2024年秋，有限元方法II，上机作业1

截至时间：2024/12/8，晚上12点

要求:

- 用TeX写上机报告(中英文均可), 包含必要的数值结果讨论, 页数上限18.
- 本次上机作业中, 须自己组装刚度矩阵, 推荐使用软件包iFEM. 请仔细阅读iFEM（或其他类似程序）中的实现方法，特别需要关注Matlab程序的向量化操作.
- 截止时间前将程序和上机报告的源码发送至snwu@math.pku.edu.cn

Consider the following second-order elliptic equation

$$\begin{cases} -\nabla \cdot (a(x)\nabla u) = f & \text{in } \Omega \subset \mathbb{R}^2, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the coefficient $a(x)$ satisfies the uniform ellipticity condition, i.e., there exist constants $\alpha_0, \alpha_1 > 0$ such that $\alpha_0 \leq a(x) \leq \alpha_1$. In this lab, you are required to implement the $\mathcal{P}_3$ Hermite element.

Problem 1. On the uniform meshes over the domain $\Omega = [-1,1]^2$, consider a smooth coefficient $a(x) = 1 + 0.5\sin(\pi x)$. Choose a smooth solution u and compute f and g accordingly based on this smooth solution. Report the errors in H^1, L^2, W^1_∞, and L^∞ norms to verify the correctness of your code.

Problem 2. On a uniform mesh over the L-shaped domain $[-1,1]^2 \setminus [0,1] \times [-1,0]$, choose $a(x) = 1$ and the exact solution

$$u = (1 - r^2)v(r,\theta), \quad v(r,\theta) = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\theta\right). \tag{2}$$

Report the errors in H^1, L^2, W^1_∞, and L^∞ norms.

Problem 3 On a uniform mesh over the L-shaped domain, consider a given right-hand side $f = 1$ and boundary condition $g = 0$ (in this case, the exact solution is unknown). Freely choose some smooth functions $a(x)$ that meet the uniform ellipticity condition, and evaluate the accuracy of your code in different norms. (In this case, a reference solution can be obtained on a very fine mesh.)

Problem 4. Use adaptive meshes (please describe your algorithms for ESTIMATE, MARK, and REFINE): Report the convergence histories for Problem 2 & 3.

2024年秋，有限元方法II，上机作业2

截至时间：2025/01/12，晚上12点

要求:

- 用TeX写上机报告(中英文均可), 包含必要的数值结果讨论, 页数上限18.
- 本次上机作业不限制程序语言与软件包.
- 截止时间前将程序和上机报告的源码发送至snwu@math.pku.edu.cn

Consider the following mixed formulation of the Poisson equation

$$\begin{cases} \boldsymbol{p} - \nabla u = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ -\operatorname{div} \boldsymbol{p} = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Use the mixed finite element spaces $\mathrm{RT}_k - \mathcal{P}_k^{-1}$ and $\mathrm{BDM}_{k+1} - \mathcal{P}_k^{-1}$ to solve (1). Here, the source term f and boundary data g are derived from the exact solution u. Perform the computations on *quasi-uniform* meshes.

Problem 1. Choose $\Omega = (-1,1)^2$ and a *smooth* solution u. Report the errors for $\boldsymbol{p}$ in the $\boldsymbol{H}(\mathrm{div})$ norm and L^2 norm, as well as the errors for u in the L^2 norm.

Problem 2. For the aforementioned setting, implement the post-processing of u and report the errors in the L^2 norm.

Problem 3. Consider a non-convex domain defined as

$$\Omega := \{(x,y) \in (-1,1)^2 : 0 < \theta < \pi/\beta\}, \quad \text{where } \frac{1}{2} \leq \beta < 1.$$

Set the exact solution to be

$$u = r^{\beta} \sin(\beta\theta). \tag{2}$$

- Report the errors for $\boldsymbol{p}$ in the $\boldsymbol{H}(\mathrm{div})$ norm and L^2 norm, as well as the errors for u in the L^2 norm for different values of β.
- Report the L^2 errors of the post-processed numerical solution.

- For the subdomain Ω' obtained by removing the region near the reentrant corner, for example, $\Omega' = \{(x, y) \in \Omega \setminus [-c_0, c_0]^2\}$ for some given $0 < c_0 < 1$, report the $L^2(\Omega')$ errors of u before and after the post-processing.

Remark: The case in which $k = 0$ (RT_0-$\mathcal{P}_0^{-1}$ and BDM_1-$\mathcal{P}_0^{-1}$) is required. At least one high-order case (e.g., RT_1-$\mathcal{P}_1^{-1}$ or BDM_2-$\mathcal{P}_1^{-1}$) is required.