2024秋，有限元方法II，作业1

交作业时间：2024/09/26

The Mathematical Theory of Finite Element Methods:

- Chapter 0: 0.x.6, 0.x.11, 0.x.12, 0.x.13
- Chapter 1: 1.x.13, 1.x.20, 1.x.21, 1.x.42

Supplementary Questions:

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exists a constant $C(n)$ depending only on n such that for any $0 \le \lambda < n$,

$$\max_{x\in\Omega} \int_\Omega |x-y|^{-\lambda} \mathrm{d}y \le C(n)(n-\lambda)^{-1}|\Omega|^{1-\lambda/n}.$$

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Show that for any $q \ge 1$,

$$\|v\|_{L^q(\Omega)} \le C(n) q^{1-\frac{1}{n}} |\Omega|^{\frac{1}{q}} \|v\|_{W^{1,n}(\Omega)} \qquad \forall v \in W_0^{1,n}(\Omega).$$

2024秋，有限元方法II，作业2

交作业时间：2024/10/24

The Mathematical Theory of Finite Element Methods:

- Chapter 3: 3.x.10, 3.x.12, 3.x.13, 3.x.15, 3.x.18, 3.x.35
- Chapter 4: 4.x.6, 4.x.12, 4.x.16

Supplementary Questions:

1. Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element satisfying

   (a) There exists $\sigma > 0$ such that $\sigma h_K \leq \rho_K \leq h_K$;
   (b) $\mathcal{P}_{m-1} \subset \mathcal{P} \leq W_\infty^m(K)$;
   (c) $\mathcal{N} \subset (C^l(\bar{K}))'$;
   (d) $(K, \mathcal{P}, \mathcal{N})$ is affine-interpolation equivalent to the reference finite element $(\widehat{K}, \widehat{\mathcal{P}}, \widehat{\mathcal{N}})$.

   Here, $m + l - n/p > 0$. For $0 \leq j \leq m-1$, the constant t_j satisfies

   $$\begin{cases} p \leq t_j \leq \frac{(n-1)p}{n-(m-j)p} & \text{if } (m-j)p < n, \\ p \leq t_j < \infty & \text{if } (m-j)p = n, \\ p \leq t_j \leq \infty & \text{if } (m-j)p > n. \end{cases}$$

   Then, there exists a constant C independent of K such that, for any $0 \leq j \leq m-1$ and $v \in W_p^m(K)$

   $$\sum_{|\alpha|=j} \|\partial^\alpha (v - Iv)\|_{L^{t_j}(\partial K)} \leq C h_K^{m-j+(n-1)/t_j - n/p} |v|_{W_p^m(K)}.$$

2024秋，有限元方法II，作业3

交作业时间：2024/11/12

The Mathematical Theory of Finite Element Methods:

- Chapter 5: 5.x.8, 5.x.9, 5.x.16, 5.x.17, 5.x.21
- Chapter 10: 10.x.3, 10.x.4, 10.x.7, 10.x.11, 10.x.15, 10.x.16

Supplementary Questions:

1. Conduct the duality argument for the IPDG scheme

$$a_h^-(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,$$

where $V_h = \{v \in L^2(\Omega) : v|_T \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}_h\}$.

2024秋，有限元方法II，作业4

交作业时间：2024/11/28

1. Given a triangle T with diameter h_T, if $r \in \mathbb{R}$, show that
$$h_T \|r\|_{L^2(T)} \lesssim \|r\|_{H^{-1}(T)},$$
where the hidden constant depends on the shape-regularity of T.

2. Let V, Q be Banach spaces, $B : V \to Q'$ be a bound linear operator, $Z := N(B)$. For any subspace $S \subset V$, define
$$S^\circ := \{f \in V' \mid \langle f, v\rangle = 0,\ \forall v \in S\}.$$
For any subspace $F \subset V'$, define
$$^\circ F := \{v \in V \mid \langle f, v\rangle = 0,\ \forall f \in F\}.$$
   - Show that S° and $^\circ F$ are closed.
   - Show that $^\circ(S^\circ) = S$ if and only if S is closed in V; And $(^\circ F)^\circ = F$ if and only if F is closed in V'.
   - Show that $^\circ R(B') = Z$.
   - Show that $R(B') = Z^\circ$ if and only if $R(B')$ is closed in V'.

3. Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \to H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then, the following indentity holds
$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}.$$

4. Define
$$\mathbb{H}_k(K) := \{\underset{\sim}{q} \in \underset{\sim}{\mathcal{P}}_k(K) \mid \mathrm{div}\underset{\sim}{q} = 0 \quad \text{and} \quad \underset{\sim}{q} \cdot \underset{\sim}{n}|_{\partial K} = 0\}.$$
Show that in 2D, $\mathbb{H}_k(K) = \mathrm{curl}(b_K \mathcal{P}_{k-2}(K))$. (Hint: You can directly utilize the dimension formula for $\mathbb{H}_k(K)$ discussed in class.)

5. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. For any $\underset{\sim}{v} \in (L^2(\Omega))^2$, show that there exists $\psi \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)$, such that
$$\underset{\sim}{v} = \nabla\psi + \mathrm{curl}\phi,$$
and
$$\|\nabla\phi\|_{L^2} + \|\nabla\psi\|_{L^2} \lesssim \|\underset{\sim}{v}\|_{L^2}.$$
(Hint: $R(\mathrm{curl}) = N(\mathrm{div})$ on simply connected domain.)

6. Let $\Omega \subset \mathbb{R}^3$ be the half space $x_3 < 0$ and Γ be the space $x_3 = 0$. Given a vector $\underset{\sim}{\chi} = (\chi_1, \chi_2, \chi_3)^T \in H(\text{curl}, \Omega)$, show that $\text{div}_\Gamma \text{Tr}\underset{\sim}{\chi} = \text{curl}\underset{\sim}{\chi} \cdot \underset{\sim}{n}|_\Gamma$, where $\text{Tr}\chi = \underset{\sim}{\chi} \times \underset{\sim}{n}$.

7. For any $\underset{\sim}{w} \in \underset{\sim}{\mathcal{H}}_k$ ($\mathcal{H}_k$ stands for the homogeneous polynomial space of order k), define

$$\underset{\sim}{\eta} = \frac{\text{curl}\underset{\sim}{w}}{k+1}, \quad \mu = \frac{\underset{\sim}{x} \cdot \underset{\sim}{w}}{k+1}.$$

show that

$$-\underset{\sim}{x} \times \underset{\sim}{\eta} + \nabla \mu = \underset{\sim}{w}.$$

8. Show that for any $\underset{\sim}{p}_k \in \underset{\sim}{\mathcal{P}}_k$, there exists a decomposition

$$\underset{\sim}{p}_k = \underset{\sim}{w}_{k-1} + \nabla \theta,$$

where $\underset{\sim}{w}_{k-1} \in \underset{\sim}{\mathcal{P}}_{k-1} + \underset{\sim}{x} \times \underset{\sim}{\mathcal{P}}_{k-1}$, and $\theta \in \mathcal{P}_{k+1}$.

2024秋，有限元方法II，作业5

交作业时间：2024/12/20

1. Let $\hat{K} \subset \mathbb{R}^n$, F be a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $K = F(\hat{K})$. Assume that F is globally invertible on K and its Jacobian DF is invertible. For any $\underset{\sim}{\hat{q}} \in (C^\infty(\hat{K}))^n$, define

$$\mathcal{G}(\underset{\sim}{\hat{q}})(x) := \frac{1}{J(\hat{x})} DF(\hat{x}) \underset{\sim}{\hat{q}}(\hat{x}), \quad \hat{x} = F^{-1}(x),$$

where $J(x) = |\det DF(\hat{x})|$. Show that

$$\mathrm{div} \underset{\sim}{q} = \frac{1}{J} \widehat{\mathrm{div}} \underset{\sim}{\hat{q}}.$$

Here, $\widehat{\mathrm{div}}$ means the derivatives on $\hat{x}$.

2. Let $v = \mathcal{F}(\hat{v}) := \hat{v}(F^{-1}(x))$, and $\underset{\sim}{q} = \mathcal{G}(\underset{\sim}{\hat{q}})$, where $F(\hat{x}) = B\hat{x} + b_0$ is an affine mapping. Show that

$$\begin{aligned} \int_K \underset{\sim}{q} \cdot \mathrm{grad} v \, \mathrm{d}x &= \int_{\hat{K}} \underset{\sim}{\hat{q}} \cdot \widehat{\mathrm{grad}} \hat{v} \, \mathrm{d}\hat{x}, \\ \int_K v \, \mathrm{div} \underset{\sim}{q} \, \mathrm{d}x &= \int_{\hat{K}} \hat{v} \, \widehat{\mathrm{div}} \underset{\sim}{\hat{q}} \, \mathrm{d}\hat{x}, \\ \int_{\partial K} \underset{\sim}{q} \cdot \underset{\sim}{n} v \, \mathrm{d}s &= \int_{\partial \hat{K}} \underset{\sim}{\hat{q}} \cdot \underset{\sim}{\hat{n}} \hat{v} \, \mathrm{d}\hat{s}. \end{aligned}$$

3. Given any $\epsilon > 0$, let $X = H(\mathrm{curl}) \cap \underset{\sim}{H}^{1/2+\epsilon}$. Given a Lipschitz domain Ω, show that there exists $\delta(\epsilon, \Omega) > 0$ so that $\mathrm{curl} X(\Omega) \subset \underset{\sim}{L}^{2+\delta(\epsilon,\Omega)}(\Omega)$. Let $W = H(\mathrm{div}) \cap \underset{\sim}{L}^{2+\delta(\epsilon)}$. Choose one of the following to prove the commutative diagram:

$$\begin{CD} X(K) @>{\mathrm{curl}}>> W(K) \\ @V{\Pi_k^N}VV @VV{\Pi_k^{RT}}V \\ N_k(K) @>{\mathrm{curl}}>> RT_k(K) \end{CD} \qquad \begin{CD} X(K) @>{\mathrm{curl}}>> W(K) \\ @V{\Pi_{k+1}^{NC}}VV @VV{\Pi_k^{RT}}V \\ NC_{k+1}(K) @>{\mathrm{curl}}>> RT_k(K) \end{CD}$$

Find the similar version for $BDM_k(K)$ (no need to show the proof).

4. Show that the following two inequalities are equivalent

$$\|p\|_{L^2(\Omega)} \lesssim \|p\|_{H^{-1}(\Omega)} + \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \quad \forall p \in L^2(\Omega), \tag{1}$$

$$\|p\|_{L^2(\Omega)} \lesssim \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}(\Omega)} \qquad \forall p \in L_0^2(\Omega). \tag{2}$$

5. Let Ω be a connected domain with a Lipschitz boundary. Assume that $\Gamma_D \subset \partial\Omega$ satisfies $\mathrm{meas}(\Gamma_D) \neq 0$. Show that

$$\|\utilde{v}\|_{\utilde{H}^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\utilde{v})\|, \quad \forall \utilde{v} \in \utilde{H}_D^1(\Omega),$$

where $\utilde{H}_D^1(\Omega) := \{\utilde{v} \in \utilde{H}^1(\Omega) : \ \utilde{v} = 0 \text{ on } \Gamma_D\}$.

6. For the Stokes pair $\utilde{\mathcal{P}}_1^{\mathrm{CR}}$-$\mathcal{P}_0^{-1}$, prove that it satisfies the following discrete inf-sup condition:

$$\inf_{q_h \in Q_h} \sup_{\utilde{v}_h \in V_h} \frac{(\mathrm{div}_h \utilde{v}_h, q_h)}{\|\utilde{v}_h\|_{1,h} \|q_h\|_{L^2}} \gtrsim 1,$$

where div_h denotes the piecewise divergence, and $\|\cdot\|_{1,h}$ represents the piecewise H^1-norm. For the aforementioned element, provide the error estimate for the following numerical scheme ($\nu = 1$):

$$\begin{cases} 2(\boldsymbol{\varepsilon}_h(\utilde{u}_h), \boldsymbol{\varepsilon}_h(\utilde{v}_h)) - (\mathrm{div}_h \utilde{v}_h, p_h) = (\utilde{f}, \utilde{v}_h) & \forall \utilde{v}_h \in V_h, \\ \qquad\qquad\qquad\qquad -(\mathrm{div}_h \utilde{u}_h, q_h) = 0 & \forall q_h \in Q_h. \end{cases}$$

7. Let $V = \utilde{H}_0^1(\Omega)$. Consider $V^0 = \{\utilde{v} \in V \mid \int_T \mathrm{div}\utilde{v}\,\mathrm{d}x = 0, \forall T \in \mathcal{T}_h\}$, define $\Pi_2 : V^0 \to B(\mathrm{gradQ_h})$ by

$$\left.\begin{cases} \Pi_2 \utilde{v}|_T \in B(\mathrm{gradQ_h})|_T, \\ \displaystyle\int_T \mathrm{div}(\Pi_2 \utilde{v} - \utilde{v})\,\mathrm{d}x = 0, \quad \forall q_h \in Q_h|_T, \end{cases}\right. \quad \forall T \in \mathcal{T}_h.$$

Show that $\|\Pi_2 \utilde{v}\|_1 \lesssim \|\utilde{v}\|_1$ for $\utilde{v} \in V^0$.

8. Consider the Stokes problem with homogeneous Dirichlet boundary condition:

$$\begin{aligned} -\Delta \utilde{u} + \nabla p &= \utilde{f} \quad \text{in } \Omega, \\ -\mathrm{div}\utilde{u} &= 0 \quad \text{in } \Omega, \\ \utilde{u}|_{\partial\Omega} &= \utilde{0}. \end{aligned}$$

Let $V = \utilde{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. Given a stable Stokes pair $V_h \times Q_h \subset V \times Q$, we can obtain the following energy energy estimate

$$\|\utilde{u} - \utilde{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \lesssim \inf_{\utilde{v}_h \in V_h} \|\utilde{u} - \utilde{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}.$$

Assume further the approximation property of $V_h \times Q_h$:

$$\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{z} - \underset{\sim}{v}_h\|_{H^1} \lesssim h \|\underset{\sim}{z}\|_{H^2} \quad \forall \underset{\sim}{z} \in \underset{\sim}{H}^2(\Omega),$$
$$\inf_{q_h \in Q_h} \|r - q_h\|_{L^2} \lesssim h \|r\|_{H^1}, \quad \forall r \in H^1(\Omega).$$

Duality argument: Find appropriate regularity assumption of the dual problem:

$$-\Delta \underset{\sim}{z} + \nabla r = \underset{\sim}{\theta} \quad \text{in } \Omega,$$
$$\text{div} \underset{\sim}{z} = 0 \quad \text{in } \Omega,$$
$$\underset{\sim}{z}|_{\partial\Omega} = \underset{\sim}{0}.$$

so that one can obtain the L^2 estimate of $\underset{\sim}{u}$:

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2} \lesssim h \left(\inf_{\underset{\sim}{v}_h \in V_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2} \right).$$