnesses, with which it was interwoven.

Both conceptions - *Grassmann*'s as well as *Plücker*'s - have their own peculiar advantages; they can be alternately employed with good results.

V. On the So-called Non-Euclidean Geometry.

The projective metrical geometry alluded to in the text is essentially coincident, as recent investigations have shown, with the metrical geometry which can be developed under non-acceptance of the axiom of parallels, and is to-day under the name of non-Euclidean geometry widely treated and discussed. The reason why this name has not been mentioned at all in the text, is closely related to the expositions given in the preceding note. With the name non-Euclidean geometry have been associated a multitude of non-mathematical ideas, which have been as zealously cherished by some as resolutely rejected by others, but with which our purely mathematical considerations have nothing to do whatever. A wish to contribute towards clearer ideas in this matter has occasioned the following explanations.

The investigations referred to on the theory of parallels, with the results growing out of them, have a definite value for mathematics from two points of views.

They show, in the first place, - and this function of theirs may be regarded as concluded once for all, - that the axiom of parallels is not a mathematical consequence of the other axioms usually assumed, but the expression of an essentially new principle of space-perception, which has not been touched upon in the foregoing investigations. Similar investigations could and should be performed with regard to every axiom (and not alone in geometry); an insight would thus be obtained into the mutual relation of the axioms.

But, in the second place, these investigations have given us an important mathematical idea, - the idea of a manifoldness of constant curvature. This idea is very intimately connected, as has already been remarked and in §10 of the text discussed more in detail, with the projective measurement which has arisen independently of any theory of parallels. Not only is the study of this measurement in itself of great mathematical interest, admitting of numerous applications, but it has the additional feature of including the measurement given in geometry as a special (limiting) case and of teaching us how to regard the latter from a broader point of view.

Quite independent of the views set forth is the question, what reasons support the axiom on parallels, i.e., whether we should regard it as absolutely given, as some claim, or only as approximately proved by experience, as others say. Should there be reasons for assuming the latter position, the mathematical investigations referred to afford us then immediately the means for constructing a more exact geometry. But the inquiry is evidently a philosophical one and concerns the most general foundations of our understanding. The mathematician as such is not concerned with this inquiry, and does not wish his investigations to be regarded as dependent on the answer given to the question from the one of the other point of view[41].

VI. Line Geometry as the Investigation of a Manifoldness of Constant Curvature.

In combining line geometry with the projective measurement in a manifoldness of five dimensions, we must remember that the straight lines represent elements of the manifoldness which, metrically speaking, are at infinity. It then becomes necessary to consider what the value of a system of a projective measurement is for the elements at infinity; and this may here be set forth somewhat at length, in order to remove any difficulties which might else seem to stand in the way of conceiving of line geometry as a metrical geometry. We shall illustrate these expositions by the graphic example of the projective measurement based on a quadric surface.

Any two points in space have with respect to the surface an absolute invariant, - the anharmonic ratio of the two points together with the two points of intersection of the line joining them with the surface. But when the two points move up to the surface, this anharmonic ratio becomes zero independently of the position of the points, except in the case where the two points fall upon a generator, when it becomes indeterminate. This is the only special case which can occur in their relative position unless they coincide, and we have therefore the theorem:

The projective measurement in space based upon a quadric surface does not yet furnish a measurement for the geometry on the surface.

This is connected with the fact that by linear transformations of the surface into itself any three points of the surface can be brought into coincidence with three others[42].

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41. [To the explanations in the text I should like to add here two supplementary remarks.

In the first place, when I say that the mathematician as such has no stand to take place on the philosophical question, I do not mean to say that the philosopher can dispense with the mathematical developments in treating the aspect of the question which interests him; on the contrary, it is my decided conviction that a study of these developments is the indispensable prerequisite to every philosophical discussion of the subject.

Secondly, I have not meant to say that my *personal* interest is exhausted by the mathematical aspect of the question. For my conception of the subject, in general, let me refer to a recent paper: "Zur Nicht-Euklidischen Geometrie" (Math. Annalen, vol. 37).]

42. These relations are different in ordinary metrical geometry; for there it is true that two points at infinity have an absolute invariant. The contradiction which might thus be found in the enumeration of the linear transformations of the surface at infinity into itself is removed by the fact that the translations and transformations of similarity

If a measurement on the surface itself be desired, we must limit the group of transformations, and this result is obtained by supposing any arbitrary point of space (or its polar plane) to be fixed. Let us first take a point not on the surface. We can then project the surface from the point upon a plane, when a conic will appear as the boundary curve. Upon this conic we can base a projective measurement in the plane, which must then be transferred back to the surface[43]. This is a measurement with constant curvature in the true sense, and we have then the theorem:

Such a measurement on the surface is obtained by keeping fixed a point not on the surface.

Correspondingly, we find[44]:

A measurement with zero curvature on the surface is obtained by choosing as the fixed point a point of the surface itself.

In all these measurements on the surface the generators of the surface are lines of zero length. The expression for the element of arc on the surface differs therefore only by a factor in the different cases. There is no absolute element of arc upon the surface; but we can of course speak of the angle formed by two directions on the surface.

All these theorems and considerations can now be applied immediately to line geometry. Line-space itself admits at the outset no measurement, properly speaking. A measurement is only obtained by regarding a linear complex as fixed; and the measurement is of constant or zero curvature, according as the complex is a general or a special one (a line). The selection of a particular complex carries with it further the acceptation of an absolute element of arc. Independently of this, the directions to adjoining lines cutting the given line arc of zero length, and we can besides speak of the angle between any two directions[45].

VII. On the Interpretation of Binary Forms.

We shall now consider the graphic illustration which can be given to the theory of invariants of binary cubics and biquadratics by taking advantage of the representation of $x + iy$ on the sphere.

A binary cubic f has a cubic covariant Q, a quadratic covariant Δ, and an invariant R[46]. From f and Q a whole system of covariant sextics $Q^2 + \lambda R f^2$ may be compounded, among them being Δ^3. It can be shown[47] that every covariant of the cubic must resolve itself into such groups of six points. Inasmuch as λ can assume complex values, the number of these covariants is ∞^2[48].

The whole system of forms thus defined can now be represented upon the sphere as follows. By a suitable linear transformation of the sphere into itself let the three points representing f be converted into three equidistant points of a great circle. let this great circle be denoted as the equator, and let the three points f have the longitudes 0°, 120°, 240°. Then Q will be represented by the points of the equator whose longitudes are 60°, 180°, 300°; Δ by the two poles. Every form $Q^2 + \lambda R f^2$ is represented by six points, whose latitude and longitude are given in the following table, where α and β are arbitrary numbers:

α	α	α	$-\alpha$	$-\alpha$	$-\alpha$
β	$120° + \beta$	$240° + \beta$	$-\beta$	$120° - \beta$	$240° - \beta$

contained in this group do not alter the region at infinity at all.

43. See §7 of the text.
44. See §4 of the text
45. See the article *Ueber Liniengeometrie und metrische Geometrie*, Math. Annalen, vol. 5, p. 271.
46. See in this connection the corresponding sections of Clebsch's "Theorie der binären Formen."
47. By considering the linear transformations of f into itself. See Math. Annalen, vol. 4, p. 352.
48. [See Beltrami, *Ricerche sulla geometria delle forme binarie cubiche*, Memorie dell' Accademia di Bologna, 1870.]

In studying the variation of these systems of points on the sphere, it is interesting to see how they give rise to f and Q (each reckoned twice) and Δ (reckoned three times).

A biquadratic f has a biquadratic covariant H, a sextic covariant T, and two invariants i and j. Particularly noteworthy is the pencil of biquadratic forms $iH + \lambda jf$, all belonging to the same T, among them being the three quadratic factors into which T can be resolved, each reckond twice.

Let the centre of the sphere now be taken as the origin of a set of rectangular axes OX, OY, OZ. Their six points of intersection with the sphere make up the form T. The four points of a set $iH + \lambda jf$ are given by the following table, x,y,z being the co-ordinates of any point of the sphere:

$$\begin{array}{rrr} x, & y, & z, \\ x, & -y, & -z, \\ -x, & y, & -z, \\ -x, & -y, & z. \end{array}$$

The four points are in each case the vertices of a symmetrical tetrahedron, whose opposite edges are bisected by the co-ordinate axes; and this indicates the rôle played by T in the theory of biquadratic equations as the resolvent of $iH + \lambda jf$.

ERLANGEN, *October*, 1872.

第一次习题课

讲义

Eudoxus 逼近法

若 $\{a, b\}$ 与 $\{a', b'\}$ 皆可公度，即存在 $\{p, q, m, n\}$ 使得

$$\frac{a}{b} = \frac{p}{q} \quad ; \quad \frac{a'}{b'} = \frac{m}{n}$$

则 $[a, b]$ 与 $[a', b']$ 大小比较即为有理数大小比较.

Q：若 $\{a, b\}$，$\{a', b'\}$ 有一组不可公度.
如何比较 $a:b$ 与 $a':b'$ 大小？

D：(1) 若仅有一组不可公度. w.l.o.g.
令 $\frac{a}{b} = \frac{p}{q}$. 则可通过如下方式比较

$$\frac{a'}{b'} \begin{cases} > \frac{p}{q} \\ < \frac{p}{q} \end{cases} \Leftrightarrow \begin{matrix} a' > \frac{p}{q} b' \\ a' < \frac{p}{q} b' \end{matrix} \Leftrightarrow \begin{matrix} qa' > pb' \\ qa' < pb' \end{matrix}$$

(II) 若 $\{a, b\}$、$\{a', b'\}$ 皆不可度公度

若存在 $\frac{m}{n}$ s.t

$$\frac{a}{b} > \frac{m}{n} > \frac{a'}{b'}$$，(用I之定义3与有理数比较)。

则可定义 $\frac{a}{b} > \frac{a'}{b'}$

则剩下的情形为：对于任意 $\frac{m}{n}$，$\frac{a}{b}$ 与 $\frac{a'}{b'}$ 有恒同大小关系，我们证明这种情形下，$\left|\frac{a}{b} - \frac{a'}{b'}\right|$ 可以任意小，故可以定义此时 $\frac{a}{b} = \frac{a'}{b'}$。

阿基米德公理：任给 $\{a, b\}$，总存在足够大的 N. s.t $n \geq N$ 时，

$nb > a$。

□

定理1：若 $\{a,b\}$ 不可公度，则任给 n，恒存在 m，s.t.

$$\frac{m}{n}<\frac{a}{b}<\frac{m+1}{n}.$$

Pf：由阿……公理，存在 N，使得 $a<N\left(\frac{b}{n}\right)$

取 $(m+1)$ 为最小的满足以上的数

即可

#

若 $\frac{a}{b}$，$\frac{a'}{b'}$ 对于任意分数有恒同大小关系，则

$$\frac{m}{n}<\frac{a}{b}\left(\frac{a'}{b'}\right)<\frac{m+1}{n}$$

故 $\left|\frac{a}{b}-\frac{a'}{b'}\right|<\frac{1}{n}$. 对于任意 $n\in\mathbb{N}$.

故可定义 $\frac{a}{b}=\frac{a'}{b'}$

利用逼近法证矩形面积公式.

若 $a:u$ 与 $b:u$ 至少有一组不可公度

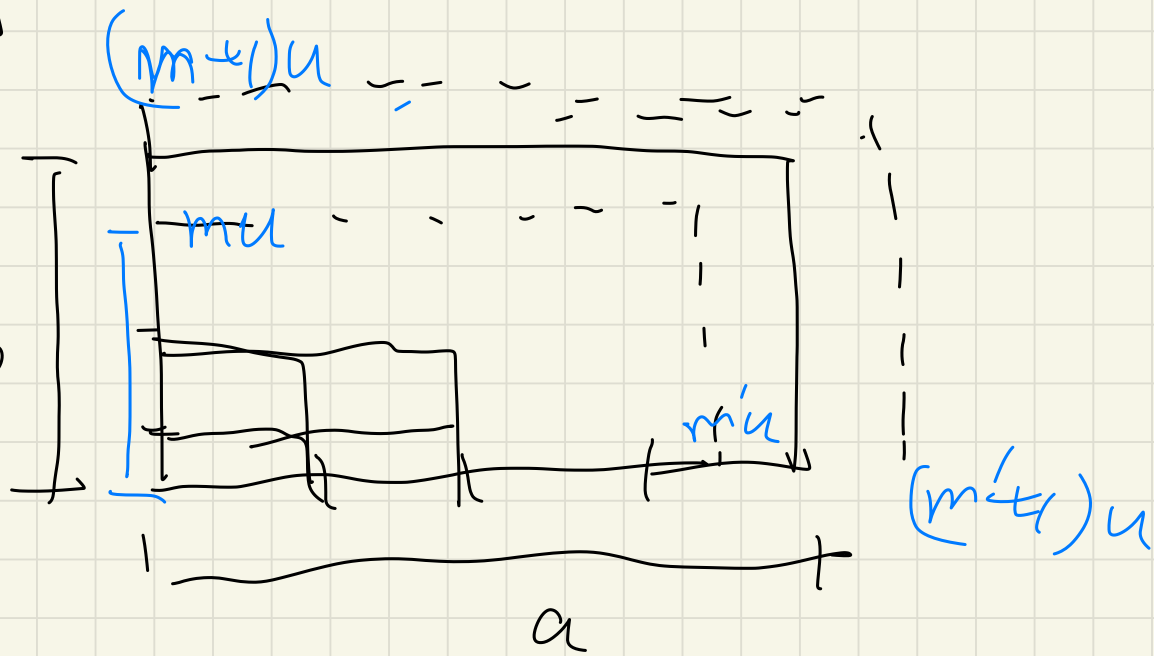
时 $\forall n$ 存在 m, m' s.t

$$\frac{m}{n} < \frac{a}{u} < \frac{m+1}{n}, \quad \frac{m'}{n} < \frac{b}{u} < \frac{m'+1}{n}$$

则

$$\frac{m}{n} \cdot \frac{m'}{n} < \frac{a}{u} \cdot \frac{b}{u} < \frac{(m+1)}{n} \cdot \frac{(m'+1)}{n}$$

同时



有

$$\frac{mm'}{n^2} < \square(a:b) : \square(u:u) < \frac{(m+1)(m'+1)}{n^2}$$

故有

$$\left| \square(a:b):\square(u,u) - (a:u)(b:u) \right|$$

$$< \frac{(m+1)(m'+1) - mm'}{n^2}$$

$$= \frac{1}{n}\left(\frac{m}{n} + \frac{m'}{n} + \frac{1}{n}\right)$$

$$< \frac{1}{n}\left(\cancel{\alpha}\ \frac{a}{u} + \frac{b}{u} + \frac{1}{n}\right)$$

故 n 足够大，相差足够小，由此得

$$\square(a;b):\square(u,u) = (a:u)\cdot(b:u)$$

矩形面积公式得证.

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